Persistence exponent for GSPs

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Joint work with N.D. Feldheim and O.N. Feldheim

Outline

1. What is the question?
2. An overview of our results
3. Outline of the proof
   - Smooth densities
   - UB for general densities
   - LB for general densities
   - Important tools
4. Conclusion

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- We want to study the decay rate of $p(T)$.

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- Comment: Slepian's Lemma applies for non stationary Gaussian processes.

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\geq &\mathbb{P}(\inf_{t\in[0,T]} X(t) > 0)\mathbb{P}(\inf_{t\in[T,T+S]} X(t) > 0) \ [\rho(.) \geq 0]
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\end{aligned}$$

- By sub-additivity, $\theta_\rho := -\lim_{T\to\infty} \frac{1}{T} \log p(T)$ exists in $[0, \infty]$.

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- Taking log and dividing by T gives

$$\frac{1}{T}\log\mathbb{P}(\inf_{t\in[0,T]} X(t) > 0) \geq k\log\mathbb{P}(\inf_{t\in[0,1/k]} X(t) > 0).$$

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- But this is a contradiction, as $\mathbb{P}(X(0) > 0) \neq 0$.

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- Finally, in Aurzada-M. 2020 it was shown that

$$\theta_\rho = 0 \quad \Leftrightarrow \quad \int_0^\infty \rho(t)dt = \infty.$$

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- On the other hand $\lim_{k\to\infty} \rho_k = \rho^{(int)}$ which is integrable, and so $\theta_{\rho^{(int)}} > 0$.

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(c) $\limsup_{t\to 0} \sup_{k\ge 1} |\log t|^\beta (1-\rho_k(t)) < \infty$ *for some* $\beta > 1$*. (tightness)*

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- Thus $\log \mathbb{P}(\sqrt{Z_1^2 + Z_2^2} < 0) = -\infty$. Consequently, $\theta_\rho = \infty$.

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- The existence of exponent for the sinc process remained open.

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- As an example, for the sinc process, μ is $\frac{1}{2}$ times Lebesgue measure on $[-1, 1]$.

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- In fact, the existence of exponent was not known in any non-trivial example (including the sinc process).
- Can we study continuity properties of such exponents, in terms of $\rho(.)$ or μ?
- Can we compare persistence exponents by comparing spectral measures?

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- Let $p_T(\mu) := \mathbb{P}(\inf_{t\in[0,T]} X_\mu(t) > 0)$.

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Theorem (Feldheim-Feldheim-M. 2021+)

The exponent

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(A2) *The limit*

$$\mu'(0) := \lim_{\varepsilon \downarrow 0} \frac{\mu[-\varepsilon, \varepsilon]}{2\varepsilon}$$

exists in $(0, \infty]$.

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- If $\mu'(0) > 0$ then $\theta(\mu) < \infty$.

Positivity and Finiteness

- Assume (A1) throughout this slide.
- If $\mu'(0) = \infty$, then $\theta(\mu) = 0$.
- Conversely, if $\mu'(0) < \infty$ and μ has a non singular component, then $\theta(\mu) > 0$.
- If $\mu'(0) > 0$ then $\theta(\mu) < \infty$.
- Conversely, if $\mu([0, t]) \sim t^{\alpha}$ for $\alpha < 1$ near 0, then $\theta(\mu) = \infty$ (Feldheim-Feldheim-Nitzan 2017).

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- Suppose μ is a spectral measure which satisfies our conditions. Then the truncated measure $\mu_{[-L,L]}$ satisfies our conditions.

- More generally, let $h(.)$ be a bounded non-negative function which is continuous near 0, with $h(0) \neq 0$.

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- If μ and ν satisfies our conditions, and one of the measures have a bounded continuous density, then so does the convolution $\mu * \nu$.

- If the correlation function $\rho(.)$ is absolutely integrable, and $1 - \rho(t)$ satisfies very mild decay conditions for $t \approx 0$, then our conditions hold.

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- We also need (A2), which demands that μ is nice near 0 ($\mu'(0)$ exists and is positive).

- To show the necessity of this, we show the existence of positive reals $A < B$ such that with

$$\mu(dx) = (A + B\cos(1/x))1\{|x| < 1\}dx,$$

the exponent $\theta(\mu)$ does not exist.

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- Further, the density is continuous in the neighborhood $(-1, 1)$, except at 0.
- This example demonstrates the special role of the origin, as a discontinuity away from the origin does not impact existence of exponent.
- We conjecture that the exponent $\theta(\mu)$ does not exist in this example for any A, B (as opposed to some A, B).

Second main result: Impact of singular measures

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Theorem (Feldheim-Feldheim-M. 2021+)

$\theta(\mu + \nu) \geq \theta(\mu)$, *with equality iff* ν *is purely singular.*

Comments on this theorem

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- Here $Z \sim N(0,1)$, and the sign $\oplus$ denotes point-wise sum of independent processes.

- In this case, we have

$$\mathbb{P}(\inf_{t\in[0,T]}\{X_\mu(t)\oplus\sigma Z\}>0) \geq \mathbb{P}(\sup_{t\in[0,T]}|X_\mu(t)|<K)\mathbb{P}(\sigma Z>K).$$

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- In the above two displays we also use stationarity.
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- This will be sketched in the second part of the talk.

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- However, the state of the art result applies only to non-negative covariances which are strictly decreasing (Li-Shao 2005).
- One of our central estimates is a change of level lemma for general GSPs, which applies under the much weaker assumptions (A1) and (A2).
- In particular, this change of level lemma applies to "any" non-negative correlation function.

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- In this case, for $T \geq 2\pi$,

$$\begin{aligned}\mathbb{P}(\inf_{t\in[0,T]}\{Z_1\cos(t) \oplus Z_2\sin(t)\} > \delta) =& 0 \text{ if } \delta \geq 0,\\ =& c > 0 \text{ if } \delta < 0.\end{aligned}$$

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- Consequently,

$$\begin{aligned}-\lim_{T\to\infty}\frac{1}{T}\log\mathbb{P}(\inf_{t\in[0,T]}\{Z_1\cos(t) \oplus Z_2\sin(t)\} > \delta) =& \infty \text{ if } \delta \geq 0,\\ =& 0 \text{ if } \delta < 0.\end{aligned}$$

Why are singular measures special?

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Theorem (Feldheim-Feldheim-M. 2021+)

$\psi(\mu,\ell) = 0$ *iff* μ *is singular.*

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for $T \geq 2\pi$.
- Since $\mathbb{P}(\sqrt{Z_1^2 \oplus Z_2^2} < \ell) > 0$, we have $\psi(\mu, \ell) = 0$.

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for $T \geq 2\pi$.
- Since $\mathbb{P}(\sqrt{Z_1^2 \oplus Z_2^2} < \ell) > 0$, we have $\psi(\mu, \ell) = 0$.
- The general proof proceeds via approximating a singular X_μ as a combination of $o(T)$ many Gaussians.

Third main result: Continuity

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- In particular, no convergence is necessary for singular component of the measure sequence (away from the origin), as they do not impact persistence.
- This result generalizes the continuity of exponent result for non-negative correlations obtained in Dembo-M. 2012.

Outline

Smooth densities

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- Since $\|f''\|_\infty < \infty$, this gives the bound $|\rho(t)| \leq \frac{C}{t^2}$ for all $t \geq 1$.

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$$\frac{1}{T}\log p_T(\mu) \leq \frac{1}{M(1+\delta)}\log p_M(\mu) + o_M(1).$$

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- The existence of the limit follows since $\delta > 0$ is arbitrary.

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- We split the proof into upper and lower bounds, the proofs of which are very different.

Justification of upper bound

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- Consequently,
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- Also, Khatri-Sidak's inequality gives

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- The second one follows by another application of the change of level lemma.

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- Using the assumption gives $\sigma'(0) = \nu'(0) - \mu'(0) \approx 0$.

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- It thus suffices to show that

$$p_T(\mu) = p_T(\mu + h^2\sigma) \overset{T}{\geq} p_T(\mu + \sigma) = p_T(\nu).$$

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- Note that if $\mu = 0$, the claim reduces to $p_T(h^2\sigma) \overset{T}{\geq} p_T(\sigma)$, which is conceptually easier.

Justification of lower bound

- Indeed, in this case we need to show that

$$\mathbb{P}(\inf_{t\in[0,T]}\int X_\sigma(t-s)H(s)ds>0)\overset{T}{\geq}\mathbb{P}(\inf_{t\in[0,T]}X_\sigma(t)>0).$$

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- Then

$$\inf_{t\in[-L,T+L]} X_\sigma(t) \geq 0 \Rightarrow \inf_{t\in[0,T]} \int X_\sigma(t-s)H(s)ds \geq 0.$$

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- This give the exact bound

$$p_T(h^2\sigma) \geq p_{T+2L}(\sigma).$$

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- For a general μ, we use log-concavity of the Gaussian measure, and the fact $\int H(s)ds = h(0) = 1$. This gives, for any $v(.)$,

$$\int H(s) \log \mathbb{P}(\inf_{t\in[0,T]} \{X_\mu(t) + v(t-s)\} > 0)ds$$

$$\leq \log \mathbb{P}\Big(\inf_{t\in[0,T]} \{X_\mu(t) + \int H(s)v(t-s)ds\} > 0\Big).$$

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- Using stationarity+compact support of H, the LHS is lower bounded by

$$\log \mathbb{P}(\inf_{t\in[-L,T+L]}\{X_\mu(t) + v(t)\} > 0).$$

Justification of lower bound

- Plugging in $v = X_\sigma$ gives

$$\mathbb{P}\big(\inf_{t\in[-L,T+L]}\{X_\mu(t)+X_\sigma(t)\}>0\big)$$
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- This is same as

$$p_{T+2L}(\mu+\sigma)\le p_T(\mu+h^2\sigma),$$

which is what we wanted.

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- Thus

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- Combining, we have $\theta(\mu) = \theta(\mu + \sigma)$.
- We also show that equality does not hold if σ is not singular.

Outline

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- Can we handle non-stationary Gaussians? One advantage of correlation function is that it makes sense even for non-stationary Gaussian processes.
- Can we say how does the GSP look like conditioned on persistence?


The End