Percolation theory: from classical to dependent models

Subhajit Goswami

Based on joint works with H Duminil-Copin, A. Raoufi, P.F. Rodriguez, F. Severo and A. Yadin

Institut des Hautes Études Scientifiques

July 23, 2019

Outline

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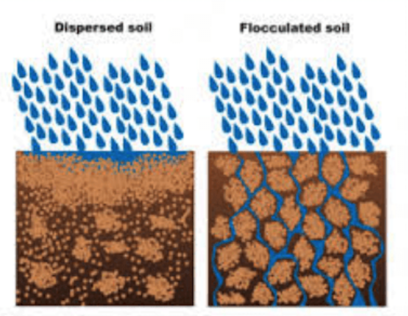
What is Percolation?

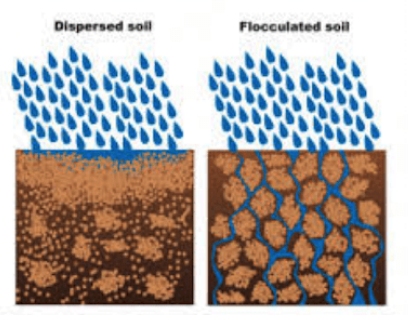
What is the likelihood that a liquid, say water, will pass through a porous medium?



(Image source: https://www.theseptictankstore.co.uk/blog/soil-percolation-porosity-test/)

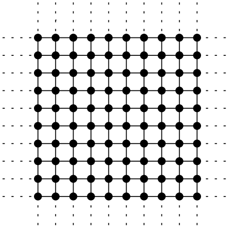
What is Percolation?

Percolation model arose as a simple stochastic model for such a situation (Broadbent and Hammersley'57)

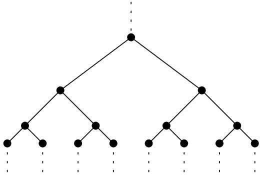


(Image source: https://www.theseptictankstore.co.uk/blog/soil-percolation-porosity-test/)

Mathematical formulation



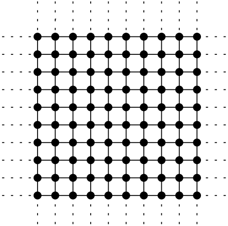
$\mathbb{Z}^2$



$\mathbb{T}_3$

Consider a connected graph $\mathcal{G}$, e.g. the square lattice $\mathbb{Z}^2$ or a tree

Mathematical formulation



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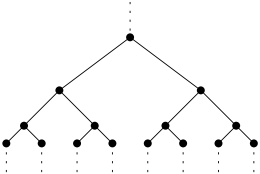
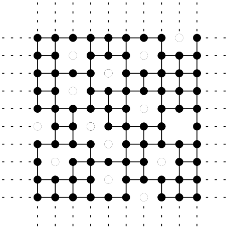


$\mathbb{T}_3$

Let p be a parameter in $[0, 1]$ (the so-called density)

Mathematical formulation



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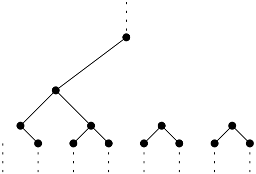
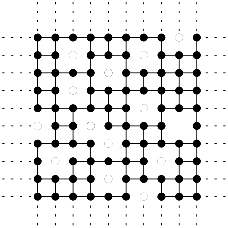


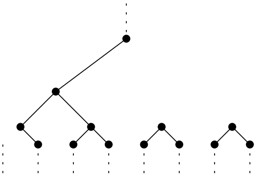
$\mathbb{T}_3$

Declare every site to be open with probability p and closed otherwise

Mathematical formulation



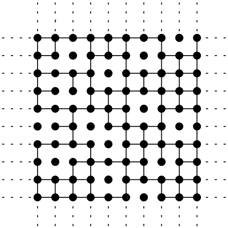
$\mathbb{Z}^2$



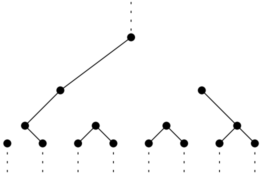
$\mathbb{T}_3$

This model is called site percolation

Mathematical formulation



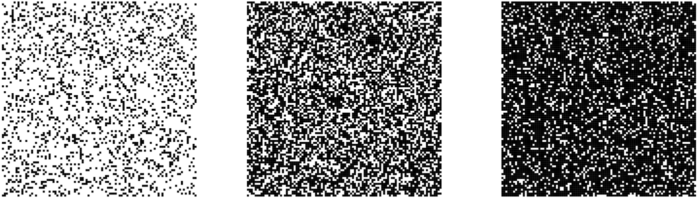
$\mathbb{Z}^2$



$\mathbb{T}_3$

In a different version (bond percolation) we open or close edges

How does it look like?



(a) $p = 0.2$ (b) $p = 0.6$ (c) $p = 0.8$

Outline

Some notations

We encode a percolation configuration by

$$\omega = (\omega_v : v \in \mathcal{G}) \in \{0,1\}^{\mathcal{G}}$$

where

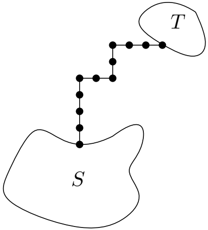
$$\omega_v = \begin{cases} 1 & \text{if } v \text{ is open} \\ 0 & \text{if } v \text{ is closed} \end{cases}$$

Some notations

The probability measure on the space Ω of percolation configurations for density p is given by

$$\mathbf{P}_p = \prod_{v \in \mathcal{G}} (p\delta_1 + (1-p)\delta_0)$$

Some notations



Primary events of interest:

$$\{S \leftrightarrow T\} = \{S, T \subset G \text{ are connected by an open path}\}$$

Some notations

From now onwards $\mathcal{G}$ will be an infinite connected graph

Notice that $\mathcal{G}$ is a metric space w.r.t. the minimum path length

$\Lambda_{n,x}$ is the ball of radius n around x in graph distance $d_{\mathcal{G}}$, i.e.

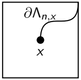
$$\Lambda_{n,x} = \{y \in \mathcal{G} : d_{\mathcal{G}}(x, y) \leq n\}$$

The boundary of Λ_n is $\partial\Lambda_{n,x} = \Lambda_{n,x} \setminus \Lambda_{n-1,x}$

The first question: does it percolate?

The quantity to look at is the one-arm probability:

$$\theta_{n,x}(p) = \mathbf{P}_p \Bigg[$$



$$\Bigg] = \mathbf{P}_p[x \leftrightarrow \partial\Lambda_{n,x}]$$

Notice that $\theta_{n,x}(p) \searrow \theta_x(p) = \mathbf{P}_p[x \leftrightarrow \infty]$ as $n \to \infty$

Also notice that $\theta_x(p) > 0$ if and only if $\theta_y(p) > 0$ for all $y \in \mathcal{G}$

The function $\theta_x(p)$

Can we compute $\theta_{n,x}(p)$ explicitly? Let's make an attempt!

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The function $\theta_x(p)$

Can we compute $\theta_{n,x}(p)$ explicitly? Let's make an attempt!

$$\theta_{n,x}(p) = \sum_{\omega \in \{0 \longleftrightarrow \partial\Lambda_{n,x}\}} \mathbf{P}_p[\omega]$$

$$\mathbf{P}_p[\omega] = p^{\sum_{y\in\Lambda_{n,x}} \omega_y} (1-p)^{\sum_{y\in\Lambda_{n,x}} 1-\omega_y}$$ (Easy!)

Evaluate the sum over admissible configurations (VERY difficult!)

The function $\theta_x(p)$

One can compare this with the difficulty of computing the partition function for models in statistical physics (e.g. the Ising model)

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In fact this analogy is far from being artificial!

Percolation provides one of the simplest yet extremely rich example of phase transition (e.g. solid-liquid-gas, ferromagnet-paramagnet)

The function $\theta_x(p)$: the phase diagram

The following properties are not difficult to see from the definition:

$\theta_x(p)$ is non-decreasing in p, $\theta(0) = 0$ and $\theta(1) = 1$

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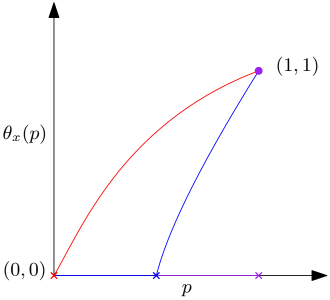
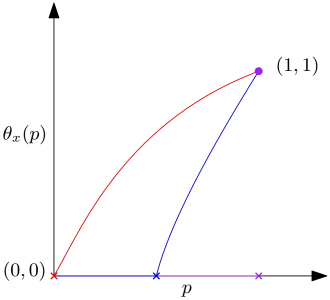


The function $\theta_x(p)$: the critical density

Therefore there exists a critical parameter $p_c = p_c(\mathcal{G})$ defined as:

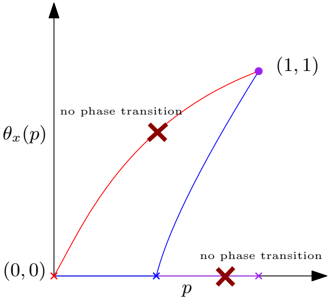
$$p_c = \sup\{p \in [0,1] : \theta(p) = 0\}$$



The function $\theta_x(p)$: existence of phase transition

Notice that p_c can be *a priori* 0 or 1 (no phase transition)

$$p_c = \sup\{p \in [0,1] : \theta(p) = 0\}$$



Existence of phase transition

p_c is always positive! In fact, $p_c(\mathcal{G}) \geq \frac{1}{\max_{v \in \mathcal{G}} \deg(v)}$ (Peierls'36)

The proof is essentially an energy-entropy type argument

Existence of phase transition

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The proof is essentially an energy-entropy type argument

$p_c(\mathbb{Z}^d) < 1$ for all $d \geq 2$ whereas $p_c(\mathbb{Z}) = 1$

Compare with the fact that the Ising model has no phase transition in dimension 1! (Ising'25)

Existence of phase transition: general graphs

Deriving a generic condition on $\mathcal{G}$ ensuring $p_c(\mathcal{G}) < 1$ is non-trivial

A natural guess would be that some "suitable" notion of dimension of $\mathcal{G}$ is strictly bigger than 1 (Benjamini-Schramm'96)

Existence of phase transition: general graphs

We say $\mathcal{G}$ has isoperimetric dimension *at least* $d > 0$ if

$$|\partial K| \geq c|K|^{\frac{d-1}{d}} \text{ for all finite } K \subset \mathcal{G}$$

Existence of phase transition: general graphs

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Theorem (Duminil-Copin, G., Raoufi, Severo and Yadin 2018)

Let $\mathcal{G}$ a graph of bounded degree with isoperimetric dimension strictly larger than 4, then $p_c(\mathcal{G}) < 1$.

Existence of phase transition: general graphs

$\mathcal{G}$ is called quasi-transitive if the action of the automorphism group $\mathrm{Aut}(\mathcal{G})$ on $\mathcal{G}$ has finitely many orbits

Typical examples include Cayley graphs of finitely generated group

We say that $\mathcal{G}$ has super-linear growth if $\limsup \frac{1}{n}|\Lambda_{n,x}| = \infty$

Existence of phase transition: general graphs

Combined with existing results our result implies:

(Trofimov'84, Lyons-Morris-Schramm'08)

Theorem (Duminil-Copin, G., Raoufi, Severo and Yadin 2018)

Let $\mathcal{G}$ be a bounded degree, quasi-transitive graph with super-linear growth, then $p_c(\mathcal{G}) < 1$.

Existence of phase transition: general graphs

We say that $\mathcal{G}$ has spectral dimension *at least* $d > 0$ if

$$p_n(x,x) = \mathbb{P}[X_n = x | X_0 = x] \leq \frac{c}{n^{d/2}}$$

where X is the simple random walk (SRW) on $\mathcal{G}$

As such spectral dimension is a dynamical property of $\mathcal{G}$

Existence of phase transition: general graphs

We say that $\mathcal{G}$ has spectral dimension *at least* $d > 0$ if

$$p_n(x,x) = \mathbb{P}[X_n = x | X_0 = x] \leq \frac{c}{n^{d/2}} \text{ for all } x \in \mathcal{G}, n \geq 1$$

where X is the simple random walk (SRW) on $\mathcal{G}$

For bounded degree graphs isoperimetric dimension $> d$ implies spectral dimension $> d$ (Varopoulos'85)

Existence of phase transition: general graphs

Theorem (Duminil-Copin, G., Raoufi, Severo and Yadin 2018)

Let $\mathcal{G}$ a graph of bounded degree with spectral dimension > 4.

Then $p_c(\mathcal{G}) < 1$.

Remark: One noteworthy feature of this result is that it connects percolation threshold, an equilibrium property of $\mathcal{G}$, with spectral dimension which is a dynamical property

Existence of phase transition: key ideas of the proof

The Gaussian free field (GFF) on $\mathcal{G}$ is a centered Gaussian field

$$\varphi = \{\varphi_x : x \in \mathcal{G}\}$$

Its law $\mathbb{P}$ is determined by its covariance kernel

$$\mathbb{E}[\varphi_x \varphi_y] = g(x,y) = \frac{1}{deg(y)} \sum_{n \geq 0} p_n(x,y)$$

$g(x,y)$ is called the Green function of the SRW on $\mathcal{G}$

Existence of phase transition: key ideas of the proof

For any centered Gaussian process ψ on $\mathcal{G}$ with covariances $\mathcal{K}(\cdot,\cdot)$,

let use define a (bond) percolation process $\eta_{\mathcal{K}}$ as follows:

Definition

Given ψ, declare an edge xy to be open with probability

$$p_{xy}(\psi) = 1 - \exp(-2(\psi_x + 1)_+(\psi_y + 1)_+)$$

independently of other edges where $a_+ = \max(a, 0)$

Notice that η_g is a dependent percolation process

Existence of phase transition: key ideas of the proof

Proposition

For every $x \in \mathcal{G}$ one has

$$\mathbb{P}[x \overset{\eta_g}{\longleftrightarrow} \infty] \geq \mathbb{E}[\mathrm{sign}(\psi_x + 1)] > 0$$

Existence of phase transition: key ideas of the proof

Let ω_p denote the standard percolation on $\mathcal{G}$ with density p

The main idea is to interpolate between ω_p and η_g

More precisely we prove (here $g_\ell(x, y) = \sum_{\ell \le n \le 2\ell} p_n(x, y)$)

$$\mathbb{P}[S \xleftrightarrow{\omega_p \cup \eta_g} T] \le \mathbb{P}[S \xleftrightarrow{\omega_{p+1/\ell^2} \cup \eta_{g-g_\ell}} T]$$

provided $p_n(x, x)$ decays sufficiently fast

Now iterate this to deduce phase transition

Existence of phase transition: open questions

- Prove under the assumption that spectral dimension > 2
- Can we get rid of the assumption of "bounded degree"?

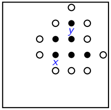
Off-critical behavior: truncated two-point function

In the remainder of the talk we will confine ourselves to the hypercubic lattice $\mathbb{Z}^d$ for $d \geq 2$

The truncated two-point function is of central importance in any model in statistical physics

In the case of percolation it is defined as

$$\tau_p(x, y) = \mathbf{P}_p \Bigg[$$



$$\Bigg] = \mathbf{P}_p[x \leftrightarrow y,\ x \not\leftrightarrow \infty]$$

Off-critical behavior: finite correlation length

It is expected that

$$\tau_p(0,x) \sim |x|^{-c} \mathrm{e}^{|x|/\xi(p)} \text{ as } x \to \infty$$

where $\xi(p)$, called the correlation length, is finite for all $p \neq p_c$

For p close to 0 or 1 (perturbative regime) finiteness of $\xi(p)$ is not very difficult to show

Off-critical behavior: finite correlation length

It is, however, very difficult to prove this for all $p \neq p_c$

In the subcritical regime $p < p_c$, this was proved by Menshikov in 1986 and by Aizenman and Barsky in 1987

In the supercritical regime $p > p_c$ this follows from a famous result by Grimmett and Marstrand in 1990

Near-critical and critical behavior: correlation length exponent

Infinite correlation length $\xi(p_c)$ is the hallmark of a critical point

It is expected that $\xi(p)$ diverges like

$$|p - p_c|^{-\nu + o(1)} \text{ as } p \to p_c \text{ for some } \nu > 0$$

Near-critical and critical behavior: known cases

Currently the correlation length exponent along with other relevant exponents are known rigorously for

- dimension 2 (Schramm, Lawler, Werner, Smirnov)
- dimension ≥ 11 (Aizenman, Barsky, Hara, Slade, Fitzner, Hofstad)

Outline

Going beyond independence

Physical and mathematical motivations abound for dependent percolation models

Physically relevant models usually involve interaction

Many models in statistical physics have representations in terms of a percolation process with certain degree of dependence

Going beyond independence: zeros of random functions

Geometry of the zero sets of random gaussian functions defined on, e.g. $\mathbb{R}^n$, is a classical topic lying at the crossroads between probability theory and geomerty

Two types of gaussian random functions have been studied:

- Random polynomials with (independent) Gaussian coefficients
- Gaussian sum of the eigenfunctions of Laplacian on a compact Riemannian manifold

Level-sets of Gaussian Free Field

In short there are many natural percolation models arising from level-sets of gaussian fields with slow decay of correlation

In the discrete set-up a canonical example is the Gaussian free field (GFF) on $\mathbb{Z}^d$ for $d \geq 3$

Indeed GFF is distributed as a Gaussian sum of the eigenfunctions of (discrete) Δ on $\mathbb{Z}^d$

Outline

GFF: definition

The GFF on $\mathbb{Z}^d$ ($d \geq 3$) is a stationary, centered Gaussian field

$$\boxed{\varphi = \{\varphi_x : x \in \mathbb{Z}^d\}}$$

Its law $\mathbb{P}$ is determined by its covariance kernel

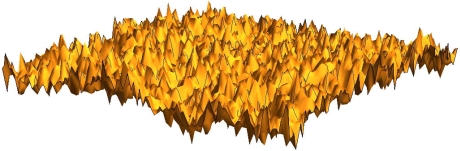
$$\boxed{\mathbb{E}[\varphi_x \varphi_y] = g(x, y) = E_x[\#\text{visits of SRW to } y]}$$

$g(x, y)$ is called the Green function of the SRW on $\mathbb{Z}^d$

GFF: correlation function

The green function is asymptotic to the newtonian potential:

$$g(x,y) = g(x-y) \sim |x-y|^{2-d} \text{ as } |x-y| \to \infty$$



A (two-dimensional) GFF. (A.Kassel)

Level-sets of GFF

We define the level-set above height h as

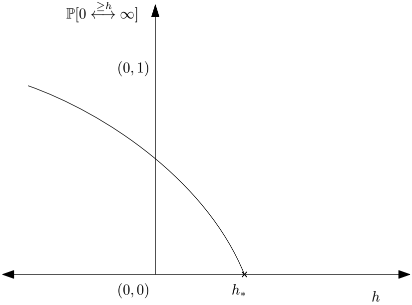
$$\{\varphi \geq h\} = \{x \in \mathbb{Z}^d : \varphi_x \geq h\}$$

These level-sets form a non-increasing family of site percolation models indexed by height

We can define the corresponding critical value $h_* = h_*(d)$ as:

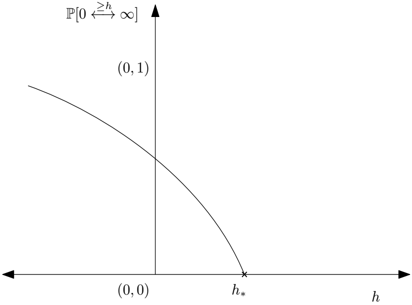
$$h_* = \inf\{h \in \mathbb{R} : \mathbb{P}[0 \overset{\geq h}{\longleftrightarrow} \infty] = 0\}$$

Level-sets of GFF: existence of phase transition



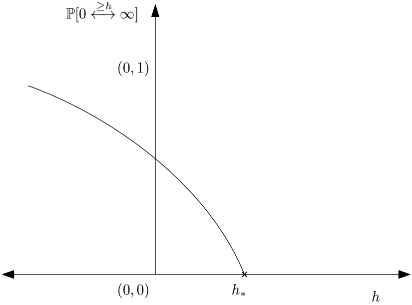
By a soft argument based on the Markov property of GFF it is possible to show that $h_* \geq 0$ (Bricmont-Lebowitz-Maes'87)

Level-sets of GFF: existence of phase transition



It is much more difficult to prove that $h_* < \infty$. It was proved for $d = 3$ in Bricmont-Lebowitz-Maes'87

Level-sets of GFF: existence of phase transition

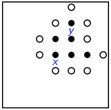


It was finally proved by Rodriguez and Sznitman (2013) for all $d \geq 3$

Level-sets of GFF: truncated two-point function

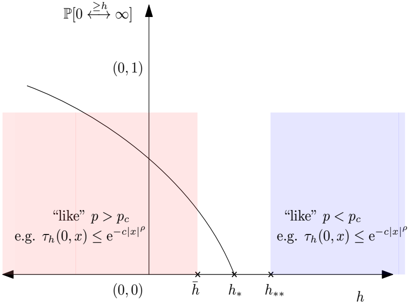
Let us recall the definition of truncated two-point function in this context:

$$\tau_h(x, y) = \mathbb{P}\Bigg[$$



$$\Bigg] = \mathbb{P}[x \overset{\geq h}{\longleftrightarrow} y,\, x \overset{\geq h}{\not\longleftrightarrow} \infty]$$

Level-sets of GFF: several possible critical points



It is a very important question to know if $\bar{h} = h_* = h_{**}$

Level-sets of GFF: uniqueness of critical point

Theorem (Duminil-Copin, G., Rodriguez, Severo 2019)

For all $d \geq 3$, $\bar{h}(d) = h_*(d) = h_{**}(d)$

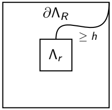
Corollary

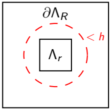
For all $d \geq 3$ *and* $h \neq h_*$, *there exists* $c > 0$ *and* $\rho \in (0, 1]$ *such that for all* $x, y \in \mathbb{Z}^d$,

$$\tau_h(x, y) \leq \mathrm{e}^{-c|x-y|^{\rho}} .$$

Level-sets of GFF: key ideas of the proof

Using a sophisticated renormalization argument we first show that

$\mathbb{P}$ [

] , $\mathbb{P}$ [

] $\geq \mathrm{poly}(R)$

for all $h \in (\bar{h}, h_{**})$

Next we show that these probabilities are superpolynomially close to those for a finitely dependent percolation process

Level-sets of GFF: open questions

However using existing techniques we can show that a wide class of finitely dependent percolation processes does not have any such phase except, possibly, at the critical point

This leads to a contradiction since $(\bar{h}, h_{**})$ can not be nonempty

Level-sets of GFF: open questions

- Is ρ actually 1? It seems that the answer might vary based on the dimension (Popov-Texeira'15, Popov-Ráth'15)
- Is it possible to obtain bounds on the exponent of correlation length?
- What happens at the critical point?

Thank you for your attention!