Generalised Gaussian Kinematic Formulae

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Integral formulae: definition via example

Consider a domain D, and the set of straight lines G in $\mathbb{R}^2$.
Parameterization of G: angle ϕ that the direction perpendicular to given line ℓ makes with a fixed direction; and distance p of line ℓ from the origin.
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What is the "cumulative" chord length?

$$\int_G \sigma_\ell(D)\, d\ell = \pi \times (\text{area of } D)$$

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- What is $\mathbb{E}(X_1)$? Clearly,

$$\mathbb{E}(X_1) = \sum_{n \geq 0} np_n = f(L_1) \quad \text{(the only parameter in the problem)},$$

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- What is $\mathbb{E}(X_1)$? Clearly,

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  Answer: $\mathbb{E}(X_1 + X_2) = f(L_1) + f(L_2)$ (by linearity).
- What if the needles were welded together? Will the mean of the total number of intersections change? No!

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  The proportionality constant can be found to be $\frac{2}{\pi d}$ (by choosing the piece of wire to be a circle with diameter d).
- This rather non-probabilistic proof of Buffon's needle problem was given by Barbier (1860).

A kinematic formula

- Consider two rectifiable curves Γ_1 and Γ_2 in $\mathbb{R}^2$, with lengths L_1 and L_2.
- Let G_2 be the group of rigid motions in $\mathbb{R}^2$, equipped with the *natural* measure ν.
- Let $\phi(\Gamma_1 \bigcap g\Gamma_2)$ be the number of points of intersection of the curves Γ_1 and $g\Gamma_2$.

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Theorem (Poincaré (1912))

$$\int_{G_2} \phi(\Gamma_1 \cap g\Gamma_2)\, d\nu(g) = 4\ L_1\ L_2$$

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Remark: Important aspect of above problems: the rigid motion invariances.

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- Natural candidates: rigid motion invariant, additive and monotone *valuations*.
- Hadwiger (1957): Consider $\mathcal{K}^n$, the family of all polyconvex sets. Then, there exist $(n+1)$ geometric functionals which form a basis for all rigid motion invariant, additive, monotone *valuations*. These geometric functionals are called Lipschitz-Killing curvatures (LKCs) / Minkowski functionals. [for proof: Klain-Rota (1997), or Beifang Chen (2004)]

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- But, how does one characterize LKCs? $\longrightarrow$ A tube formula

- For an m-dimensional subset $A \subset \mathbb{R}^n$, $\mathcal{L}_0(A)$ is its Euler–Poincaré characteristic, and $\mathcal{L}_m(A)$ is its m-dimensional volume.
- $\mathcal{L}_i$, of say a set A, is an intrinsic, integral geometric characteristics of the set.
- LKCs for a smooth Riemannian manifold M can be defined as

$$\mathcal{L}_k(M) = c(n,k) \int_M \mathrm{Tr}\left(R^{\frac{n-k}{2}}\right) Vol_g$$

whenever $\frac{n-k}{2}$ is an integer, and it is zero otherwise.
- Scaling: $\mathcal{L}_k(\lambda A) = \lambda^k \mathcal{L}_k(A)$.

Lipschitz–Killing curvatures (LKCs): examples

- A box B with dimensions (a, b, c): $\mathcal{L}_0(B) = 1$, $\mathcal{L}_1(B) = (a + b + c)$, $\mathcal{L}_2(B) = (ab + bc + ac)$, $\mathcal{L}_3(B) = abc$.
- A ball $B_n(r)$ of radius r in $\mathbb{R}^n$:

$$\mathcal{L}_j(B_n(r)) = r^j \binom{n}{j} \frac{\omega_n}{\omega_{n-j}}$$

- A sphere $S^{n-1}(r)$ of radius r in $\mathbb{R}^n$:

$$\mathcal{L}_j(S^{n-1}(r)) = 2r^j \binom{n}{j} \frac{\omega_n}{\omega_{n-j}},$$

  for even values of $(n - j - 1)$, and 0 otherwise.
- For a unit codimensional manifold, every alternate $\mathcal{L}_i$ vanishes.

Euclidean Kinematic Fundamental Formula (KFF)

Bröcker & Kuppe (2000)

- G_n: isometry group on $\mathbb{R}^n$; isomorphic to $\mathbb{R}^n \times O(n)$.
- ν_n: a normalized measure on G_n, such that for any $A \in \mathcal{B}(\mathbb{R}^n)$, $\nu_n(\omega \in G_n : \omega(x) \in A) = \mathcal{H}_n(A)$, for any $x \in \mathbb{R}^n$.
- Then for *smooth* M_1 and M_2, writing $M_2(\omega) = \{\omega(x) : x \in M_2\}$, we have

$$\int_{G_n} \mathcal{L}_i(M_1 \cap M_2(\omega))\, d\nu_n(\omega) = \sum_{j=0}^{n-i} \frac{s_{i+1}s_{n+1}}{s_{i+j+1}s_{n-j+1}} \mathcal{L}_{i+j}(M_1)\mathcal{L}_{n-j}(M_2)$$

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- An earlier version in two dimensions was proved by Blaschke.

Gaussian Kinematic Fundamental Formula

Preliminaries

- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a random field defined on $\mathbb{R}^d$, and M be a smooth manifold embedded in $\mathbb{R}^d$.
- Consider the sets: $N_u^f(\omega) = \{x \in \mathbb{R}^d : f(x, \omega) \geq u\}$

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Interest is in computing

$$\int_\Omega \mathcal{L}_0 \left(M \cap N_u^f(\omega) \right) \mu(d\omega)$$

A Gaussian Kinematic Formula (GKF)

Taylor (2006)

- Let M be an m-dimensional smooth manifold.
- Let $y_1, \ldots, y_k$ be i.i.d. Gaussian random fields on M.
- Let $F : \mathbb{R}^k \to \mathbb{R}$ be twice differentiable, and define $f = F(y_1, y_2, \ldots, y_k)$. Then

$$\mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right)\right) = \sum_{j=0}^{n} c_j \, \mathcal{L}_j^{y}(M) \, \mathcal{M}_j^{\gamma_k}\left(F^{-1}[u,\infty)\right)$$

where $\mathcal{L}_j^{y}(\cdot)$ are the LKCs defined w.r.t. the induced metric given by

$$g^{y}(X,Y) = \mathbb{E}\left(Xy_1 \cdot Yy_1\right),$$

(The metric induced by any y_i is the same due to i.i.d. nature of y_i's); and $\mathcal{M}_j^{\gamma_k}$ are the Gaussian Minkowski functionals (GMFs).

Gaussian geometric characteristics via a Gaussian tube formula

Gaussian Minkowski functionals (GMFs): $\mathcal{M}_j^{\gamma_n}$

- Let A be *smooth* subset of $\mathbb{R}^n$, with $\gamma_n(dx) = (2\pi)^{-n/2} e^{-\|x\|^2/2} dx$, then the GMFs can be defined as

$$\gamma_n(\mathrm{Tube}(A,\rho)) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma_n}(A),$$

  where $\mathrm{Tube}(A,\rho)$ is a tube of radius ρ around A.

- One can also define the GMFs as integral of some Hermite polynomials with respect to the measures induced by $\mathcal{L}_i$'s, called the generalized curvature measures.

Discussion

- Recall that

$$\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right) = \sum_{k=0}^{m}(-1)^k \mu_k$$

where

$$\mu_k = \#\{x \in M : f(x) \geq u,\ \ \nabla f(x) = 0,\ \ \text{index}\left(\nabla^2 f(x)\right) = k\}.$$

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- Using this relationship and a generalized Kac-Rice formula we can try and compute $\mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right)\right)$.
- Once we have a simplified expression, the goal is to identify various terms involved, and finally get

$$\mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right)\right) \quad = \quad \sum_{j=0}^{n} c_j\ \mathcal{L}_j^y(M)\ \mathcal{M}_j^{\gamma_k}\left(F^{-1}[u,\infty)\right)$$

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- A natural question then is if this result can be generalized to possibly open a new class of kinematic formuae.

Testing the Limits of Gaussian Kinematic Fundamental Formula

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The non IID case

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where $D = (\lambda_1, \ldots, \lambda_k)$. Here D represented the covariance amongst the random fields, while I denotes the spatial covariance.

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where $D = (\lambda_1, \ldots, \lambda_k)$. Here D represented the covariance amongst the random fields, while I denotes the spatial covariance.
- We assume that each y_p induces a metric g^p on the manifold M such that $g^p_{i,j} = g^p(E_i, E_j) = \lambda_p g(E_i, E_j)$ where $\{E_i\}$ is an ONB w.r.t. the base spatial metric g.

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- We also assume that each y_p is sufficiently smooth.
- Let $F : \mathbb{R}^k \to \mathbb{R}$ be **smooth**. Define $f = F(y_1, y_2, \ldots, y_k)$.

A generalized GKF

Theorem

Writing $\mathcal{L}_0$ for the Euler-Poincaré, and setting $\mathcal{K} = F^{-1}[u, \infty)$ we have

$$\mathbb{E}\left(\mathcal{L}_0(M \cap f^{-1}[u, \infty))\right) = \sum_{j=0}^{d} c_j \, \mathcal{L}_j(M) \, \mathcal{M}_j^*(\mathcal{K}),$$

where $\mathcal{M}_j^*(\mathcal{K})$ are coefficients appearing in the Taylor series expansion of Gaussian volume of ellipsoidal tubes

$$T^D(\mathcal{K}, \epsilon) = \mathcal{K} \oplus B_D(\epsilon),$$

with $B_D(\epsilon) = \{x \in \mathbb{R}^k : x^T D^{-1} x \leq \epsilon^2\}$.

A peek into the proof

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$$\mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right)\right)$$

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and using the definition of Euler-Poincaré characteristic via critical points,

$$\mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right)\right) = \mathbb{E}\left(\sum_{k=0}^{m} (-1)^k \mu_k\right)$$

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$$\begin{aligned}
\mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right)\right) &= \mathbb{E}\left(\sum_{k=0}^{m}(-1)^k \mu_k\right) \\
= & \int_M \mathbb{E}\left\{ \mathrm{Tr}\left(-\nabla^2 f(x)\right)^m 1_{(f(x)\geq u)} \middle| \nabla f(x) = 0\right\} p_{\nabla f(x)}(0)\ dx \\
= & \int_M \mathbb{E}\left[1_{(f(x)\geq u)} \mathbb{E}\left\{ \mathrm{Tr}\left(-\nabla^2 f(x)\right)^m \middle| f(x), \nabla f(x) = 0\right\}\right] \\
& \times p_{\nabla f(x)}(0)\ dx
\end{aligned}$$

- Notice that $\{\nabla^2 f \,|\, y, \nabla y\}$ is a Gaussian $(1, 1)$ form and we have neat formulae available for its moments.

Sneak peek contd...

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- In general, if W is a $(1,1)$ Gaussian form with mean and covariance given by μ and C, respectively, then

$$\mathbb{E}[W^k] = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{(k-2j)!j!2^j} \mu^{k-2j} C^j.$$

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- In our case:
$\mu_{y,\nabla y} = \mathbb{E}\{\nabla^2 f \,\big|\, y, \nabla y\} = y^* \nabla^2 F - I \langle D \nabla F(y), y \rangle$

- For a *smooth* Gaussian random field z defined on a manifold M, we usually have

$$-2R_z = \mathbb{E}\left[(\nabla^2 z)^2\right],$$

where R_z is the Riemannian curvature tensor w.r.t. the metric induced by z.

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- In our case: the conditional (co)variance $\mathbb{E}\left((y-\mu_{y,\nabla y})^2 \middle| y, \nabla y\right)$ is given by

$$-\|D\nabla F(y)\|^2 I^2 - 2\|D^{1/2}\nabla F(y)\|R,$$

where R is the Riemannian curvature tensor with respect to the base metric g.

- Then need to go from conditioning on $(y, \nabla y)$ to conditioning on $(f, \nabla f)$, which involves another Gaussian computation (majorly technical).

A simplification

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\begin{aligned}
&\mathbb{E}\left(\mathcal{L}_0(M \cap f^{-1}[u,\infty))\right) \\
= \; & \left(\sum_{\nu=1}^{k} \frac{1}{\lambda_{2,\nu}} \mathbb{E}\left[1_{(f>u)} \left(\frac{\partial F(y)}{\partial y_\nu}\right)^2\right]\right) p_{\nabla f}(0) 4\pi \mathcal{L}_0(M) \\
& + \frac{1}{2} \sum_{i,j=1}^{2} \mathbb{E}\Big[1_{(f>0)} \Big(\mu^2(y,\nabla y)(E_i,E_j,E_i,E_j) \\
& - S^T_{\nabla F}(E_i,E_i) \Sigma_{M,(y,\nabla y)} \Sigma^{-1}_{(y,\nabla y)} \Sigma_{(y,\nabla y),M} S_{\nabla F}(E_j,E_j) \\
& + S^T_{\nabla F}(E_i,E_j) \Sigma_{M,(y,\nabla y)} \Sigma^{-1}_{(y,\nabla y)} \Sigma_{(y,\nabla y),M} S_{\nabla F}(E_j,E_i)\Big)\Big] p_{\nabla f}(0) \mathcal{L}_2(M
\end{aligned}
$$

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Analysis

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Masterstroke: The coefficients match with the ellipsoidal Gaussian tube formula, thus proving the result.

Thanks