Large deviations of mean-field interacting particle systems in a fast varying environment

Sarath Yasodharan
Joint work with Rajesh Sundaresan

ECE Department, Indian Institute of Science

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Background and motivation

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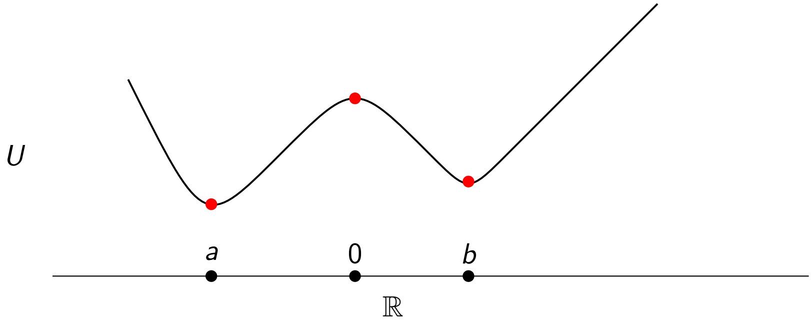

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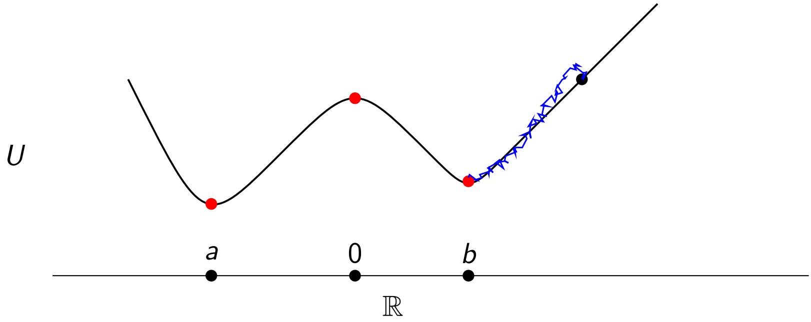

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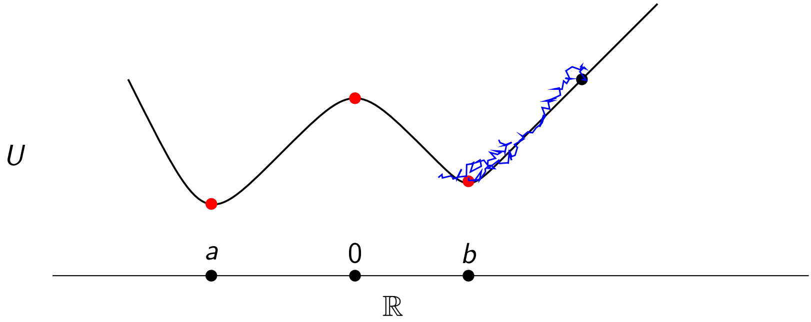

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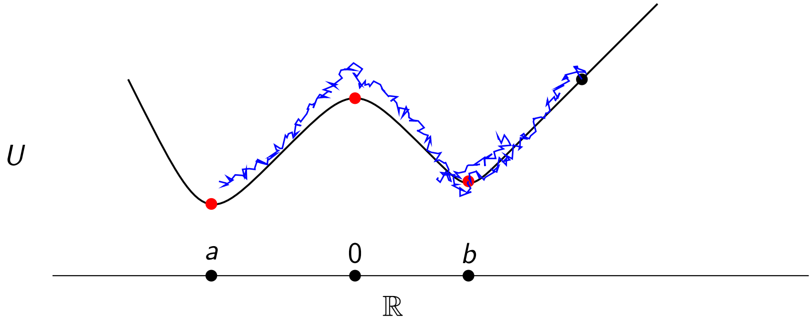

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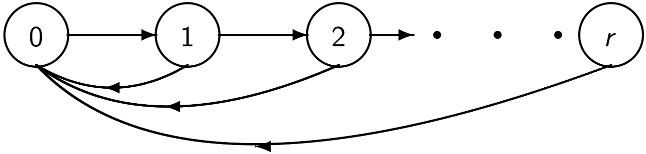
- Channel state: idle, collision, successful transmission
- State of a node represents aggressiveness of packet transmission.
- 

- Evolution of the state of a node:
  - Becomes less aggressive after a collision.
  - Moves to the most aggressive state after a successful packet transmission.

A sample path of the macroscopic behaviour

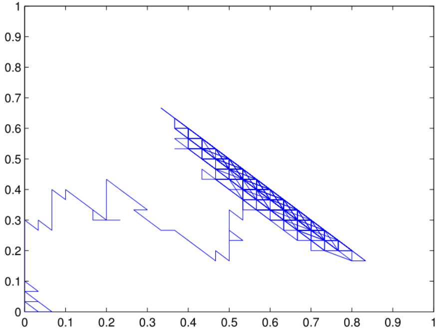


Figure: Evolution of states in a WiFi network under the MAC protocol

- Metastability phenomenon: Multiple stable regions in the system. Transition between two stable regions occur over large time durations.

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$$\mu_N(t) := \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n^N(t)} \in M_1^N(\mathcal{X}) \subset M_1(\mathcal{X}).$$

- We are given functions $\lambda_{x,x'}(\cdot, y)$, $(x, x') \in \mathcal{E}_{\mathcal{X}}$, $y \in \mathcal{Y}$ and $\gamma_{y,y'}(\cdot)$, $(y, y') \in \mathcal{E}_{\mathcal{Y}}$ on $M_1(\mathcal{X})$.

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- Markovian evolution at time t:
  - Particles: $x \to x'$ at rate $\lambda_{x,x'}(\mu_N(t), Y_N(t))$;
  - Environment: $y \to y'$ at rate $N\gamma_{y,y'}(\mu_N(t))$.

System model

- (μ_N, Y_N) is a Markov process with infinitesimal generator

$$f \mapsto \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}} N\xi(x)\lambda_{x,x'}(\xi,y)\left[f\left(\xi+\frac{\delta_{x'}}{N}-\frac{\delta_x}{N},y\right)-f(\xi,y)\right]$$
$$+N\sum_{y':(y,y')\in\mathcal{E}_{\mathcal{Y}}}(f(\xi,y')-f(\xi,y))\gamma_{y,y'}(\xi),$$

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- We consider the process (μ_N, θ_N) with sample paths in $D([0,T], M_1(\mathcal{X})) \times D_\uparrow([0,T], M(\mathcal{Y}))$.

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Suppose that $\mu_N(0) \to \nu$ in $M_1(\mathcal{X})$. Then μ_N converges in probability, in $D([0, T], M_1(\mathcal{X}))$, to the solution to the ODE

$$\dot{\mu}_t = \bar{\Lambda}^*_{\mu_t,\pi_{\mu_t}}\mu_t,\ 0 \le t \le T,\ \mu_0 = \nu.$$

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- μ_N is a small random perturbation of the above ODE. We study fluctuations of (μ_N, θ_N).

Large deviations

- S: a metric space. $\{X_N\}_{N\geq 1}$ is a sequence of S-valued random variables.
- Roughly, $P(X_N \in A) \sim \exp\{-NI_A\}$ where $I_A = \inf_{x\in A} I(x)$.

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- Roughly, $P(X_N \in A) \sim \exp\{-NI_A\}$ where $I_A = \inf_{x\in A} I(x)$.
- $\{X_N\}_{N\geq 1}$ is said to satisfy the large deviation principle (LDP) with rate function $I : S \to [0, +\infty]$ if
  - for each $M > 0$, $\{x \in S : I(x) \leq M\}$ is a compact subset of S;
  - for each open set $G \subset S$,

$$\liminf_{N\to\infty} \frac{1}{N} \log P(X_N \in G) \geq -\inf_{x\in G} I(x);$$

  - for each closed set $F \subset S$,

$$\limsup_{N\to\infty} \frac{1}{N} \log P(X_N \in F) \leq -\inf_{x\in F} I(x).$$

Large deviations: contraction principle

- S, T are metric spaces. $f : S \to T$ is continuous.
- $\{X_N\}$s are S-valued random variables. Define $Y_N = f(X_N)$.

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- Compactness of level sets:
  $\{y \in T : J(y) \leq M\} = f(\{x \in S : I(x) \leq M\})$.
- Upper and lower bounds:
  $P(Y_N \in A) = P(X_N \in f^{-1}(A))$.

Main result

Theorem

Suppose that $\{\mu_N(0)\}_{N\geq 1}$ *satisfies the LDP on* $M_1(\mathcal{X})$ *with rate function* I_0*. Then the sequence* $\{(\mu_N(t), \theta_N(t)), 0 \leq t \leq T\}_{N\geq 1}$ *satisfies the LDP on* $D([0,T], M_1(\mathcal{X})) \times D_{\uparrow}([0,T], M(\mathcal{Y}))$ *with rate function*

$$I(\mu, \theta) := I_0(\mu(0)) + J(\mu, \theta).$$

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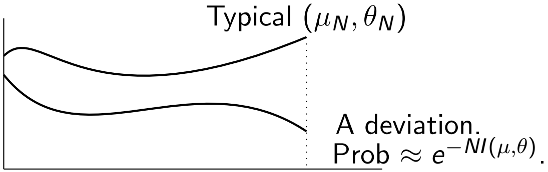

The rate function J

$$
\begin{aligned}
J(\mu,\theta) := \int_{[0,T]} \Bigg\{ \sup_{\alpha\in\mathbb{R}^{|\mathcal{X}|}} \Bigg(& \langle \alpha, (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t)\rangle \\
& - \sum_{(x,x')\in\mathcal{E}_\mathcal{X}} \tau(\alpha(x')-\alpha(x))\bar{\lambda}_{x,x'}(\mu_t,m_t)\mu_t(x)\Bigg) \\
+ \sup_{g\in\mathbb{R}^{|\mathcal{Y}|}} \sum_{y\in\mathcal{Y}} \Bigg(& -L_{\mu_t} g(y) \\
& - \sum_{y':(y,y')\in\mathcal{E}_\mathcal{Y}} \tau(g(y')-g(y))\gamma_{y,y'}(\mu_t)\Bigg) m_t(y) \Bigg\}\, dt
\end{aligned}
$$

whenever the mapping $[0,T] \ni t \mapsto \mu_t \in M_1(\mathcal{X})$ is absolutely continuous, where $\theta(dtdy) = m_t(dy)dt$, and $J(\mu,\theta) = +\infty$ otherwise.

- $\tau(u) = e^u - u - 1, u \in \mathbb{R}$.

Some remarks about the rate function

- $J(\mu, \theta) \geq 0$ with equality iff (μ, θ) satisfies the mean-field limit.
- Two parts. The mean-field part (slow component) and occupation measure part (fast component).
  - For the slow component, the average of the fast variable appears.
  - For the fast component, the slow variable is frozen.
- For occupation measure of Markov processes, the canonical form of the rate function is $\int_{[0,T]} \sup_{h>0} \sum_{\mathcal{Y}} -\frac{L_{\mu_t} h(y)}{h(y)} m_t(y) dt$ (Donsker and Varadhan, 1973). This can be obtained by taking $h = e^g$.

Large deviations of μ_N

Corollary

$\{\mu_N\}$ satisfies the LDP on $D([0,T], M_1(\mathcal{X}))$ with rate function

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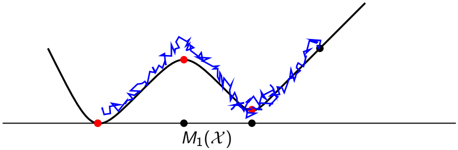
- Can quantify metastable transitions, mean exit time from a domain etc.



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Outline of the proof

- Usual techniques:
  - Discretisation and change of measure: Freidlin and Wentzell (1984), Liptser (1996), Veretennikov (1999), Dawson and Gärtner (1987), Léonard (1995).
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  - Unique identification any subsequential rate function (regardless of the subsequence) implies the LDP.
- Also used in the context of invariant measure LDP, Borkar and Sundaresan (2012).

Exponential tightness

Theorem

The sequence $\{(\mu_N(t), \theta_N(t)), t \in [0, T]\}_{N \geq 1}$ *is exponentially tight in* $D([0, T], M_1(\mathcal{X})) \times D_{\uparrow}([0, T], M(\mathcal{Y}))$*, i.e., given any* $M > 0$*, there exists a compact set* K_M *such that*

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For $\beta > 0$ and $\alpha \in \mathbb{R}^{|\mathcal{X}|}$, with $X_{N,t} = \langle \alpha, \mu_N(t) \rangle$,

$$\exp\Bigg\{ N\Bigg(\beta X_{N,t} - \beta X_{N,0} - \beta \int_0^t \Phi_{Y_{N,s}} f(\mu_{N,s}) ds - \int_0^t \sum_{(x,x')} \tau(\beta(\alpha(x') - \alpha(x))) \lambda_{x,x'}(\mu_{N,s}, Y_{N,s}) \mu_{N,s}(x) ds \Bigg) \Bigg\}, t \geq 0,$$

is an exponential martingale. Use Doob's inequality and a condition for exponential tightness in $D([0, T], \mathbb{R})$ (Puhalskii, 1994).

An equation for the subsequential rate function

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- Define $U_t^{\alpha,g}(\mu,\theta)$ by

$$\int_{[0,t]} \Big\{ \langle \alpha_s(\mu_s), \dot{\mu}_s - \bar{\Lambda}^*_{\mu_s,m_s}\mu_s \rangle - \sum_{(x,x')} \tau(\alpha_s(\mu_s)(x') - \alpha_s(\mu_s)(x))\bar{\lambda}_{x,x'}(\mu_s,m_s)\mu_s(x) + \sum_y \Big(-L_{\mu_s} g_s(\mu_s,\cdot)(y) - \sum_{y:(y,y')\in\mathcal{E}_\mathcal{Y}} \tau(g_s(\mu_s,y') - g_s(\mu_s,y))\gamma_{y,y'}(\mu_s) \Big) m_s(y) \Big\} ds.$$

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- On one hand, for a smaller class of α and g,

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  This can be extended to (1).
- Moreover, the supremum in (1) is attained.

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- It can be shown that $I^* = J$.
- Note that $\tilde{I} \geq I^*$ on Γ. Outside Γ, I^* can be shown to be $+\infty$.
- Goal: show that $\tilde{I} \leq I^*$ whenever $I^* < +\infty$. Once this is established, the LDP follows.

Identification of $\tilde{I}$ on "nice" elements

- Suppose $(\hat{\mu}, \hat{\theta})$ is such that $I^*(\hat{\mu}, \hat{\theta}) < +\infty$, and
  - $\inf_{t \in [0,T]} \min_{x \in \mathcal{X}} \hat{\mu}_t(x) > 0$,
  - the mapping $[0, T] \ni t \mapsto \hat{\mu}_t \in M_1(\mathcal{X})$ is Lipschitz continuous,
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- With this $(\hat{\alpha}, \hat{g})$, get $(\tilde{\mu}, \tilde{\theta})$ that attains the supremum in $\sup_{(\mu,\theta)\in\Gamma}(U_T^{\hat{\alpha},\hat{g}}(\mu, \theta) - \tilde{I}(\mu, \theta)) = 0$.
- Hence, $I^*(\tilde{\mu}, \tilde{\theta}) \geq U_T^{\hat{\alpha},\hat{g}}(\tilde{\mu}, \tilde{\theta}) = \tilde{I}(\tilde{\mu}, \tilde{\theta})$.

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- It follows that $\tilde{I}(\hat{\mu}, \hat{\theta}) = I^*(\hat{\mu}, \hat{\theta})$.

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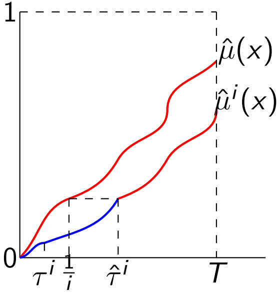

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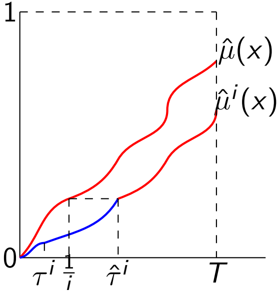

- Other conditions are relaxed using suitable approximations. We finally get $\tilde{I} = I^*$ for all elements.

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Reference: arXiv: 2008.06855

Thank You