Convex Integration Solutions for Stochastic Transport Equation [1]

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[1] joint work with Kazuo Yamazaki (Texas).

Deterministic Transport Equation

- Transport equation: $\partial_t \varrho + \mathrm{div}(\varrho u) = 0$, $\varrho|_{t=0} = \varrho_0$.
- Assume that u is incompressible, i.e., $\mathrm{div}\, u = 0$.
- Diperna-Lions'89:
  - Let $p, \tilde{p} \in [1, \infty]$, and $u \in L^1_t W^{1,\tilde{p}}_x(\mathbb{T}^d), \mathrm{div}\, u = 0$.
  - For any $\varrho_0 \in L^p(\mathbb{T}^d)$, there exists unique distributional solution $\varrho \in L^\infty_t L^p_x(\mathbb{T}^d)$, provided $\frac{1}{p} + \frac{1}{\tilde{p}} \leq 1$.
- What happens if

$$\frac{1}{p} + \frac{1}{\tilde{p}} > 1, \quad (\text{or } \frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d}).$$

- Non-uniqueness results by Modena-Sattig-Szekelyhidi, Brue-Colombo-De Lellis.

Deterministic Transport Equation

- Result:
  - Let $p, \tilde{p} \in [1, \infty)$, and $\frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d}$.
  - Given any smooth, mean-zero $\bar{\varrho}$, and a smooth divergence free $\bar{u}$: consider the set

$$E = \{t \in [0, T] : \partial_t \bar{\varrho} + \operatorname{div}(\bar{\varrho}\bar{u}) = 0\}.$$

  Then there exists $\varrho \in C_t^0 L_x^p$, and $u \in C_t^0 L_x^{p'} \cap C_t^0 W_x^{1,\tilde{p}}$ s.t
  - $\partial_t \varrho + \operatorname{div}(\varrho u) = 0, \operatorname{div} u = 0$ in $\mathcal{D}'$.
  - $\varrho(t) = \bar{\varrho}(t)$, and $u(t) = \bar{u}(t)$ for all $t \in E$.
- Non-uniqueness:
  - Pick a smooth, mean-zero function a, and

$$\bar{u} \equiv 0, \quad \bar{\varrho}(t,x) = \begin{cases} 0, & t \in [0, T/3] \\ a(x), & t \in [2T/3, T]. \end{cases}$$

  - Note that $(\bar{\varrho}, \bar{u})$ is a solution on $[0, T/3] \cup [2T/3, T] \subset E$.
  - Existence of a solution s.t. $\varrho(0) = 0$, and $\varrho(T) = a$.

Stochastic Transport-Diffusion Equation

- Stochastic transport equation:

$$d\rho(t,x) + \mathrm{div}(u(t,x)\rho(t,x))\,dt = \Delta\rho\,dt + G(\rho)dB(t), \quad \nabla\cdot u = 0,$$
$$\rho(0,x) = \rho^{\mathrm{in}}(x),\ x\in\mathbb{T}^d$$

- Three different types of noise:
  1. additive noise; i.e., $G(\rho)dB = dB$ where B is a certain GG^*-Wiener process;
  2. linear multiplicative noise in Itô's interpretation; i.e., $G(\rho)dB = \rho dB$ where B is a $\mathbb{R}$-valued Wiener process;
  3. transport noise in Stratonovich's interpretation; i.e., $G(\rho)dB = -\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\rho\circ dB_i$ where $B=(B_1,\dots,B_d)$ is a Brownian motion.

- Challenges:
  - Stochastic perturbation may provide a regularizing effect on problems.
  - Whether the noise makes the critical exponent of uniqueness different.
  - Establish probabilistically strong solutions.

Typical Result: Additive noise

- Suppose that $d \geq 3$, B is a GG^*-Wiener process, and

$$Tr((-\Delta)^{\frac{d}{2}+2\varsigma} GG^*) < \infty \quad \text{for some} \quad \varsigma > 0.$$

- Given $T > 0$, $K > 1$, and $\kappa \in (0,1)$, there exists a $\mathbb{P}$-a.s. strictly positive stopping time T_L such that

$$\mathbb{P}(\{T_L \geq T\}) > \kappa$$

- There exist an $(\mathcal{F}_t)_{t\geq 0}$-adapted process u that is divergence-free such that

$$u \in L^\infty(\Omega; C([0, T_L]; L^{p'}(\mathbb{T}^d))) \cap L^\infty(\Omega; C([0, T_L]; W^{1,\tilde{p}}(\mathbb{T}^d))),$$

an $(\mathcal{F}_t)_{t\geq 0}$-adapted process

$$\rho \in L^\infty(\Omega; C([0, T_L]; L^p(\mathbb{T}^d))),$$

and $\rho^{\text{in}} \in L^p(\mathbb{T}^d)$ that is deterministic such that ρ solves the underlying equation.

Proposition

- Iteration Scheme: For all $\delta > 0$, for all (ρ_0, u_0, R_0) smooth such that

$$\begin{aligned} \partial_t \rho_0 + \operatorname{div}(\rho_0 u_0) &= -\operatorname{div} R_0 \\ \operatorname{div} u_0 &= 0, \end{aligned}$$

  there exists (ρ_1, u_1, R_1) smooth solves

$$\begin{aligned} \partial_t \rho_1 + \operatorname{div}(\rho_1 u_1) &= -\operatorname{div} R_1 \\ \operatorname{div} u_1 &= 0. \end{aligned}$$

- Additionally, for all $t \in [0, T]$

$$\begin{aligned} \|\rho_1(t) - \rho_0(t)\|_p &\leq M \|R_0(t)\|_1^{1/p}, \\ \|u_1(t) - u_0(t)\|_{p'} &\leq M \|R_0(t)\|_1^{1/p}, \\ \|u_1(t) - u_0(t)\|_{W^{1,\tilde{p}}} &\leq \delta, \\ \|R_1(t)\|_1 &\leq \delta. \end{aligned}$$

- Moreover, if $R_0(t, \cdot) \equiv 0$ for some $t \in [0, T]$, then $R_1(t, \cdot) \equiv 0$ and $(\rho_1, u_1)(t) \equiv (\rho_0, u_0)(t)$.
- Idea: use highly oscillatory perturbation to reach ρ_1 from ρ_0.

Sketch of the proof

- Add perturbation:

$$\rho_1 = \rho_0 + \theta + \theta_c, \quad u_1 = u_0 + w + w_c$$

  Here θ_c is the mean value corrector, and w_c is the divergence free corrector.

- Note that

$$\begin{aligned}\partial_t \rho_1 + \mathrm{div}(\rho_1 u_1) = & -\underbrace{\mathrm{div}(\theta w - R_0)}_{\mathrm{R}^{quad}} \\ & + \underbrace{\mathrm{div}(\rho_0 w + \theta u_0) + \partial_t \theta}_{\mathrm{R}^{lin}} \\ & + \underbrace{\text{terms containing}\, \theta_c, w_c}_{\mathrm{R}^{corr}} := -\,\mathrm{div}\, R_1.\end{aligned}$$

- Ansatz: Note that $R_0(t,x) = \sum_{j=1}^d R_{0,j}(t,x) e_j$

$$\theta(t,x) = \sum_{j=1}^d \mathrm{sign}(R_{0,j}) |R_{0,j}|^{1/p} \Theta^j(\lambda x)$$

$$w(t,x) = \sum_{j=1}^d |R_{0,j}|^{1/p'} W^j(\lambda x)$$

Sketch of the proof

- Here $\Theta^j : \mathbb{T}^d \to \mathbb{R}$, and $W^j : \mathbb{T}^d \to \mathbb{R}^d$ are smooth fixed profiles such that
  - $\operatorname{div}(\Theta^j W^j) = 0$, and $\operatorname{div} W^j = 0$
  - $\fint \Theta^j(x)\,dx = \fint W^j(x)\,dx = 0$, $\fint \Theta^j(x) W^j(x)\,dx = e_j$.
  - For $i \neq j$, $\operatorname{Supp}\Theta^i \cap \operatorname{Supp} W^j = \phi$.
- In literature usually people use Beltrami or Mikado flows.
- Construction of Mikado flow:
  - Choose $\varphi \in C_c^\infty(\mathbb{R}^{d-1})$ with $\operatorname{Supp}(\varphi) \subset (0,1)^{d-1}$, $\int \varphi = 0$, and $\int \varphi^2 = 1$. Periodise and still call it φ.
  - Define

$$\Theta^d(x) := \varphi(x_1, x_2, \cdots, x_{d-1})$$
$$W^d(x) := \varphi(x_1, x_2, \cdots, x_{d-1}) e_d$$

Errors

- First compute:

$$\begin{aligned}\operatorname{div}(\theta w - R_0) &= \operatorname{div}\left(\sum_{j=1}^{d} R_{0,j}(\Theta^j(\lambda x)W^j(\lambda x) - e_j)\right)\\ &= \sum_{j=1}^{d} \underbrace{\nabla R_{0,j}}_{\text{Slow amplitude}} \cdot \underbrace{(\Theta^j(\lambda x)W^j(\lambda x) - e_j)}_{\text{Fast oscillating with zero mean value}}\\ &:= \operatorname{div}(R^{\text{quad}})\end{aligned}$$

- A suitable inverse divergence operator gives $\|R^{\text{quad}}\|_1 \approx \mathcal{O}(1/\lambda)$.
- Sobolev Estimates:

$$\nabla w = \lambda \sum_{j=1}^{d} |R_{0,j}|^{1/p'} \nabla W^j(\lambda x) + \text{“another lower order term”}$$

- Intermittency and difference with Onsager conjecture.
- Observation: Let $\psi \in C_c^\infty(\mathbb{R}^d)$, $\operatorname{Supp}(\psi) \subset (0,1)^d$. Define $\psi_\mu(x) = \psi(\mu x)$. Then

$$\|\mu^a \psi_\mu\|_r = \mu^{a-d/r}\|\psi\|_r, \quad \|\mu^a \nabla^k \psi_\mu\|_r = \mu^{a+k-d/r}\|\psi\|_r$$

- Change the ansatz: We shall fix a and b such that

$$\Theta_\mu^d(x) := \mu^a \varphi_\mu(x_1, x_2, \cdots, x_{d-1})$$
$$W_\mu^d(x) := \mu^b \varphi_\mu(x_1, x_2, \cdots, x_{d-1}) e_d$$

Errors

- It turns out that $a = (d-1)/p$ and $b = (d-1)/p'$
- It is easy to see that

$$\|\Theta^j_\mu(x)\|_p = \mu^{a - \frac{d-1}{p}}, \quad \|W^j_\mu(x)\|_{p'} = \mu^{b - \frac{d-1}{p'}}$$

- Then we can estimate

$$\begin{aligned}\|\nabla w\|_{\tilde{p}} &\leq C(R_0)\lambda \|\nabla W^j_\mu\|_{\tilde{p}} \\ &\lesssim \lambda \mu^{\frac{d-1}{p'} - \frac{d-1}{\tilde{p}} + 1} \approx \lambda \mu^{-\gamma}, \text{ for some } \gamma > 0.\end{aligned}$$

- Here we first assume that $\frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d-1}$.
- Note that if $\frac{d-1}{p'} - \frac{d-1}{\tilde{p}} + 1 < 0$, then μ can kill λ. This is where we need the assumption on p and $\tilde{p}$.
- It is not difficult to verify that other error terms + L^p-type estimates can be controlled by the above choice of building blocks.

General Case

- How to deal with the case $\frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d}$? We need to change Mikado construction.
- Naive choice $\rightsquigarrow$ Not divergence free!

$$\Theta_\mu^d(x) := \mu^{d/p} \varphi_\mu(x_1, x_2, \cdots, x_{d-1}, x_d)$$
$$W_\mu^d(x) := \mu^{d/p'} \varphi_\mu(x_1, x_2, \cdots, x_{d-1}, x_d) e_d$$

- Add another highly oscillatory term: Fix $\psi^d : \mathbb{T}^d \to \mathbb{R}$ such that $\partial_d \psi^d = 0$, $\fint \psi = 0$, and $\fint \psi^2 = 1$.

$$\theta^d(x) := \mu^{d/p} \varphi_\mu(\lambda x) \psi^d(\gamma x)$$
$$w^d(x) := \mu^{d/p'} \varphi_\mu(\lambda x) \psi^d(\gamma x) e_d$$

- We need $\lambda \mu << \gamma$.
- Calculate

$$\operatorname{div} w^d = \lambda \mu^{d/p'} (\partial_d \varphi_\mu)(\lambda x) e_d \qquad \underbrace{\psi^d(\gamma x)}_{\text{fast oscillatory with zero mean}}$$

- Then define the divergence corrector

$$w^c = \operatorname{div}^{-1}(\operatorname{div} w)$$

General Case

- Estimate for the divergence corrector (Note that $\mathrm{div}^{-1}(fg_\lambda) \approx \frac{1}{\gamma} fg$)

$$\|w^c\|_r \approx \lambda \mu^{d/p'} \frac{1}{\gamma} \mu^{1-d/r} \approx \frac{\lambda\mu}{\gamma} \|w\|_r$$

- Main problem: $\Theta W = \mu^d \varphi_\mu^2(\lambda x)(\psi^d)^2(\gamma x) e_d$ is not divergence free!
- Note that $(\psi^d)^2(\gamma x)$ has non-zero mean and we can't play the usual anti-divergence corrector trick!
- Idea: Allow error to move in time and cancel it with time derivative $\rightsquigarrow$ change Mikado density $\rightsquigarrow$ introduce phase speed ω:

$$\Theta_\mu^j(x) := \mu^{d/p} \varphi_\mu(\lambda(x - t\omega e_j))\psi^j(\gamma x)$$
$$W_\mu^j(x) := \mu^{d/p'} \varphi_\mu(\lambda(x - t\omega e_j))\psi^j(\gamma x) e_j$$

- Observe that $\partial_t Q^j + \mathrm{div}(\Theta^j W^j) = 0$, with $Q^j := \frac{\mu^d}{\omega} \varphi_\mu^2(\lambda(x - t\omega e_j))(\psi^j)^2(\gamma x)$.

Final Building Block

- New Mikado density: $\Theta^j(x,t) + Q^j(x,t)$, and velocity $W^j(x,t)$.
- In this case

$$\begin{aligned}&\partial_t(\Theta + Q) + \operatorname{div}((\Theta + Q)W)\\ &= \partial_t\Theta + \underbrace{\partial_t Q + \operatorname{div}(\Theta W)}_{=0} + \operatorname{div}(\Theta Q)\\ &= \operatorname{div}(\operatorname{div}^{-1}(\partial_t\Theta + \operatorname{div}(\Theta Q)))\end{aligned}$$

- We can estimate:

$$\|\operatorname{div}^{-1}(\partial_t\Theta)\|_1 \approx \frac{\lambda\omega}{\gamma}\mu^{1-d+d/p}$$

- Application in NSE: $p = p' = 2$

$$\Delta w = \operatorname{div}(\nabla w)$$

- Run the above construction with $p = 2$ and $\tilde{p} = 1$ $\rightsquigarrow$ $1/2 + 1 > 1 + 1/d$, i.e., $d \geq 3$.

Stochastic Transport-Diffusion

- Additive noise:
  - Consider the heat equation forced by the same noise

$$dz(t,x) = \Delta z(t,x)dt + dB(t,x), \quad z(0,x) \equiv 0$$

  - Consider the random PDE solved by $\theta(t,x) := \rho(t,x) - z(t,x)$

$$\partial_t \theta(t,x) + \mathrm{div}(u(t,x)\theta(t,x)) = \Delta\theta(t,x) - \mathrm{div}(u(t,x)z(t,x)),$$
$$\nabla \cdot u = 0,\ \theta(0,x) = \rho^{\mathrm{in}}(x).$$

- Linear multiplicative noise:
  - Consider random PDE solved by $\theta(t,x) := \rho(t,x)e^{-B(t)}$

$$\partial_t \theta(t,x) + \mathrm{div}(u(t,x)\theta(t,x)) + \frac{1}{2}\theta(t,x) = \Delta\theta(t,x),\ \nabla\cdot u = 0,\ \theta(0,x) = \rho^{\mathrm{in}}(x).$$

- Transport noise:
  - Consider random PDE solved by $\theta(t,x) := \rho(t,x+B(t))$

$$\partial_t \theta(t,x) + \mathrm{div}(u(t,x+B(t))\theta(t,x)) = \Delta\theta(t,x),\ \nabla\cdot u = 0,\ \theta(0,x) = \rho^{\mathrm{in}}(x).$$

Main Features

- Idea is to produce infinitely many solutions that break the energy inequality.
  - Change the iteration scheme: $M_0(t) \triangleq L^4 e^{4Lt}$

$$\|(\theta_1 - \theta_0)(t)\|_{L^p_x} \leq M(\delta M_0(t))^{\frac{1}{p}},$$
$$\|(u_1 - u_0)(t)\|_{W^{1,\tilde{p}}_x} \leq \delta M_0(t),$$
$$\|R_1(t)\|_{L^1_x} \leq \delta M_0(t).$$

  - Need to work with $R_l(t,x) = (R_0 *_x \phi_l *_t \varphi_l)(t,x) = \sum_{j=1}^d R_l^j(t,x) e_j$.
  - Need to iterate also $R_0 \in C_t^0 C_x^1 \cap C_t^{\frac{1}{2}-2\varpi} C_x^0$.
- Need to introduce a stopping time T_L to control the noise terms in the iteration.
  - For all $t \in [0, T_L]$,

$$\|z(t)\|_{L^\infty_x} \leq L^{\frac{1}{4}}, \quad \|z(t)\|_{W^{1,\infty}_x} \leq L^{\frac{1}{4}}, \quad \|z\|_{C_t^{\frac{1}{2}-2\varpi} L^\infty_x} \leq L^{\frac{1}{2}}.$$

- Add a convex integration solution to a weak solution to produce solutions on the entire time interval.
  - To extend this convex integration solution to the interval $[0, T]$, we can glue an appropriate weak solution of stochastic transport to this convex integration solution.

Thank you!