Total variation cutoff for the flip-transpose top with random shuffle

Subhajit Ghosh
Department of Mathematics
Postdoctoral Fellow
Indian Institute of Technology Bombay
India

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Brief overview of the presentation

- Background and motivation.
- The flip-transpose top with random shuffle.
- Background theory to study the flip-transpose top with random shuffle.
- Formulation as random walk on the hyperoctahedral group.
- Spectrum of the transition matrix.
- Total variation cutoff phenomenon.
- Biased variant.

Background and motivation

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- This has application in many subjects including mathematics, computer science, statistical physics and biology.
- This theory took a new direction in 1981, when Diaconis and Shahshahani introduced the use of non-commutative Fourier analysis techniques.
- Our model is mainly inspired by the *transpose top with random shuffle* studied by Flatto, Odlyzko and Wales in 1985. Diaconis proved the total variation cutoff for the transpose top with random shuffle in 1987.

The flip-transpose top with random shuffle

- Given an arrangement of n distinct oriented cards in a row, the shuffling scheme is the following: *Choose a card uniformly at random from the row and transpose it with the last* (*or* n*th*) *card after a random decision of flipping* (*both*) *the card*(*s*) *or not.*

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- Typical transition for the flip-transpose top with random shuffle of 9 distinct oriented cards:

$$\begin{array}{ccc} & & 123456987 \\ & \nearrow & \\ 123456\,7\,89 & & \\ & \searrow & \\ & & 123456987 \end{array}$$

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- Typical transition for the flip-transpose top with random shuffle of 9 distinct oriented cards:

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- This can be described as a random walk on the hyperoctahedral group B_n.

Aim: Study the mixing time for the flip-transpose top with random shuffle on B_n.

- A *signed permutation* is a bijection π from $\{-n, \ldots, -1, 1, \ldots, n\}$ to itself satisfying $\pi(-i) = -\pi(i)$ for all $1 \leq i \leq n$.
- A signed permutation is completely determined by its image on the set $[n] := \{1, \ldots, n\}$. Given a signed permutation π, we write it in window notation by $[\pi_1, \ldots, \pi_n]$, where π_i is the image of i under π.
- The set of all signed permutations forms a group under composition and is known as the *hyperoctahedral group* and is denoted by B_n.

Signed permutations and arrangements of oriented cards

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- We associate a signed permutation $[\pi_1, \pi_2, \ldots, \pi_n]$ to an arrangement of n oriented cards in a row in the following way: The ith card (counting started from left) has label $|\pi_i|$, and its orientation is determined from the sign of π_i.

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- Thus every arrangement of the oriented cards in a row represents a signed permutation in its window notation.

Mixing time and cutoff phenomenon

- Consider a discrete time Markov chain with finite state space Ω and transition matrix M. Assumption: The chain is irreducible and aperiodic.
- Π be the stationary distribution of the chain.
- $\lim\limits_{t \to \infty} \Pi_0 M^t = \Pi$ for any initial distribution Π_0.

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- Total variation distance: $||\mu - \nu||_{\mathrm{TV}} := \sup_{A\subset\Omega} |\mu(A) - \nu(A)|$.
- Mixing time: The ε-mixing time $(0 < \varepsilon < 1)$ is defined as follows,

$$t_{\mathsf{mix}}(\varepsilon) := \min\{t : d(t) < \varepsilon\}, \text{ where } d(t) = \max_{x\in\Omega} ||M^t(x,\cdot) - \Pi||_{\mathrm{TV}}.$$

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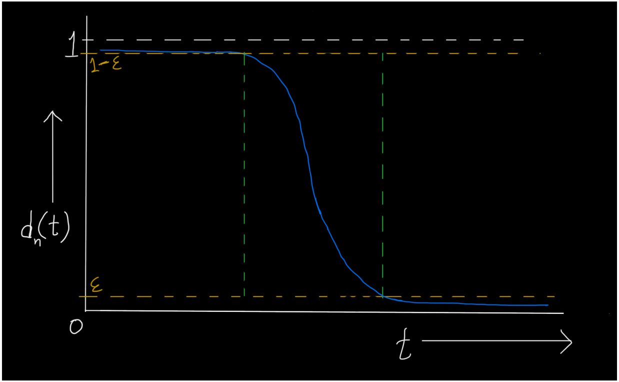
- Cutoff phenomenon: Let $\{X^{(n)}\}_n$ be a sequence of Markov chains and $t^{(n)}_{\mathsf{mix}}(\varepsilon)$ denote the ε-mixing time for $X^{(n)}$. Then the sequence is said to satisfy the cutoff phenomenon if

$$\lim_{n\to\infty} t^{(n)}_{\mathsf{mix}}(\varepsilon) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{t^{(n)}_{\mathsf{mix}}(\varepsilon)}{t^{(n)}_{\mathsf{mix}}(1-\varepsilon)} = 1 \text{ for all } 0 < \varepsilon < 1.$$

Cutoff phenomenon (continued)



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Representation theory background

- Linear representation of a finite group: $\rho : G \xrightarrow{\text{Hom.}} GL(V)$, V is a finite-dimensional vector space and $GL(V)$ is the set of all invertible linear maps from V to itself. The vector space V is called a G-module in this case. Dimension d_ρ is the dimension of V.

- Right regular representation: $R : G \xrightarrow{\text{Hom.}} GL\left(\mathbb{C}[G]\right)$ defined by,

$$g \mapsto \left(\sum_{h \in G} C_h h \mapsto \sum_{h \in G} C_h hg \right),\ C_h \in \mathbb{C},\ g \in G.$$

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- Decomposition $\mathbb{C}[G]$ into irreducible G-modules:

$$\mathbb{C}[G] \cong \bigoplus_{\sigma \in \widehat{G}} \dim(V^\sigma)\, V^\sigma.$$

Non-commutative Fourier analysis techniques

- Convolution of probability measures on G: $\quad (p * q)(x) = \sum_{y \in G} p(xy^{-1})q(y).$
- Fourier transformation of p at a representation ρ: $\quad \widehat{p}(\rho) = \sum_{x \in G} p(x)\rho(x).$

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- Fourier transformation of p at a representation ρ: $\widehat{p}(\rho) = \sum_{x \in G} p(x)\rho(x)$.
- Random walk on G driven by p: Markov chain on G with transition probabilities $M_p(x, y) = \mathbb{P}(X_1 = y | X_0 = x) := p(x^{-1}y),\ x, y \in G$.

$$M_p = (M_p(x,y))_{x,y \in G} = \left(\widehat{p}(R)\right)^T .$$

- The distribution after k^{th} transition will be p^{*k}, more precisely $\mathbb{P}(X_k = y | X_0 = x) = p^{*k}(x^{-1}y)$.
- Irreducible if and only if the support of p generates G. In that case the stationary distribution is the uniform distribution on G.

Our model as random walk on B_n

- The flip-transpose top with random shuffle on B_n is the random walk on B_n driven by P, defined on B_n by

$$P(\pi) = \begin{cases} \frac{1}{2n}, & \text{if } \pi = \mathrm{id}, \text{ the identity element of } B_n, \\ \frac{1}{2n}, & \text{if } \pi = (i, n) \text{ for } 1 \leq i \leq n-1, \\ \frac{1}{2n}, & \text{if } \pi = (-i, n) \text{ for } 1 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

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- Transition matrix: $\widehat{P}(R)$.

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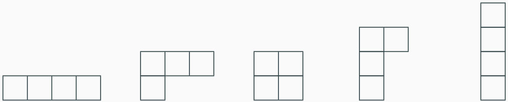
Definitions of useful combinatorial objects

- Partition: $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ if $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ and $|\lambda| := \sum_{i=1}^{\ell} \lambda_i = n$.
- Young diagrams of shape λ:



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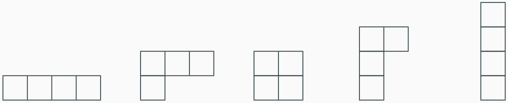

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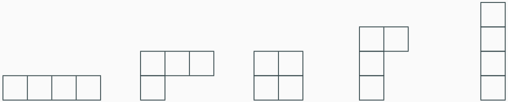

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- $\mathrm{tab}_\mathcal{D}(n, \mu)$: set of all standard (Young) double-tableau of shape μ.

$$\left(\begin{array}{|c|c|c|} \hline 3 & 4 & 8 \\ \hline 6 & 7 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} \, , \, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 5 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right)$$

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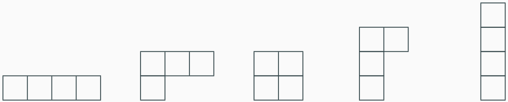

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- $c(b_T(i))$: content of the box in T $(\in \mathrm{tab}_{\mathcal{D}}(n, \mu))$ containing i.

Theorem (—)
For each $\mu = \left(\mu^{(1)}, \mu^{(2)}\right) \in \mathcal{D}_{\mathrm{n}}$ satisfying $m := |\mu^{(1)}| \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, let $T \in \mathrm{tab}_{\mathcal{D}}(n, \mu)$. Then $\frac{c(b_T(n))+1}{n}$ and $\frac{c(b_T(n))}{n}$ are eigenvalues of $\widehat{P}(R)$ with multiplicity $M(\mu)$ each, where

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Illustration for $n = 2$: The eigenvalues of the transition matrix for the flip-transpose top with random shuffle on B_2 are the following: 1 , $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, 0 , 0 , 0 , $-\frac{1}{2}$

$$\left(\phi \, , \, \boxed{1}\boxed{2} \right), \left(\phi \, , \, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right), \left(\boxed{1} \, , \, \boxed{2} \right), \left(\boxed{2} \, , \, \boxed{1} \right)$$

Proof idea: Vershik-Okounkov approach to the representation theory of B_n

- The sequence $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n$ forms a multiplicity free chain.
  - Thus irreducible B_n-modules V has a canonical decomposition into irreducible B_1-modules.
  - This decomposition is known as the Gelfand-Tsetlin decomposition and the irreducible B_1-modules are known as the Gelfand-Tsetlin vectors of V.

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- GT_n is a maximal commuting subalgebra of $\mathbb{C}[B_n]$ generated by $\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n$, where $\mathcal{Z}_i$ denotes the center of $\mathbb{C}[B_i]$.
  - GT_n is known as the Gelfand-Tsetlin subalgebra of $\mathbb{C}[B_n]$.
  - Elements of GT_n act by scalars on the Gelfand-Tsetlin vectors of all irreducible representations of B_n.
  - The Gelfand-Tsetlin subalgebra for this chain is generated by the Young-Jucys-Murphy elements $X_1, \ldots, X_n$ of $\mathbb{C}[B_n]$.

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- The Young-Jucys-Murphy elements of $\mathbb{C}[B_n]$ are given by $X_1 = 0$ and

$$X_i = \sum_{j=1}^{i-1} ((j,i) + (-j,i)) \in \mathbb{C}[B_i] \text{ for all } 2 \leq i \leq n.$$

Upper bound for $\left|\left|P^{*k} - U_{B_n}\right|\right|_{\text{TV}}$

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For the flip-transpose top with random shuffle on B_n, we have the following:

$$\left|\left|P^{*k} - U_{B_n}\right|\right|_{\text{TV}} < \sqrt{2(e+1)}\, e^{-c} + o(1),$$

for $k \geq n \log n + cn$ and $c > 0$.

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Proof sketch: Using Diaconis-Shahshahani upper bound lemma we have

$$4\left|\left|P^{*k}-U_{B_n}\right|\right|_{\mathsf{TV}}^2$$

$$\leq \left(\frac{n-1}{n}\right)^{2k} + \sum_{\substack{\lambda\vdash n\\ \lambda\neq(n)}} d_\lambda \left(\sum_{T\in\mathrm{tab}(\lambda)} \left(\left(\frac{c(b_T(n))+1}{n}\right)^{2k} + \left(\frac{c(b_T(n))}{n}\right)^{2k}\right)\right)$$

$$+\sum_{m=1}^{\lfloor\frac{n}{2}\rfloor} \sum_{\substack{\mu^{(1)}\vdash m\\ \mu^{(2)}\vdash(n-m)\\ \mu=\left(\mu^{(1)},\,\mu^{(2)}\right)}} M(\mu)\left(\sum_{T\in\mathrm{tab}_{\mathcal{D}}(n,\mu)} \left(\left(\frac{c(b_T(n))+1}{n}\right)^{2k} + \left(\frac{c(b_T(n))}{n}\right)^{2k}\right)\right).$$

Recall: For each $\mu = \left(\mu^{(1)},\mu^{(2)}\right) \in \mathcal{D}_{\mathrm{n}}$ satisfying $m := |\mu^{(1)}| \in \{0,1,\ldots,\lfloor\frac{n}{2}\rfloor\}$, let $T \in \mathrm{tab}_{\mathcal{D}}(n,\mu)$. Then $\frac{c(b_T(n))+1}{n}$ and $\frac{c(b_T(n))}{n}$ are eigenvalues of $\widehat{P}(R)$ with multiplicity $M(\mu)$ each.

Upper bound for $\left|\left|P^{*k} - U_{B_n}\right|\right|_{\mathrm{TV}}$ (continued)

- For $k \geq n \log n$, we have

$$\sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{\mu^{(1)} \vdash m \\ \mu^{(2)} \vdash (n-m) \\ \mu = \left(\mu^{(1)},\, \mu^{(2)}\right)}} M(\mu) \left(\sum_{T \in \mathrm{tab}_{\mathcal{D}}(n,\mu)} \left(\left(\frac{c(b_T(n)) + 1}{n} \right)^{2k} + \left(\frac{c(b_T(n))}{n} \right)^{2k} \right) \right)$$

$$< 4e \left(e^{n^2 e^{-\frac{2k}{n}}} - 1 \right).$$

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$$\sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{\mu^{(1)} \vdash m \\ \mu^{(2)} \vdash (n-m) \\ \mu = \left(\mu^{(1)},\, \mu^{(2)}\right)}} M(\mu) \left(\sum_{T \in \mathrm{tab}_{\mathcal{D}}(n,\mu)} \left(\left(\frac{c(b_T(n)) + 1}{n} \right)^{2k} + \left(\frac{c(b_T(n))}{n} \right)^{2k} \right) \right)$$

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$$\sum_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} d_\lambda \left(\sum_{T \in \mathrm{tab}(\lambda)} \left(\left(\frac{c(b_T(n)) + 1}{n} \right)^{2k} + \left(\frac{c(b_T(n))}{n} \right)^{2k} \right) \right)$$

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- Therefore

$$4||P^{*k} - U_{B_n}||_{\mathrm{TV}}^2 \leq 2e^{-\frac{2k}{n}} + (4+4e) \left(e^{n^2 e^{-\frac{2k}{n}}} - 1 \right) + e^{-\frac{4k}{n}}, \text{ for } k \geq n \log n.$$

Lower bound for $\left|\left|P^{*k} - U_{B_n}\right|\right|_{\mathrm{TV}}$

- Let us define a surjective homomorphism f from B_n onto S_n as follows:

$$f : \pi \mapsto \left(f(\pi) : i \mapsto |\pi(i)|, \text{ for } 1 \leq i \leq n\right), \text{ for } \pi \in B_n.$$

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- If Q is the generator of the transpose top with random shuffle on S_n, then $Pf^{-1} = Q$. For any positive integer k we can prove that $\left(Pf^{-1}\right)^{*k} = P^{*k}f^{-1}$ (induction on k). We also have $U_{B_n}f^{-1} = U_{S_n}$.

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- We have $\left|\left|P^{*k} - U_{B_n}\right|\right|_{\mathsf{TV}} \geq \left|\left|Q^{*k} - U_{S_n}\right|\right|_{\mathsf{TV}}$ using the following lemma:

Lemma ([4, Lemma 7.9])
Given two probability distributions μ and ν on Ω and a mapping $\psi : \Omega \to \Lambda$, we have $||\mu - \nu||_{\mathsf{TV}} \geq ||\mu\psi^{-1} - \nu\psi^{-1}||_{\mathsf{TV}}$, where Λ is finite.

Lower bound for $\left|\left|P^{*k} - U_{B_n}\right|\right|_{\mathrm{TV}}$ (continued)

- Diaconis (1987) proved the following inequality for large n.

$$\left|\left|Q^{*k} - U_{S_n}\right|\right|_{\mathsf{TV}} \geq 1 - \frac{2\left(3 + 3(n-2)e^{-\frac{k}{n}} - 2(n-1)e^{-\frac{2k}{n}} + o(1)\right)}{\left(1 + (n-2)e^{-\frac{k}{n}}\right)^2}, \quad \text{for } k > 1.$$

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Theorem (—)
For the flip-transpose top with random shuffle on B_n, for large n, we have the following:

$$\left|\left|P^{*k} - U_{B_n}\right|\right|_{\mathsf{TV}} \geq 1 - \frac{2\left(3 + 3e^{-c} + o(1)(e^{-2c} + e^{-c} + 1)\right)}{(1 + (1 + o(1))e^{-c})^2},$$

when $k = n \log n + cn$ and $c \ll 0$.

Total variation Cutoff

- For appropriate choice of a positive integer N_0, $c_1 > 0$ and $c_0 \ll 0$ depending on ε, we have
$$n \log n + c_0 n \leq t_{\text{mix}}^{(n)}(\varepsilon) \leq n \log n + c_1 n, \text{ for all } n \geq N_0.$$

Recall: $t_{\text{mix}}^{(n)}(\varepsilon) := \min\{k : \left|\left|P^{*k} - U_{B_n}\right|\right|_{\mathrm{TV}} < \varepsilon\},\ 0 < \varepsilon < 1.$

- $\left|\left|P^{*k} - U_{B_n}\right|\right|_{\mathrm{TV}} < \sqrt{2(e+1)}\ e^{-c} + o(1)$, for $k \geq n \log n + cn$ and $c > 0$.
- For large n, $\left|\left|P^{*k} - U_{B_n}\right|\right|_{\text{TV}} \geq 1 - \frac{2\left(3+3e^{-c}+o(1)(e^{-2c}+e^{-c}+1)\right)}{\left(1+(1+o(1))e^{-c}\right)^2}$, when $k = n \log n + cn$ and $c \ll 0$.

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Biased variant

- Let $0 \leq \alpha \leq 1$. Given an arrangement of n distinct oriented cards in a row, choose a card uniformly at random and choose the last card. Then perform one of the following moves:
  1. Transpose the chosen cards with probability $\frac{\alpha}{2}$.
  2. Transpose the chosen cards after flipping both the cards with probability $\frac{\alpha}{2}$.
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- This is the random walk on B_n driven by the probability measure P_α on B_n, defined below.

$$P_\alpha(\pi) = \begin{cases} \frac{1}{n} \cdot \frac{\alpha}{2}, & \text{if } \pi = (i, n) \text{ or } (-i, n) \text{ for } 1 \leq i \leq n, \quad \text{here } (n, n) := \mathrm{id}, \\ \frac{1}{n} \cdot \frac{1-\alpha}{2}, & \text{if } \pi = (-n, n)(i, n) \text{ or } (-i, i)(i, n) \text{ for } 1 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Biased variant (continued)

Theorem (—)
For each $\mu = \left(\mu^{(1)}, \mu^{(2)}\right) \in \mathcal{D}_\mathrm{n}$ satisfying $m := |\mu^{(1)}| \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, let $T \in \mathrm{tab}_\mathcal{D}(n, \mu)$. Then $\frac{c(b_T(n))+1}{n}$ and $\frac{c(b_T(n))}{n}(2\alpha - 1)$ are eigenvalues of $\widehat{P}_\alpha(R)$ with multiplicity $M(\mu)$ each.

Recall from the flip-transpose top with random shuffle: For each $\mu = \left(\mu^{(1)}, \mu^{(2)}\right) \in \mathcal{D}_\mathrm{n}$ satisfying $m := |\mu^{(1)}| \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, let $T \in \mathrm{tab}_\mathcal{D}(n, \mu)$. Then $\frac{c(b_T(n))+1}{n}$ and $\frac{c(b_T(n))}{n}$ are eigenvalues of $\widehat{P}(R)$ with multiplicity $M(\mu)$ each.

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- The same mapping f defined by $f : \pi \mapsto (f(\pi) : i \mapsto |\pi(i)|, \text{ for } 1 \leq i \leq n)$, for $\pi \in B_n$ projects this biased variant to the transpose top with random shuffle on S_n. Therefore $||P^{*k} - U_{B_n}||_{\mathsf{TV}}$ and $||P_\alpha^{*k} - U_{B_n}||_{\mathsf{TV}}$ have the same lower bound.

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- $-1 \leq 2\alpha - 1 \leq 1$ implies that $||P^{*k} - U_{B_n}||_{\mathsf{TV}}$ and $||P_\alpha^{*k} - U_{B_n}||_{\mathsf{TV}}$ have the same upper bound.
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- The biased variant have total variation cutoff at $n \log n$.

Recall from the flip-transpose top with random shuffle: For each $\mu = \left(\mu^{(1)}, \mu^{(2)}\right) \in \mathcal{D}_\mathrm{n}$ satisfying $m := |\mu^{(1)}| \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, let $T \in \mathrm{tab}_{\mathcal{D}}(n, \mu)$. Then $\frac{c(b_T(n))+1}{n}$ and $\frac{c(b_T(n))}{n}$ are eigenvalues of $\widehat{P}(R)$ with multiplicity $M(\mu)$ each.

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Thank You!