Bangalore Probability Seminar June 24th

Small Ball Question ::

Let $\{X_t\}_{t\geq 0}$ be a stochastic process on $\mathbb{R}^d$

Fix $T>0$.

Small ball Probability question is the study of ::

$$\mathbb{P}\Big(\sup_{0\leq t\leq T} |X_s| < \varepsilon \Big) \quad \text{as } \varepsilon \to 0$$

- Typically hard to do ; No general Theory

$X \equiv$ Brownian motion in $\mathbb{R}$:- $X_0 = 0$

[Mogulskii '74]

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log P\Big(\sup_{0\leq s\leq 1} |X_s| < \varepsilon \Big) = -\frac{\pi^2}{8}$$

$$\boxed{c_1 e^{-\frac{c_2 \pi^2}{8\varepsilon^2}} \leq \mathbb{P}\Big(\sup_{0\leq s\leq 1} |X_s| < \varepsilon \Big) \leq c_4 e^{-\frac{c_3 \pi^2}{8\varepsilon^2}} \qquad c_1, c_2, c_3, c_4 > 0}$$

[Lifshits — Small ball Reference papers till Dec 2016]

Connections

$$\mathbb{P}\left(\sup_{0 \le s \le T} |X_s| < \varepsilon \right)$$ — Small ball

$$\left(\frac{X_{s/\varepsilon^2}}{\varepsilon} \overset{d}{=} X_s \quad X_0 = 0 \right) = \mathbb{P}\left(\sup_{0 \le s \le T/\varepsilon^2} |X_s| < 1 \right)$$ — Confinement

$$= \mathbb{P}\left(\tau > T/\varepsilon^2 \right)$$ — Exit time

where $\tau = \inf \{ s \ge 0 \mid |X_s| \ge 1 \}$

$$= \mathbb{P}\left(\frac{1}{T/\varepsilon^2} \int_0^{T/\varepsilon^2} \mathbb{1}_{(-1,1)}(X_s)\, ds = 1 \right)$$

"$\frac{\pi^2}{8}$" — Principal Eigen value of Laplacian in $[-1,1]$

Donsker – Varadhan

$\hookrightarrow$ occupation measure (Large Deviation)

Brownian Sheet [Talagrand 94, Bass 88]

$\{ B_{(s,t)} : s \in [0,T], t \in [0,T] \}$ – Gaussian Process.

- $E[B_{(s,t)}] = 0$
- $E[B_{(s,t)} B_{(s',t')}] = \min(s,s') \min(t,t')$

[Talagrand '94]

$$e^{-\frac{C_2}{\varepsilon^2}\left(\log\frac{1}{\varepsilon}\right)^3} \leq \mathbb{P}\left(\sup_{0\leq s\leq 1} |B_{s,t}| < \varepsilon\right) \leq e^{-\frac{C_1}{\varepsilon^2}\left(\log\frac{1}{\varepsilon}\right)^3}$$

$$\mathbb{P}\left(\|B\|_2 < \varepsilon\right) \leq e^{-\frac{C_3}{\varepsilon^2}\left(\log\frac{1}{\varepsilon}\right)^2}$$

[Bass 88] $B_{s,t} = \sum_m \sum_j Z_{jm}\,\tilde{\varphi}_{jm}(s,t)$

i.i.d Normal

integrals of Haar Basis

Applications of Small-Ball :-

- $\{X_k\}_{k\geq 1}$ i.i.d F on $[0,1]^2$

$$F_n(s,t) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{[0,s]\times[0,t]}(X_k)$$

$$\Rightarrow \underbrace{\sqrt{n}\,(F_n - F)}_{\text{Power of K-S test}} \xrightarrow{d} B \quad \underbrace{(\text{"Brownian sheet"})}_{\text{Small ball Probability}}$$

- law of iterated logarithms for Brownian sheet

- [GIRSANOV] - Support Theorem

class :- $\varphi_{s,t}$ $\quad \mathbb{P}\left(\sup_{s,t\in[0,1]^2} |B_{s,t} - \varphi_{s,t}| < \varepsilon\right) = ?$

as $\varepsilon \to 0$

Proof for Brownian motion – Small ball estimate

Upper bound :- $|B_0| < \varepsilon$

$$P\left(\sup_{0\le s\le T} |B_s| < \varepsilon\right) \le \cdots \; ?$$

Idea :- • Partition $[0,T]$ into intervals of lengths ε^2

• Scaling • Markov Property

$p_i = i\varepsilon^2 \qquad i = 0,1,2,\ldots [T/\varepsilon^2]$

$$\mathbb{P}\left(\sup_{0\le s\le T} |B_s| < \varepsilon\right) \le \mathbb{P}\left(|B_{p_i}| < \varepsilon \;\forall i = 0, \ldots\right)$$

$$F_i = \{ |B_{p_i}| < \varepsilon \}. \qquad = \prod_{j=1}^{[T/\varepsilon^2]} \mathbb{P}\left(F_j \,\Big|\, \bigcap_{i=0}^{j-1} F_i\right)$$

$x = B_0,$

$$\mathbb{P}(F_1) = \mathbb{P}_x\left(|B_{\varepsilon^2}| < \varepsilon\right)$$

$$\le \sup_{|x|<\varepsilon} \mathbb{P}_x\left(|B_{\varepsilon^2}| < \varepsilon\right)$$

$\tilde{B}$ – B.M

$\tilde{B}_0 = 0.$

$$= \sup_{|x|<\varepsilon} \underbrace{\mathbb{P}_0\left(|x + \tilde{B}_{\varepsilon^2}| < \varepsilon\right)}_{\text{maximised at } x=0}$$

$$\underbrace{\begin{matrix} 0 < x < \varepsilon \\ -\varepsilon < x < 0 \end{matrix}} \equiv \text{Symmetry + Scaling} \le \mathbb{P}_0(-1 < \tilde{B}_1 < 2) \equiv \eta$$

$$\Rightarrow \mathbb{P}(F_1) < \eta$$

By Markov Property

$$P\left(F_j \,\middle|\, \bigcap_{i=0}^{j-1} F_i \right) < \eta$$

$$\Rightarrow \mathbb{P}\left(\sup_{0 \le s \le T} |B_s| < \varepsilon \right) < \eta^{C/\varepsilon^2}$$

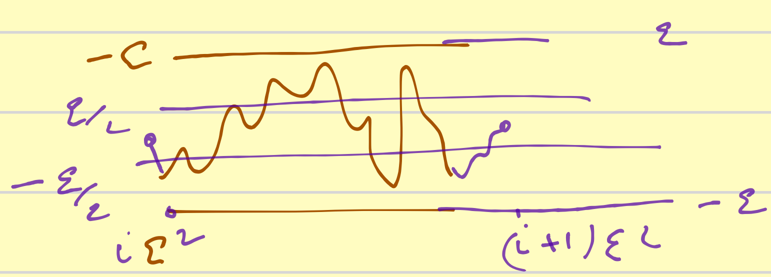
$$= c_2 e^{-c_3/\varepsilon^2}$$

Lower Bound

$$? \le \mathbb{P}\left(\sup_{0 \le s \le T} |B_s| < \varepsilon \right)$$

Intervals $[i\varepsilon^2, (i+1)\varepsilon^2]$ $i = 0, 1, \ldots, [T/\varepsilon^2] - 1$

$$A_i = \left\{ \sup_{i\varepsilon^2 \le s \le (i+1)\varepsilon^2} |B_s| < \varepsilon, \ |B_{(i+1)\varepsilon^2}| < \frac{\varepsilon}{2} \right\}$$



It is enough $P(A_0) \ge \delta$, for some $0 < \delta < 1$
(Same as upper bound)

$|B_0| < \varepsilon/2$

$x = B_0$ $\quad P_x\left(\sup_{0 \le s \le \varepsilon^2} |B_s| < \varepsilon, \ |B_{\varepsilon^2}| < \frac{\varepsilon}{2} \right)$

Say $x = 0$: $\quad \ge \mathbb{P}_0\left(\sup_{0 \le s \le \varepsilon^2} |B_s| \le \frac{\varepsilon}{2} \right)$

$> \delta$ (Brownian scaling)

General case: $\tilde{B}_s = B_s + \frac{s x}{\varepsilon^2}$

$$\frac{dQ}{dP} = \exp\left(-\frac{x}{\varepsilon^2} \int_0^{\varepsilon^2} dB_s - \frac{x^2}{2\varepsilon^2} \int_0^{\varepsilon^2} ds \right)$$

work with Q and estimate $Q(A_0)$

- to get $Q(A_0) > \delta_1$ for some $\delta_1 > 0$

- $C_2 \le E\left(\frac{dQ}{dP}\right)^2 \le C_1$ ✓ easy to see

$\Rightarrow$ $P(A_0) > \delta$. $\square$