Fluctuations in the number of level sets of planar Gaussian fields

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joint work with
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Recall that $N_{LS}(R;\ell)$ and $N_{ES}(R;\ell)$ are the number of level/ excursion set components of a Gaussian field f inside a ball $B(R)$, with $c_{LS}(\ell) := \lim_{R\to\infty} N_{LS}(R;\ell)/(\pi R^2)$ and c_{ES} similar.

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Theorem (Belyaev, McAuley, M., '19)

For the BF field

$$Var[N_{LS}(R;\ell)] \gtrsim R^2$$

for all $\ell \in \mathbb{R}$ such that $c'_{LS}(\ell) \neq 0$, and similarly for $N_{ES}(R;\ell)$.

For the RPW

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The orders R^2/R^3 for the BF/RPW are quite natural and we expect them to be tight (at least for generic levels).

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Corollary (Belyaev, McAuley, M.)

The variance of N_{LS}/N_{ES} is of 'full order' for the following levels:

- *BF field ('full order' $= R^2$)*
  - *LS:* $\ell \in (-\infty, -1.38) \cup (1.38, \infty)$
  - *ES:* $\ell \in (-\varepsilon, 0.64) \cup (1.02, \infty)$
- *RPW ('full order' $= R^3$)*
  - *LS:* $\ell \in (-\infty, -1] \cup [1, \infty)$
  - *ES:* $\ell \in (-\infty, 0) \cup (0, 0.87) \cup [1, \infty)$

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1) One can bound global topological events by local observables.

2) Gaussian fields can 'breathe', i.e. f and $f + \varepsilon$ are statistically indistinguishable on a compact domain $D \subset \mathbb{R}^2$ for small enough $\varepsilon > 0$ (depending on D).

3) By a coupling argument, one can deduce variance lower bounds for the number of level sets at a fixed level by considering the **change as the level varies**.

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Lemma

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Hence, by the Kac-Rice formula

$$\begin{aligned}
&\mathbb{E}[N_{LS}(R;\ell)^2] \\
&\qquad \leq \mathbb{E}[N_{LS}(R;\ell)(N_{LS}(R;\ell)-1)] \\
&\qquad = \int_{x,y\in B(R)} \mathbb{E}\left[|det(\nabla^2 f(x)\nabla^2 f(x))|\big|\nabla f(x) = \nabla f(y) = 0\right] \\
&\qquad\qquad \times \varphi_{\nabla f(x),\nabla f(y)}(0,0)\, dxdy
\end{aligned}$$

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&\qquad\quad \times \varphi_{\nabla f(x),\nabla f(y)}(0,0)\, dxdy \\
&\quad \lesssim R^4.
\end{aligned}$$

We need a slightly refined version valid for small height windows:

Lemma (M. 20)

For every $\ell \in \mathbb{R}$, $R > 0$ *and* $a < b$,

$$|N_{LS}(R; b) - N_{LS}(R; a)| \lesssim \#\ \textit{critical points in } B(R) \textit{ with level in } [a, b].$$

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$$\mathbb{E}[(N_{LS}(R; b) - N_{LS}(R; a))^2]$$
$$\leq \int_{\substack{x,y\in B(R)\\ s,t\in[a,b]}} \mathbb{E}\left[|det(\nabla^2 f(x)\nabla^2 f(x))| \middle| (\nabla f(x), \nabla f(y), f(x), f(y)) = (0,0,s,t)\right]$$
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2) Gaussian fields can 'breathe'

We next consider how small $\varepsilon > 0$ needs to be, as a function of R, such that the fields

$$f|_{B(R)} \qquad \text{and} \qquad (f+\varepsilon)|_{B(R)}$$

are close in total variation distance (and hence any functionals of the fields are also close in TV-distance).

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Nevertheless, an **approximate** version of this comparison is true and this is sufficient for our purposes.

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Fact 1: Since the covariance kernel K is in $L^1(\mathbb{R}^2)$, f has a 'moving average representation'

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where $q \in L^2(\mathbb{R}^2)$ is a kernel such that $q \star q = K$, and W is the white noise on $\mathbb{R}^2$.

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Fact 2: Let $W_D = W \mathbb{1}_D$ be white noise on a compact domain $D \subset \mathbb{R}^2$. Then $f_D = q \star W_D$ has an 'orthogonal decomposition'

$$f_D = q \star W_D = \sum_{i \geq 1} Z_i (q \star \varphi_i)$$

where Z_i is a sequence of independent standard Gaussians, and φ_i is any orthonormal basis of $L^2(D)$.

Fact 3: For a standard Gaussian vector $Z = (Z_1, \dots Z_n)$,

$$d_{TV}(Z, Z + \varepsilon) \lesssim \varepsilon\sqrt{n}.$$

Proof. Use Pinsker's inequality $d_{TV}(\mu, \nu) \lesssim \sqrt{d_{KL}(\mu||\nu)}$, and then the additivity of relative entropy for product measures.

Putting these together we have:

Proposition

For the BF field

$$d_{TV}(f|_{B(R)}, (f + \varepsilon g|_{B(R)})) \lesssim \varepsilon R$$

where $g = (q \star \mathbb{1}_{B(2R)})|_{B(R)} \approx (\int q) \times \mathbb{1}_{B(R)}$.

Proof. Divide the plane into unit boxes D_i, then decompose orthogonally each $W\mathbb{1}_{D_i}$ with $\varphi_1 = c\mathbb{1}_{D_i}$, and then shift the Gaussians Z_1 by ε in each box.

Upshot: Since $\int q > 0$ for the BF field, the number of level/excursion set components of

$$f|_{B(R)} \qquad \text{and} \qquad (f + \varepsilon)|_{B(R)}$$

are close in total variation distance as soon as $\varepsilon \ll 1/R$.

The proceeding argument works for any field f whose spectral density ρ does not vanish at the origin (the condition $\int q > 0$ is equivalent to $\rho(0) > 0$).

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However, for the RPW this argument does not work ($K \notin L^1$, and the spectral measure is Lebesgue on the unit circle).

There is a good reason for this – the RKHS of the RPW consists of solutions of the Helmholtz equation $\nabla f = -f$ which does not contain constant functions (or any approximation of them).

So one cannot expect f and $f + \varepsilon$ to be comparable in total variation distance.

For the RPW we use a slightly different approach. Recall the orthogonal decomposition of the RPW

$$f(x) = f(r, \theta) = \Re\Big(\sum_{n=-\infty}^{n=\infty} Z_n e^{2\pi i n\theta} J_{|n|}(r) \Big)$$

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Fact 3^***:** For a standard Gaussian vector $Z = (Z_1, \dots Z_n)$,

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Proof. Again use Pinsker's inequality.

Upshot: Fix $\ell \neq 0$. Then the fields

$$f_n|_{B(R)} \qquad \text{and} \qquad \left(f_n \times \frac{\ell+\varepsilon}{\ell}\right)\Big|_{B(R)}$$

are close in total variation distance as soon as $\varepsilon \ll 1/\sqrt{R}$.

Hence so are the number of components of the level sets

$$\{f|_{B(R)} = \ell\} \qquad \text{and} \qquad \{f|_{B(R)} = \ell + \varepsilon\}.$$

To sum-up, we have:

BF: The number of level/excursion set components of $f|_{B(R)}$ at levels

$$\ell \qquad \text{and} \qquad \ell+\varepsilon$$

are close in total variation distance for $\varepsilon \ll 1/R$.

RPW: The number of level/excursion set components of $f|_{B(R)}$ at levels

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are close in total variation distance for $\varepsilon \ll 1/\sqrt{R}$.

The fact that we can shift levels by $1/\sqrt{R}$ ($\gg 1/R$) for the RPW is a manifestation of the **strong degeneracies** in the RPW.

3) A coupling method for variance lower bounds

Recently Chatterjee proposed a general coupling method for proving lower bounds on variances:

Lemma (Chatterjee 2017)

Let X and Y be two random variables defined on the same probability space. Then for every $a \leq b$,

$$P(a \leq X \leq b) \leq \frac{1}{2}\Big(1 + P(|X - Y| \leq b - a) + d_{TV}(X, Y)\Big).$$

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Hence, if X_n is a sequence of random variables and there exist another sequence Y_n, and constants $\sigma_n, \delta > 0$, such that:

- $P(|X_n - Y_n| > \delta\sigma_n) > 1/4$
- $d_{\mathrm{TV}}(X_n, Y_n) < 1/8$.

then X_n fluctuates on the scale $\gtrsim \sigma_n$, and hence $\mathrm{Var}(X_n) \gtrsim \sigma_n^2$.

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By controlling the error in the above convergence, if $c'_{ES}(\ell) \neq 0$ we can prove that

$$\mathbb{E}[Y_R - X_R] = \varepsilon_R R^2 c'_{ES}(\ell) + o(\varepsilon_R R^2).$$

Moreover, by considering critical points with heights in $[\ell, \ell + \varepsilon_R]$, we have

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To conclude, for the BF we have $d_{TV}(X_R, Y_R) \lesssim 1$ as long as $\varepsilon_R \lesssim 1/R$, and so the coupling method gives $\mathrm{Var}[X_R] \gtrsim (\varepsilon_R R^2)^2 \gtrsim R^2$.

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On the other hand, if $\ell \neq 0$ then for the RPW we have $d_{TV}(X_R, Y_R) \lesssim 1$ as long as $\varepsilon_R \lesssim 1/\sqrt{R}$, and so the coupling method gives $\mathrm{Var}[X_R] \gtrsim (\varepsilon_R R^2)^2 \gtrsim R^3$.

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Recall that we can prove

$$c_{ES}(\ell) = \int_{\ell}^{\infty} p_{m^+}(x) - p_{s^-}(x)\,dx = \int_{\ell}^{\infty} p_{m^+}(x) - p_s(x)g(x)\,dx.$$

where $p_s(\ell)$ is the density of saddle points, and

$g(\ell) := \mathbb{P}[0 \text{ is a lower connected saddle} \mid 0 \text{ is a saddle with height } \ell]$.

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$g(\ell) := \mathbb{P}[0 \text{ is a lower connected saddle} \mid 0 \text{ is a saddle with height } \ell].$

Hence $c'_{ES}(\ell) \neq 0$ is equivalent to

$$g(\ell) \neq p_{m^+}(\ell)/p_s(\ell)$$

and the right-hand side is explicitly computable.

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This is surprisingly hard to prove in general. But by considering the field conditioned to have a saddle at 0 at height x (and using explicit properties of the BF/RPW) we can prove:

Lemma

For both the BF and RPW, $g(\ell)$ is strictly increasing in ℓ.

Since $g(0) = 1/2$ (by symmetry), this yields

$$g(\ell) = \begin{cases} > 1/2 & \text{if } \ell > 0, \\ < 1/2 & \textit{if } \ell < 0, \end{cases}$$

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which gives regions where $g(\ell) \neq p_{m^+}(\ell)/p_s(\ell)$ and so $c'_{ES}(\ell) \neq 0$.

A similar argument works for $c'_{LS}(\ell)$.

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2) For the RPW, there are no local maxima at levels $\ell < 0$.

Hence $c'_{ES}(\ell) > 0$ for all $\ell \leq 0$.

The result of this argument is explicit, but somewhat modest:

Corollary (Belyaev, McAuley, M.)

We have $c'_{LS}(\ell) \neq 0$ and $c'_{ES}(0) \neq 0$ (and hence 'full order' variance, except at $\ell = 0$ for the RPW) for the following levels:

- *BF field*
  - *LS:* $\ell \in (-\infty, -1.38) \cup (1.38, \infty)$
  - *ES:* $\ell \in (-\varepsilon, 0.64) \cup (1.02, \infty)$
- *RPW*
  - *LS:* $\ell \in (-\infty, -1] \cup [1, \infty)$
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2. Are fluctuations of the number of level/excursion sets genuinely of lower order if $c'_{LS}(\ell) = 0$ and $c'_{ES}(\ell) = 0$? We think this is **not** true for the BF field, but might be true for the RPW (by analogy with the length of the nodal set, where this phenomena occurs and is known as **Berry cancellation**). If it's true, how small are the fluctuations in these cases?

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3. Matching upper bounds / leading order constants / CLTs?

Thank you!

References:

D. Beliaev, M. McAuley and S. Muirhead, Fluctuations in the number of level and excursion set components of a planar Gaussian field, preprint, 2019

D. Beliaev, M. McAuley and S. Muirhead, Smoothness and monotonicity of the excursion set density of planar Gaussian fields, to appear in *Electron. J. Probab*

D. Beliaev, M. McAuley and S. Muirhead, On the number of excursion sets of planar Gaussian fields, to appear in *Probab. Theory Related Fields*

S. Muirhead, A second moment bound for critical points of planar Gaussian fields in shrinking height windows, to appear in *Stat. Prob. Lett.*