Anomalous diffusion on the GFF landscape

Subhajit Goswami

Based on joint work with Marek Biskup and Jian Ding

Institut des Hautes Études Scientifiques

July 22, 2019

Diffusion in a log-correlated potential

Model for various phenomena of interest in

- Statistical mechanics (e.g. single vortex in a XY spin model with gaussian random gauge disorder)
- Condensed matter physics (e.g. vacancies in pancake lattices of layered $3D$ superconductors)
- Population biology

among others

Diffusion in a log-correlated potential

Formally the model on $\mathbb{R}^2$ is described by a Langevin equation:

$$dX(t) = -\nabla \eta(X(t))dt + \sqrt{T}dB(t)$$

where $B(t)$ is a standard Brownian motion independent of the potential η satisfying

$$\mathbb{E}\left(\eta(x) - \eta(y)\right)^2 \sim 2\sigma^2 \ln|x-y| \text{ as } |x-y| \to \infty$$

Diffusion in a log-correlated potential

Physics literatures predict that $X(t)$ is subdiffusive, i.e.

$$\boxed{\mathbb{E}\, X(t)^2 \sim t^{2/Z}}$$

where the diffusive exponent $Z = Z(T)$ is strictly larger than 2

Moreover $Z(T)$ is predicted to undergo a dynamic phase transition around the critical temperature $T_c = \sigma$

Diffusion in a log-correlated potential

In fact there is a precise prediction about the exact value of

$Z(T)$ †

$$\begin{aligned} Z(T) &= 2 + 2(T_c/T)^2 \quad \text{when } T > T_c \\ &= 4T_c/T \qquad\qquad \text{otherwise} \end{aligned}$$

These exponents emerge from the intensive free energy of the equilibrium measure of the dynamics which is (formally) given by

$$L^2(\mathrm{e}^{-2\eta/T}dx)$$

A discrete version of the dynamics

Consider a random walk on $\mathbb{Z}^2$ in a log-correlated environment

A canonical example of a discrete log-correlated field in two dimensions is the discrete Gaussian free field (GFF)

More precisely we work with the centered Gaussian process $\eta = \{\eta_x : x \in \mathbb{Z}^2\}$ satisfying

$$\eta_0 = 0 \text{ and } \mathbb{E}[\eta_x \eta_y] = G_{\mathbb{Z}^2 \setminus \{0\}}(x, y) \ \forall x, y \in \mathbb{Z}^2$$

A discrete version of the dynamics

We work with the GFF pinned to 0 at the origin, i.e.

$$\eta_0 = 0 \text{ and } \mathbb{E}[\eta_x \eta_y] = G_{\mathbb{Z}^2\setminus\{0\}}(x, y)\ \forall x, y \in \mathbb{Z}^2$$

Here $G_{\mathbb{Z}^2\setminus\{0\}}$ is the Green function of simple random walk in $\mathbb{Z}^2 \setminus \{0\}$

It follows from standard random walk estimates that for some σ

$$\mathbb{E}\left(\eta_x - \eta_y\right)^2 \sim 2\sigma^2 \ln|x-y| \text{ as } |x-y| \to \infty$$

A discrete version of the dynamics

A natural dynamics on $\mathbb{Z}^2$ with equilibrium measure $\mathrm{e}^{2\eta_x/T}$ is a Markov chain with transition rates given by

$$\lambda_\eta(x,y) = \mathrm{e}^{\gamma(\eta_y - \eta_x)} \mathbf{1}_{|x-y|=1} \quad \forall\, x, y \in \mathbb{Z}^2$$

where $\gamma = 1/T$ is the inverse temperature

A discrete version of the dynamics

However we will consider the constant speed version of the walk

So conditionally on η, let $\{X_t\}_{t\geq 0}$ be a discrete-time Markov chain on $\mathbb{Z}^2$ with transition probabilities given by

$$p_\eta(x,y) = \frac{\mathrm{e}^{\gamma(\eta_y - \eta_x)}}{\sum_{z:|z-x|=1} \mathrm{e}^{\gamma(\eta_z - \eta_x)}} \mathbf{1}_{|x-y|=1}$$

A discrete version of the dynamics

The transition probabilities are given by

$$p_\eta(x,y) = \frac{e^{\gamma(\eta_y - \eta_x)}}{\sum_{z:|z-x|=1} e^{\gamma(\eta_z - \eta_x)}} \mathbf{1}_{|x-y|=1}$$

A discrete version of the dynamics

A trivial manipulation then gives us

$$p_\eta(x, y) = \frac{\mathrm{e}^{\gamma(\eta_y + \eta_x)}}{\sum_{z:|z-x|=1} \mathrm{e}^{\gamma(\eta_z + \eta_x)}} \mathbf{1}_{|x-y|=1}$$

It is reversible with respect to the measure π_η defined as

$$\pi_\eta(x) = \sum_{z:|z-x|=1} \mathrm{e}^{\gamma(\eta_z + \eta_x)}$$

A discrete version of the dynamics

In particular we have

$$\pi_\eta(x)p_\eta(x,y) = \pi_\eta(y)p_\eta(y,x) = \mathrm{e}^{\gamma(\eta_x+\eta_y)} \quad \forall\, |x-y| = 1$$

Due to reversibility there is an associated electrical network $\mathbb{Z}^2_\eta$

$\mathbb{Z}^2_\eta$ is $\mathbb{Z}^2$ where each edge is equipped with a conductance $c_\eta(\cdot,\cdot)$

$$c_\eta(x,y) = \pi_\eta(x)p_\eta(x,y) = \pi_\eta(y)p_\eta(y,x) = \mathrm{e}^{\gamma(\eta_x+\eta_y)}$$

A discrete version of the dynamics

This formulation allows us to use the theory of electric networks (or discrete potential theory) to analyze such processes

$\{X_t\}_{t\geq 0}$ is a random walk on random conductances to which a large body of literature has been devoted in recent years

A crucial difference is that the law of conductances is NOT shift invariant which makes it difficult to apply classical techniques

A discrete version of the dynamics

One possible caveat is that the equilibrium measure is

$$\pi_\eta(x) = \sum_{z:|z-x|=1} \mathrm{e}^{\gamma(\eta_x+\eta_z)} \text{ instead of } e^{2\gamma\eta_x}$$

However as we will see later this does not affect the intensive free energy (asymptotically) which is believed to drive the exponents

Main results: heat kernel and recurrence

In the rest of the talk we denote by P_η^x the law of $\{X_t\}_{t \geq 0}$ starting from x given η whereas by $\mathbb{P}$ the law of the field η

Main results: heat kernel and recurrence

In the rest of the talk we denote by P_η^x the law of $\{X_t\}_{t\geq 0}$ starting from x given η whereas by $\mathbb{P}$ the law of the field η

Theorem (Biskup, Ding, G. 2016)

There exists $g(t) = t^{o(1)}$ as $t \to \infty$ such that for each $\gamma > 0$,

$$\lim_{t\to\infty} \mathbb{P}[\, t^{-1} g(t)^{-1} \leq P_\eta^0(X_{2t} = 0) \leq t^{-1} g(t)\,] = 1$$

Furthermore, $\{X_t\}_{t\geq 0}$ is recurrent for $\mathbb{P}$-almost every η

Main results: subdiffusivity and diffusive exponent

Theorem (Biskup, Ding, G. 2016)

Let $\tau_{B(N)^c}$ *denote the first exit time of* $\{X_t\}_{t\geq 0}$ *from* $B(N) = [-N, N]^2 \cap \mathbb{Z}^2$ *and* g *be as before. Then we have*

$$\lim_{N\to\infty} \mathbb{P}[\, g(N)^{-1} N^{Z} \leq E^0_\eta(\tau_{B(N)^c}) \leq g(N) N^{Z}\,] = 1$$

Here $Z = Z(\gamma^{-1})$ *is same as before with* $\gamma_c = 1/T_c = 1/\sigma$ †

Main results: diffusive exponent

Theorem (Biskup, Ding, G. 2016)

For $\mathbb{P}$-almost every η,

$$P_\eta^0[|X_t| \geq g(t)^{-1} t^{1/Z}] \longrightarrow 1 \text{ as } t \to \infty$$

Main ideas of the proof: the *statics* part

The exponent Z emerges from the following asymptotic

$$\pi_\eta(B(N)) = N^{Z+o(1)} \text{ in probability as } N \to \infty$$

where let us recall that

$$\pi_\eta(B(N)) = \sum_{x \in B(N)} \pi_\eta(x) = \sum_{x \in B(N)} \sum_{z : |z-x|=1} \mathrm{e}^{\gamma(\eta_x + \eta_z)}$$

Main ideas of the proof: the *statics* part

$$\pi_\eta(B(N)) = \sum_{x \in B(N)} \pi_\eta(x) = \sum_{x \in B(N)} \sum_{z:|z-x|=1} \mathrm{e}^{\gamma(\eta_x+\eta_z)}$$

Up to a factor of $N^{o(1)}$ with high probability this is equal to

$$\mu_{\tilde{\eta}}(B(N)) = \sum_{x \in B(N)} \mathrm{e}^{2\gamma\tilde{\eta}_x}$$

where $\tilde{\eta}$ is a GFF on B_N with zero boundary condition

Main ideas of the proof: the *statics* part

$$\mu_{\tilde{\eta}}(B(N)) = \sum_{x \in B(N)} \mathrm{e}^{2\gamma \tilde{\eta}_x}$$

Here $\tilde{\eta}$ is a GFF on B_N with zero boundary condition

More precisely $\tilde{\eta} = \{\tilde{\eta}_x : x \in B(N)\}$ is a centered Gaussian process satisfying

$$\mathbb{E}[\tilde{\eta}_x \tilde{\eta}_y] = G_{B(N)}(x, y) \ \forall x, y \in B(N)$$

where $G_{B(N)}$ is the Green function of simple random walk in $B(N)$

Main ideas of the proof: the *statics* part

It follows from standard estimates that for some constant c

$$\mathbb{E}\tilde{\eta}_x^2 \le \sigma^2 \log N + c \ \ \forall x \in B(N)$$

Using the Gaussian formula $\mathbb{E}\mathrm{e}^{2\gamma X} = \mathrm{e}^{2\gamma^2 \mathbb{E}X^2}$ we get

$$\mathbb{E} \sum_{x \in B(N)} \mathrm{e}^{2\gamma\tilde{\eta}_x} \preccurlyeq N^{2+2\gamma^2\sigma^2} = N^{2+2(\gamma/\gamma_c)^2}$$

Recall that we defined $\gamma_c = 1/\sigma$

Main ideas of the proof: the *statics* part

$$\mathbb{E} \sum_{x \in B(N)} \mathrm{e}^{2\gamma \tilde{\eta}_x} \preccurlyeq N^{2+2\gamma^2\sigma^2} = N^{2+2(\gamma/\gamma_c)^2} = N^{Z(\gamma^{-1})} \text{ for } \gamma \leq \gamma_c$$

From Markov's inequality we therefore get the desired upper bound for $\gamma \leq \gamma_c$

Main ideas of the proof: the *statics* part

Roughly speaking the value of $\tilde{\eta}_x$ that contributes "most" to the expected value of $\mathrm{e}^{2\gamma\tilde{\eta}_x}$ is $2\gamma\mathbb{E}\tilde{\eta}_x^2$

In particular the largeness of our expected measure is attributable to a value around

$$2\gamma\sigma^2 \log N = (\gamma\sigma)\, 2\sigma \log n = 2\sigma \log n\, \gamma/\gamma_c$$

Main ideas of the proof: the *statics* part

Largeness of the expected measure is attributable to a value around

$$2\gamma\sigma^2 \log N = (\gamma\sigma)\, 2\sigma \log n = 2\sigma \log n\, \gamma/\gamma_c$$

On the other hand from the standard estimates we get

$$\max_{x \in B(N)} \tilde{\eta}_x \leq \sqrt{2 \log |B(N)|} \sqrt{\max_{x \in B(N)} \mathbb{E} \tilde{\eta}_x^2}\, (1 + o(1))$$

with high probability (w.h.p.) as $N \to \infty$

Main ideas of the proof: the *statics* part

Largeness of the expected measure is attributable to a value around

$$2\gamma\sigma^2 \log N = (\gamma\sigma)\, 2\sigma \log n = 2\sigma \log n\, \gamma/\gamma_c$$

Plugging the upper bound on variance we deduce

$$\max_{x \in B(N)} \tilde{\eta}_x \leq 2\sigma \log N \; (1 + o(1)) = m_N \; (say)$$

w.h.p. as $N \to \infty$

Main ideas of the proof: the *statics* part

It is therefore clear that for $\gamma > \gamma_c$ the typical order is smaller

Indeed if we consider the truncated expected measure

$$\mathbb{E}[\mathrm{e}^{2\gamma\tilde{\eta}_x}; \tilde{\eta}_x \le m_N] = \mathrm{e}^{2\gamma^2\mathbb{E}\tilde{\eta}_x^2}\,\mathbb{P}[\tilde{\eta}_x \le m_N - 2\gamma\mathbb{E}\tilde{\eta}_x^2]$$

then we get the correct exponent at all temperatures

Main ideas of the proof: the *statics* part

Notice that we did not use any information about the covariances of the field to derive these upper bounds

The corresponding lower bounds, on the other hand, involve second moment computations which exploit the covariance structure of the GFF

Main ideas of the proof: exit times of the random walk

For $S \subset \mathbb{Z}^d$, we denote the first hitting time of S by X_t as τ_S, i.e.

$$\tau_S = \inf\{t \geq 0 : X_t \in S\}$$

Recall that $\tau_{B(N)^c}$ is the first exit time of $\{X_t\}_{t \geq 0}$ from $B(N)$

An expression for $\mathbb{E}^0_\eta \tau_{B(N)^c}$ is given by the hitting time identity

$$\mathbb{E}^0_\eta \tau_{B(N)^c} = R_\eta(0, \partial B(N)) \sum_{x \in B(N)} \pi_\eta(x) P^x_\eta[\tau_0 < \tau_{B(N)^c}]$$

Main ideas of the proof: exit times of the random walk

An expression for $\mathbb{E}_\eta^0 \tau_{B(N)^c}$ is given by the hitting time identity

$$\mathbb{E}_\eta^0 \tau_{B(N)^c} = R_\eta(0, \partial B(N)) \sum_{x \in B(N)} \pi_\eta(x) P_\eta^x[\tau_0 < \tau_{B(N)^c}]$$

where $\partial B(N) = B(N+1) \setminus B(N)$

$R_\eta(\cdot, \cdot)$ is the effective resistance in the network $B(N+1)_\eta$, i.e.

$$R_\eta(S, T) = \inf_{\substack{\theta \text{ unit flow}, \\ src(\theta) = S,\, sink(\theta) = T}} \sum_{\substack{x, y \in B(N+1), \\ |x-y|=1}} \theta_{x,y}^2 / c_\eta(x, y)$$

Main ideas of the proof: exit times of the random walk

An expression for $\mathbb{E}_\eta^0 \tau_{B(N)^c}$ is given by the hitting time identity

$$\mathbb{E}_\eta^0 \tau_{B(N)^c} = R_\eta(0, \partial B(N)) \sum_{x \in B(N)} \pi_\eta(x)\, P_\eta^x[\tau_0 < \tau_{B(N)^c}]$$

Therefore an immediate upper bound is

$$\mathbb{E}_\eta^0 \tau_{B(N)^c} \leq R_\eta(0, \partial B(N)) \sum_{x \in B(N)} \pi_\eta(x)$$

Main ideas of the proof: exit times of the random walk

An expression for $\mathbb{E}_\eta^0 \tau_{B(N)^c}$ is given by the hitting time identity

$$\mathbb{E}_\eta^0 \tau_{B(N)^c} = R_\eta(0, \partial B(N)) \sum_{x \in B(N)} \pi_\eta(x)\, P_\eta^x[\tau_0 < \tau_{B(N)^c}]$$

Therefore an immediate upper bound is

$$\mathbb{E}_\eta^0 \tau_{B(N)^c} \leq R_\eta(0, \partial B(N))\, \pi_\eta(B(N))$$

Main ideas of the proof: exit times of the random walk

An upper bound on the expected first exist time is

$$\mathbb{E}^0_\eta \tau_{B(N)^c} \le R_\eta(0, \partial B(N))\, \pi_\eta(B(N))$$

However we already know

$$\pi_\eta(B(N)) \le N^{Z+o(1)} \;\; w.h.p.$$

Hence to show $\mathbb{E}^0_\eta \tau_{B(N)^c} \le N^{Z+o(1)}$ w.h.p. it suffices to prove

$$R_\eta(0, \partial B(N)) \le N^{o(1)} \;\; \text{w.h.p.}$$

Main ideas of the proof: exit times of the random walk

An expression for $\mathbb{E}^0_\eta \tau_{B(N)^c}$ is given by the hitting time identity

$$\mathbb{E}^0_\eta \tau_{B(N)^c} = R_\eta(0, \partial B(N)) \sum_{x \in B(N)} \pi_\eta(x)\, P^x_\eta[\tau_0 < \tau_{B(N)^c}]$$

To prove the corresponding lower bound it suffices to show that w.h.p. uniformly for all $x \in B(N^{1-o(1)})$ we have

$$R_\eta(0, \partial B(N))\, P^x_\eta[\tau_0 < \tau_{B(N)^c}] \geq N^{o(1)}$$

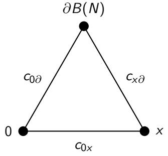
Main ideas of the proof: exit times of the random walk

We want to obtain an expression for

$$\boxed{R_\eta(0, \partial B(N))\, P_\eta^x[\tau_0 < \tau_{B(N)^c}]}$$

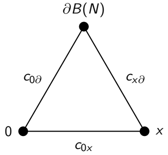
in terms of effective resistances

By the network reduction principle the trace of $\{X_t\}$ restricted to $\{0, x, \partial B(N)\}$ corresponds to a three-node network



Main ideas of the proof: exit times of the random walk

The trace of $\{X_t\}$ restricted to $\{0, x, \partial B(N)\}$ corresponds to



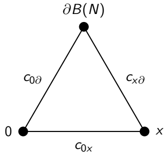
in a way that the pairwise effective resistances remain same

However in terms of this network

$$P_\eta^x[\tau_0 < \tau_{B(N)^c}] = \frac{c_{0x}}{c_{0x} + c_{x\partial}}$$

Main ideas of the proof: exit times of the random walk

The trace of $\{X_t\}$ restricted to $\{0, x, \partial B(N)\}$ corresponds to



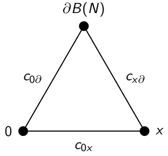
in a way that the pairwise effective resistances remain same

By series and parallel laws

$$\frac{c_{0x}}{c_{0x} + c_{x\partial}} = \frac{R_\eta(0, \partial B(N)) + R_\eta(x, \partial B(N)) - R_\eta(0, x)}{2R_\eta(0, \partial B(N))}$$

Main ideas of the proof: exit times of the random walk

The trace of $\{X_t\}$ restricted to $\{0, x, \partial B(N)\}$ corresponds to

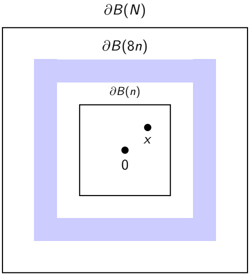


in a way that the pairwise effective resistances remain same

Therefore we get the identity

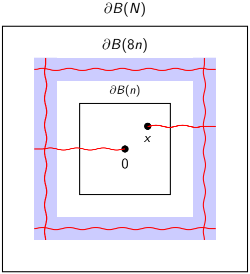
$$2R_\eta(0,\partial B(N))\, P_\eta^x[\tau_0 < \tau_{B(N)^c}] = R_\eta(0, \partial B(N)) + R_\eta(x, \partial B(N)) - R_\eta(0, x)$$

Main ideas of the proof: exit times of the random walk



$$D_{N,\eta}(x) = R_\eta(0, \partial B(N)) + R_\eta(x, \partial B(N)) - R_\eta(0, x)$$

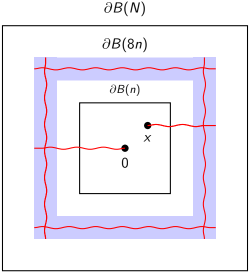
Main ideas of the proof: exit times of the random walk



By the metric property of effective resistance and planarity

$$R_\eta(0,x) \le R_\eta(0,\partial B(8n)) + R_\eta(x,\partial B(8n)) + R_\eta(\Box; n)$$

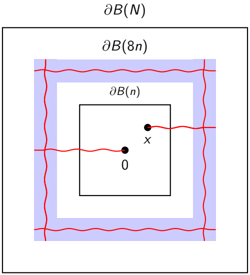
Main ideas of the proof: exit times of the random walk



Since $\partial B(8n)$ is a cut-set between $y \in \{0, x\}$ and $\partial B(N)$

$$R_\eta(y, \partial B(N)) \geq R_\eta(y, \partial B(8n)) + R_\eta(\partial B(8n), \partial B(N))$$

Main ideas of the proof: exit times of the random walk



Therefore a lower bound on the difference $D_{N,\eta}(x)$ is given by

$$\boxed{D_{N,\eta}(x) \geq 2R_\eta(\partial B(8n), \partial B(N)) - R_\eta(\square; n)}$$

Main ideas of the proof: exit times of the random walk

Showing that there exists some n between $N^{1-o(1)}$ and N w.h.p.

such that

$$R_\eta(\partial B(8n), \partial B(N)) - R_\eta(\Box; n) \geq N^{o(1)}$$

requires delicate analysis involving the Markov property of GFF

However it says that we definitely need lower bounds like

$$R_\eta(\partial B(n), \partial B(N)) \geq N^{o(1)} \;\; w.h.p.$$

Main ideas of the proof: bounding resistances

We need bounds of the form

$$N^{o(1)} \geq R_\eta(\partial B(n), \partial B(N)) \geq N^{-o(1)} \;\; w.h.p.$$

The idea is to show that the law of effective resistance is symmetric with respect to inversion $x \to 1/x$

In other words the laws of the effective resistance and effective conductance are close to each other

Main ideas of the proof: bounding resistances

Proposition (generalized series law)

For any network $(\mathcal{G}, c)$ we have

$$R_{\mathcal{G}}(x, y) = \min_{\substack{\mathcal{P} \\ \text{a set of paths btw } x,\, y}} \; \min_{\{r_{e,P} : e \in E(\mathcal{G}), P \in \mathcal{P}\} \in \mathfrak{R}_{\mathcal{P}}} \Big(\sum_{P \in \mathcal{P}} \frac{1}{\sum_{e \in P} r_{e,P}} \Big)^{-1}$$

where $\mathfrak{R}_{\mathcal{P}}$ is the set of all possible choices of $r_{e,P}$'s such that

$$\sum_{P \in \mathcal{P}} \frac{1}{r_{e,P}} \le \frac{1}{r_e} = c_e \text{ for all } e \in E(\mathcal{G}).$$

Main ideas of the proof: bounding resistances

Proposition (generalized parallel law)

For any network $(\mathcal{G}, c)$ *we have*

$$C_{\mathcal{G}}(x,y) = \min_{\substack{\Pi \\ \text{a set of cutsets btw } x,\, y}} \; \min_{\{c_{e,\pi}: e \in E(\mathcal{G}), \pi \in \Pi\} \in \mathfrak{C}_\Pi} \Big(\sum_{\pi \in \Pi} \frac{1}{\sum_{e \in \pi} c_{e,\pi}}\Big)^{-1}$$

where $\mathfrak{C}_\Pi$ *is the set of all possible choices of* $r_{e,P}$*'s such that*

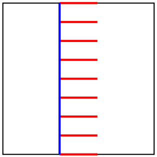
$$\sum_{\pi \in \Pi} \frac{1}{c_{e,\pi}} \le \frac{1}{c_e} \text{ for all } e \in E(\mathcal{G})\,.$$

Main ideas of the proof: bounding resistances

However since $\eta \overset{\Delta}{\sim} -\eta$, we have

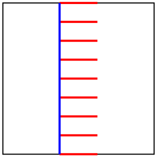
$$\{c_\eta(x,y)\} = \{\mathrm{e}^{\gamma(\eta_x+\eta_y)}\} \overset{\Delta}{\sim} \{\mathrm{e}^{-\gamma(\eta_x+\eta_y)}\} = \{r_\eta(x,y)\}$$

By planar duality a cut-set between two facing walls of a square corresponds to a path between the other two walls



Main ideas of the proof: bounding resistances

By planar duality a cut-set between two facing walls of a square corresponds to a path between the other two walls



The local smoothness of η then implies that

$$C_\eta(\partial_{\mathrm{left}} B, \partial_{\mathrm{right}} B) \overset{\text{roughly}}{\sim} N^{o(1)} R_\eta(\partial_{\mathrm{top}} B, \partial_{\mathrm{bottom}} B)$$

Main ideas of the proof: bounding resistances

The local smoothness of η implies that

$$C_\eta(\partial_{\mathrm{left}} B, \partial_{\mathrm{right}} B) \overset{\text{roughly}}{\sim} N^{o(1)} R_\eta(\partial_{\mathrm{top}} B, \partial_{\mathrm{bottom}} B)$$

Since the law of η is invariant with respect $\pi/2$-rotations, we get

$$C_\eta(\partial_{\mathrm{left}} B, \partial_{\mathrm{right}} B) \overset{\text{roughly}}{\sim} N^{o(1)} R_\eta(\partial_{\mathrm{left}} B, \partial_{\mathrm{right}} B)$$

Main ideas of the proof: bounding resistances

Since the law of η is invariant with respect $\pi/2$-rotations, we get

$$C_\eta(\partial_{\mathrm{left}} B, \partial_{\mathrm{right}} B) \overset{\text{roughly}}{\sim} N^{o(1)} R_\eta(\partial_{\mathrm{left}} B, \partial_{\mathrm{right}} B)$$

By a simple application of Gaussian concentration inequality we get

$$N^{o(1)} \geq R_\eta(\partial_{\mathrm{left}} B, \partial_{\mathrm{right}} B) \geq N^{-o(1)} \;\; w.h.p.$$

Main ideas of the proof: bounding resistances

So we have it across squares

$$N^{o(1)} \geq R_\eta(\partial_{\mathrm{left}} B, \partial_{\mathrm{right}} B) \geq N^{-o(1)} \ \ w.h.p.$$

But we want it across long rectangles and eventually across annuli

Sounds like Russo-Seymour-Welsh theory in planar percolation

Thank you for your attention!