Genealogy and spatial distribution of the N-particle branching random walk with polynomial tails

Sarah Penington

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Joint work with Matt Roberts and Zsófia Talyigás

Branching-selection systems

- Particle systems: particles branch (produce offspring) and move in space

  killing rule keeps total number of particles constant.

- Toy models for a population under selection.

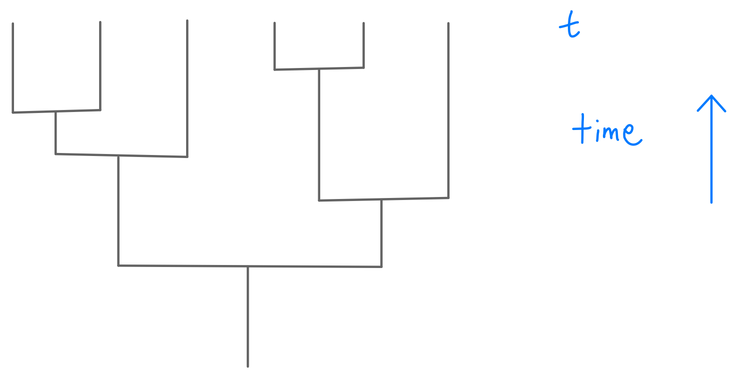
  Location of a particle (= individual) represents its evolutionary fitness.

- Introduced by Brunet and Derrida in 1990s.

  Recent results and open conjectures about long-term behaviour.

Genealogy:

Coalescent process



N-particle branching random walk (N-BRW)

Discrete-time branching-selection system.

N particles with locations in $\mathbb{R}$ at each timestep.

Let X be a real-valued random variable (jump distribution).

At each time $n \in \mathbb{N}_0$, each particle has two offspring.

Each of the $2N$ offspring particles makes an independent jump from its parent's location, with the same law as X.

The N rightmost particles (of the $2N$ offspring particles) form the population at time $n+1$.

Notation: $X_1^{(N)}(n) \leq X_2^{(N)}(n) \leq \ldots \leq X_N^{(N)}(n)$ ordered particle positions at time n.

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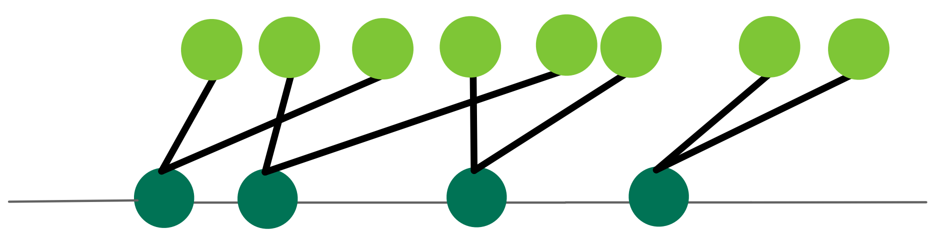

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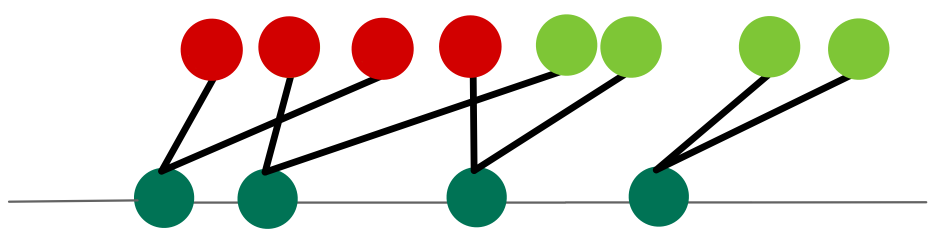

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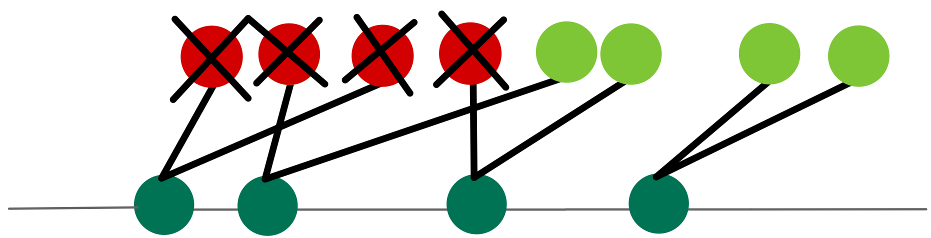

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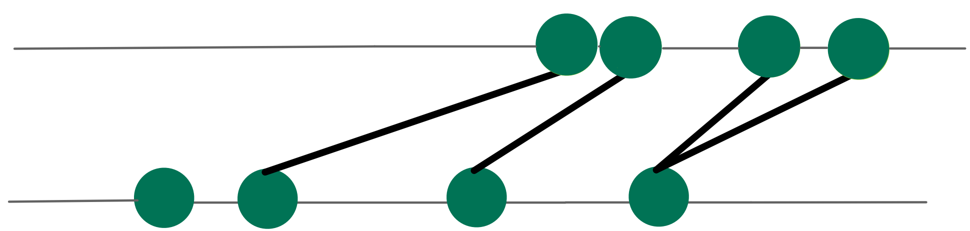

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Light-tailed jump distribution

Asymptotic speed

If $\mathbb{E}[X] < \infty$ then $\exists\, v_N \in (0,\infty)$ s.t. $\displaystyle\lim_{n\to\infty} \frac{X_N^{(N)}(n)}{n} = v_N = \lim_{n\to\infty} \frac{X_1^{(N)}(n)}{n}$ a.s. and in L^1.

Theorem (Bérard and Gouéré 2010) If $\mathbb{E}[e^{\lambda X}] < \infty$ for some $\lambda > 0$ (+ technical assumptions) then $\displaystyle\lim_{N\to\infty} v_N = v_\infty$ exists and $v_\infty - v_N \sim c(\log N)^{-2}$ as $N \to \infty$.

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Genealogy

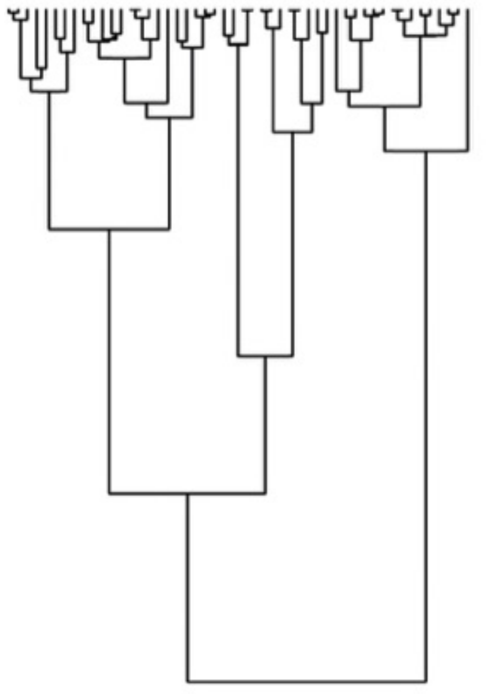
Sample k particles from the N particles and trace their ancestry backwards in time $\to$ coalescent process.

Conjecture (Brunet, Derrida, Mueller, Munier)
If X is light-tailed then the genealogy of a sample on a $(\log N)^3$ timescale converges to a Bolthausen-Sznitman coalescent as $N \to \infty$.

Coalescent processes

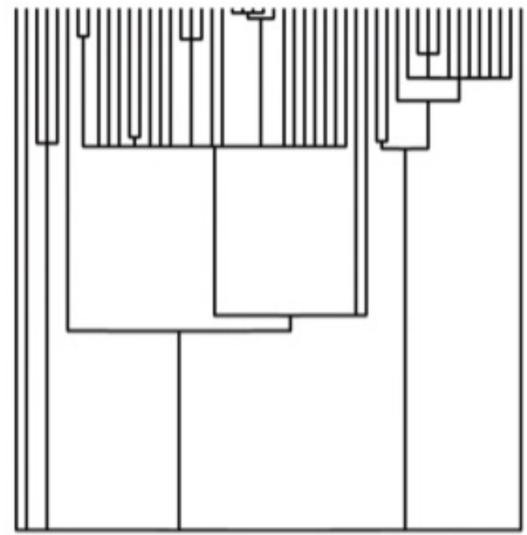
Kingman's coalescent

Neutral population: choose particles to kill uniformly at random in each generation.



Bolthausen-Sznitman coalescent

Population under selection.



Thanks to Götz Kersting and Anton Wakolbinger

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If X is light-tailed then the genealogy of a sample on a $(\log N)^3$ timescale converges to a Bolthausen-Sznitman coalescent as $N \to \infty$.

N-BRW with heavy-tailed jump distribution

Suppose $\mathbb{P}(X > x) \sim x^{-\alpha}$ as $x \to \infty$, for some $\alpha > 0$.

Asymptotic speed

Theorem (Bérard and Maillard 2014)

If $\mathbb{E}[X] < \infty$, $\lim_{n\to\infty} \frac{X_N^{(N)}(n)}{n} = v_N$ where $v_N \sim c_\alpha N^{1/\alpha} (\log N)^{1/\alpha - 1}$ as $N \to \infty$.

If $\mathbb{E}[X] = \infty$, cloud of particles accelerates.

Genealogy

Conjecture (Bérard and Maillard)

The genealogy on a $\log N$ timescale is approximately given by a star-shaped coalescent when N is large.

Time and space scales

Let $\mathbb{P}(X > x) = \dfrac{1}{h(x)}$ for $x \geq 0$.

Assume h is regularly varying with index $\alpha > 0$

i.e. for any $y > 0$, $\dfrac{h(xy)}{h(x)} \longrightarrow y^\alpha$ as $x \to \infty$.

and $\mathbb{P}(X \geq 0) = 1$ (no negative jumps).

Let $\ell_N = \lceil \log_2 N \rceil$ time scale

Let $a_N = h^{-1}(2N\ell_N)$, where $h^{-1}(x) := \inf\{y \geq 0 : h(y) > x\}$. space scale

$\mathbb{E}[\#$ jumps of size $> c_1 a_N$ in a time interval of length $c_2 \ell_N]$

$$= 2N \cdot c_2 \ell_N \, \mathbb{P}(X > c_1 a_N)$$

$$= \frac{2N c_2 \ell_N}{h(c_1 a_N)} \sim \frac{2N c_2 \ell_N}{c_1^\alpha \, 2N \ell_N} = \frac{c_2}{c_1^\alpha}$$

as $N \to \infty$.

Main result

w.h.p. = with probability $\to 1$ as $N \to \infty$.

Theorem (P., Roberts, Talyigás 2021)

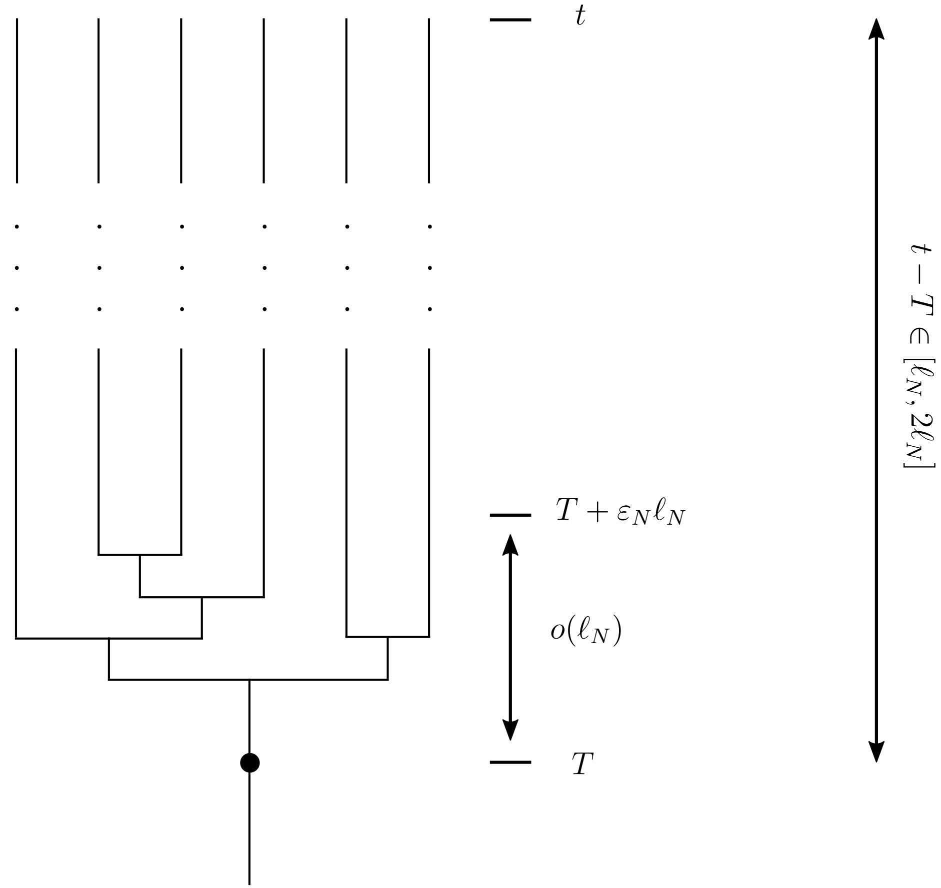
For $\eta > 0$, $k \in \mathbb{N}$ and $t > 4\ell_N$, the following occurs w.h.p.:

- Spatial distribution: At time t, there are $N - o(N)$ particles in

$$[X_1^{(N)}(t), X_1^{(N)}(t) + \eta a_N].$$

- Genealogy: The genealogy on an ℓ_N-timescale is asymptotically given by a star-shaped coalescent.

i.e. $\exists\, T \in [t - 2\ell_N, t - \ell_N]$ s.t. w.h.p., for a uniform sample of k particles at time t, every particle is descended from the rightmost particle at time T and no pair of particles in the sample has a common ancestor after time $T + \varepsilon_N \ell_N$, for any $(\varepsilon_N)_N$ with $\varepsilon_N \to 0$ and $\varepsilon_N \ell_N \to \infty$ as $N \to \infty$.



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Spatial distribution

At time t, there are $N - o(N)$ particles in $[X_1^{(N)}(t), X_1^{(N)}(t) + \eta a_N]$ w.h.p.

Proposition (PRT 2021) There exist $0 < p_r \leq q_r < 1$ s.t. $q_r \to 0$ as $r \to \infty$ and $p_r \to 1$ as $r \to 0$ s.t. for $r > 0$, for N sufficiently large and $t > 3\ell_N$,

$$\mathbb{P}\left(X_N^{(N)}(t) - X_1^{(N)}(t) \geq r a_N \right) \in [p_r, q_r].$$

Genealogy

w.h.p. $\exists\, T \in [t - 2\ell_N, t - \ell_N]$ s.t. w.h.p., for a uniform sample of k particles at time t, every particle is descended from the rightmost particle at time T and no pair of particles has a common ancestor after time $T + o(\ell_N)$.

Proposition (PRT 2021) For $0 \leq s_1 < s_2 \leq 1$, $\exists\, p > 0$ s.t. for N sufficiently large and $t > 4\ell_N$,

$$\mathbb{P}\left(T \in [t - 2\ell_N + s_1 \ell_N, t - 2\ell_N + s_2 \ell_N] \right) > p.$$

N-BRW genealogy

Jump distribution X.

		Time to coalesce	Coalescent
Light-tailed	$\mathbb{P}(X > x) \leq e^{-cx}, \ c > 0$	$(\log N)^3$	Bolthausen-Sznitman
Heavy-tailed	$\mathbb{P}(X > x) \sim x^{-\alpha}, \ \alpha > 0$	$\log N$	Star-shaped

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Simulation by Z. Talyigás.

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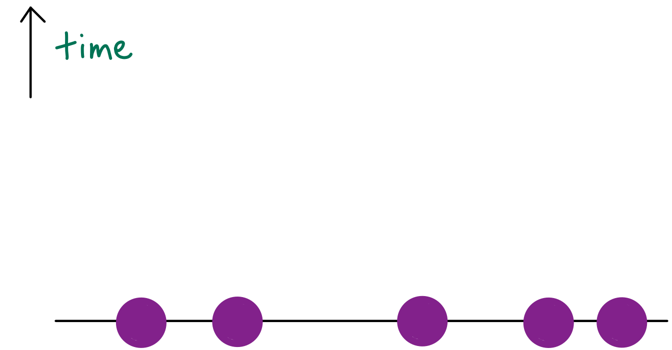
- N particles move in $\mathbb{R}$ according to independent Brownian motions.
- Each particle, independently, branches into two particles after an Exp(1) time.
- Each time a particle branches, the leftmost particle in the system is killed.

N particles in the system at all times.



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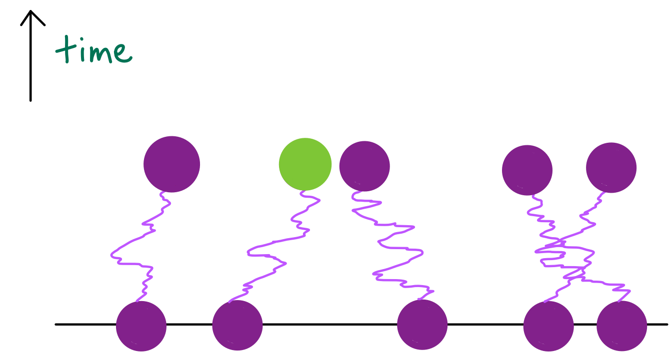

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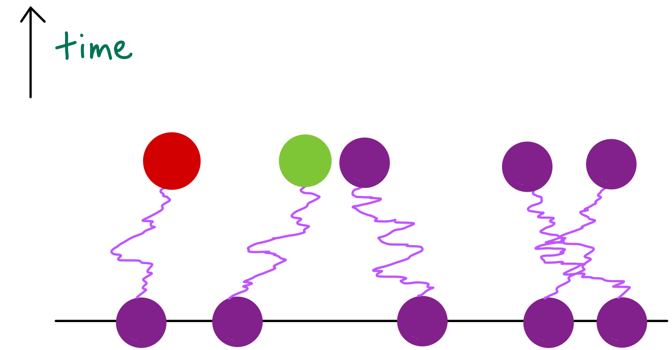

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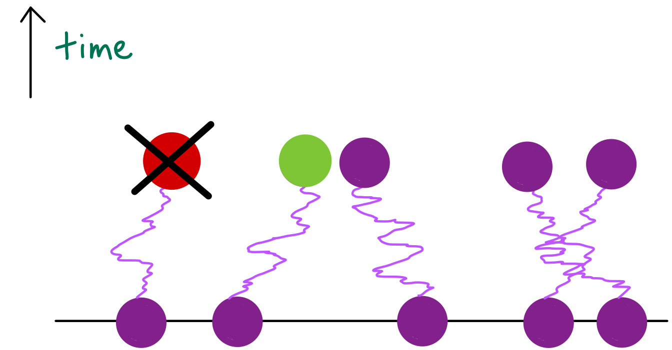

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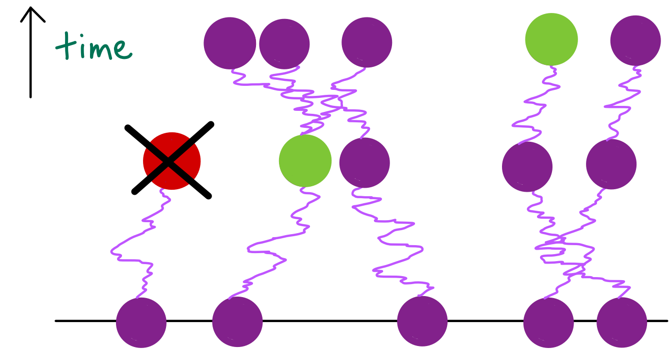

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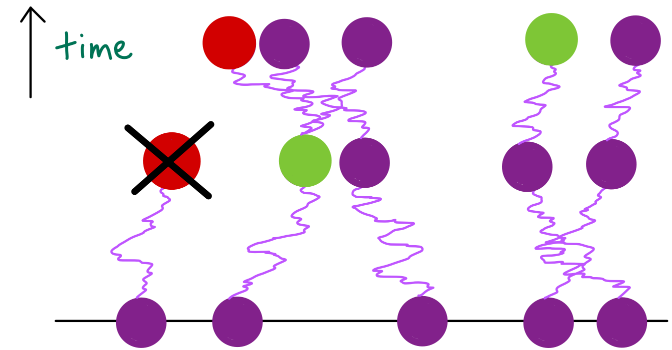

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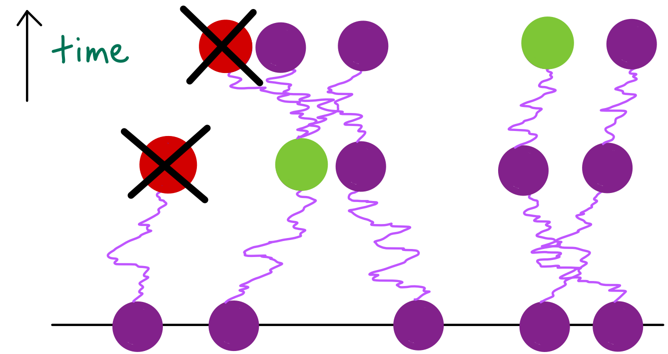

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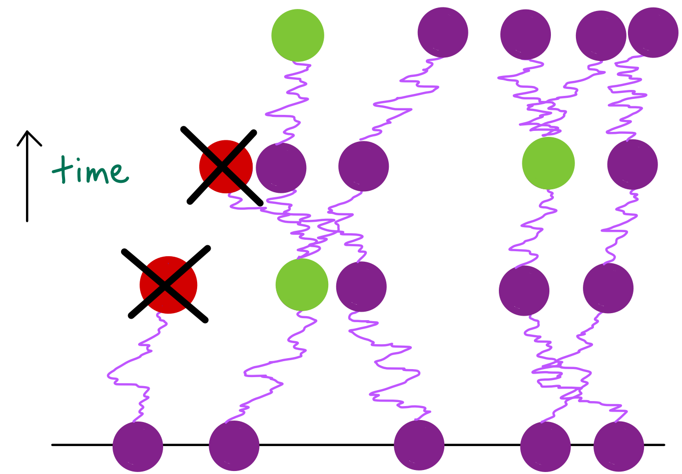

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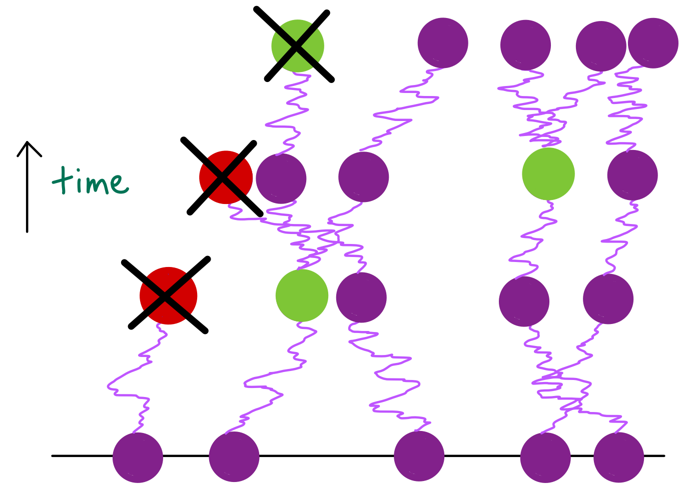

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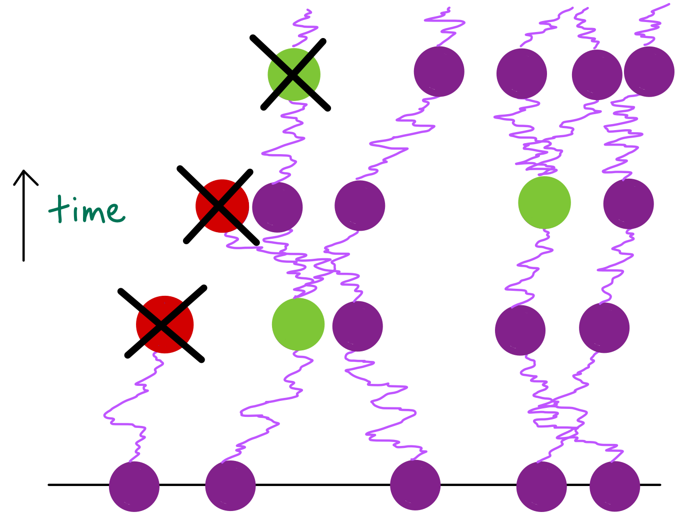

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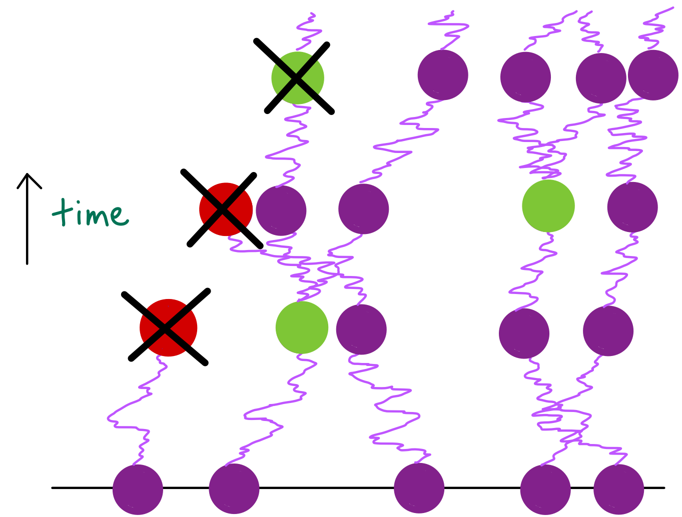

N particles in the system at all times.

Introduced by Maillard (2012).

Conjecture (Brunet/Derrida, Maillard): Genealogy of a sample on a $(\log N)^3$ timescale converges to a Bolthausen-Sznitman coalescent as $N \to \infty$.

One tool: over a fixed timescale, as $N \to \infty$, density converges to solution of a free boundary problem. (Hydrodynamic limit: De Masi/Ferrari/Presutti/Soprano-Loto '17)

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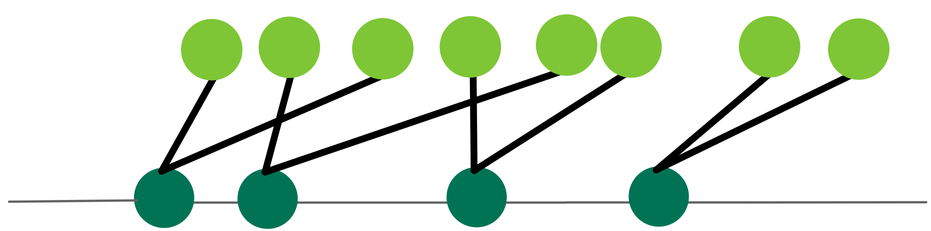

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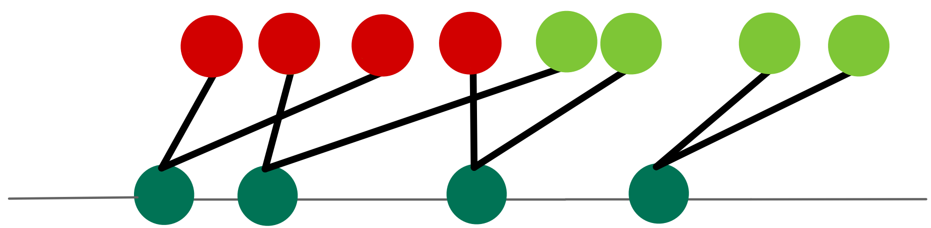

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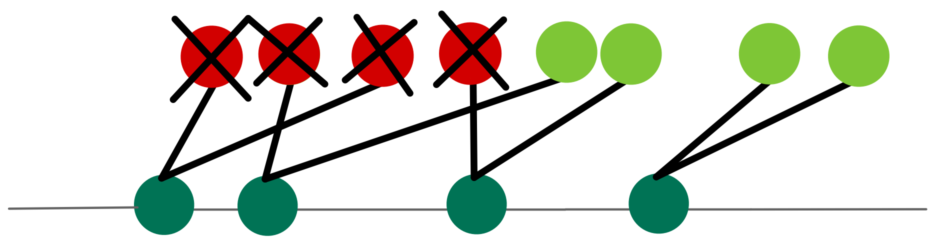

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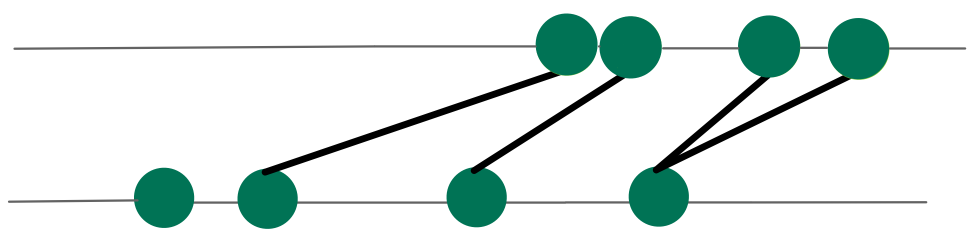

Notation: $X_1^{(N)}(n) \leq X_2^{(N)}(n) \leq \ldots \leq X_N^{(N)}(n)$ ordered particle positions at time n.

Time and space scales

Let $\mathbb{P}(X > x) = \frac{1}{h(x)}$ for $x \geq 0$.

Assume h is regularly varying with index $\alpha > 0$

i.e. for any $y > 0$, $\frac{h(xy)}{h(x)} \longrightarrow y^{\alpha}$ as $x \to \infty$.

and $\mathbb{P}(X \geq 0) = 1$ (no negative jumps).

Let $\ell_N = \lceil \log_2 N \rceil$ time scale

Let $a_N = h^{-1}(2N\ell_N)$, where $h^{-1}(x) := \inf\{y \geq 0 : h(y) > x\}$. space scale

Main result

w.h.p. = with probability $\to 1$ as $N \to \infty$.

Theorem (P., Roberts, Talyigás 2021)

For $\eta > 0$, $k \in \mathbb{N}$ and $t > 4\ell_N$, the following occurs w.h.p.:

- Spatial distribution: At time t, there are $N - o(N)$ particles in
$$[X_1^{(N)}(t), X_1^{(N)}(t) + \eta a_N].$$
- Genealogy: The genealogy on an ℓ_N-timescale is asymptotically given by a star-shaped coalescent.

i.e. $\exists\, T \in [t - 2\ell_N, t - \ell_N]$ s.t. w.h.p., for a uniform sample of k particles at time t, every particle is descended from the rightmost particle at time T and no pair of particles in the sample has a common ancestor after time $T + \varepsilon_N \ell_N$, for any $(\varepsilon_N)_N$ with $\varepsilon_N \to 0$ and $\varepsilon_N \ell_N \to \infty$ as $N \to \infty$.

Warm up lemma

Recall $\ell_N = \lceil \log_2 N \rceil$.

Recall $X_1^{(N)}(t) \le X_2^{(N)}(t) \le \dots \le X_N^{(N)}(t)$ ordered particle positions at time t.

Lemma For $s \in \mathbb{N}_0$, $\quad X_1^{(N)}(s+\ell_N) \geqslant X_N^{(N)}(s)$.

Warm up lemma

Recall $\ell_N = \lceil \log_2 N \rceil$.

Recall $X_1^{(N)}(t) \le X_2^{(N)}(t) \le \dots \le X_N^{(N)}(t)$ ordered particle positions at time t.

Lemma For $s \in \mathbb{N}_0$, $\quad X_1^{(N)}(s+\ell_N) \geq X_N^{(N)}(s)$.

Proof: Suppose (for a contradiction) that $X_1^{(N)}(t) < X_N^{(N)}(s) \quad \forall t \in [s, s+\ell_N] \cap \mathbb{N}_0$.

Then since all jumps are non-negative, the rightmost particle at time s has 2^{ℓ_N} descendants at time $s+\ell_N$.

Since $2^{\ell_N} \geq N$, this implies $X_1^{(N)}(s+\ell_N) \geq X_N^{(N)}(s)$. $\#$

So $\exists\, s^* \in [s, s+\ell_N] \cap \mathbb{N}_0$ s.t. $X_1^{(N)}(s^*) \geq X_N^{(N)}(s)$.

All jumps are ≥ 0, so $X_1^{(N)}(s+\ell_N) \geq X_1^{(N)}(s^*)$. $\square$

Construction of N-BRW from BRWs.

BRW: Initial particle at $x \in \mathbb{R}$ at time 0.

At each time $n \in \mathbb{N}_0$, each time-n particle has two offspring, each of which makes an independent jump from its parent's location with the same distribution as X.

The time-$(n+1)$ particles are these offspring particles.

Number of time-t particles is 2^t.

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Construction of N-BRW with initial particle locations $x_1, \dots, x_N$:

Take N independent BRWs with initial particles at $x_1, \dots, x_N$.

Colour BRW particles blue or red. All time-0 particles are blue.

For $n \in \mathbb{N}_0$, the N rightmost offspring particles of time-n blue particles are coloured blue. All other time-$(n+1)$ particles are coloured red.

Blue particles form an N-BRW.

Path of jumps from ancestor to descendant in N-BRW = path in one of the BRWs.

Random walk with heavy tailed jump distribution

$\mathbb{P}(X > x) = \frac{1}{h(x)}$, h regularly varying with index $\alpha > 0$.

$X_1, X_2, X_3, \ldots$ i.i.d. with $X_1 \overset{d}{=} X$.

Fix $c > 0$ small. For x v. large, unlikely that $\sum_{k=1}^{n} X_k \geq x$ and

$X_k \leq cx \quad \forall k \leq n$.

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Lemma (Durrett '83, Gantert '00)

For $m \in \mathbb{N}$, $q > 0$, $\lambda > 0$ and $0 < r < 1 \wedge \frac{\lambda(1 \wedge \alpha)}{8q}$, for N sufficiently large, if $x_N > N^{\lambda}$ then

$$\mathbb{P}\Big(\sum_{j=1}^{m\ell_N} X_j \mathbb{1}_{X_j \leq r x_N} \geq x_N \Big) \leq N^{-q}.$$

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Proof: Take $2q \log 2 < c < \frac{\lambda(1\wedge\alpha)\log 2}{r}$.

By Markov's inequality,

$$\mathbb{P}\Big(\sum_{j=1}^{m\ell_N} X_j \mathbb{1}_{X_j \le r x_N} \geq x_N\Big) = \mathbb{P}\Big(e^{c\ell_N x_N^{-1} \sum_{j=1}^{m\ell_N} X_j \mathbb{1}_{X_j \le r x_N}} \geq \underbrace{e^{c\ell_N}}_{= N^{c/\log 2}} \Big)$$
$$\leq N^{-\frac{c}{\log 2}} \, \mathbb{E}\big[e^{c\ell_N x_N^{-1} X \mathbb{1}_{X \le r x_N}} \big]^{m\ell_N}.$$

Use identity

$$\mathbb{E}[e^{vY} \mathbb{1}_{Y \le K_2} \mathbb{1}_{Y \geq K_1}] = \int_{K_1}^{K_2} v e^{vu} \, \mathbb{P}(Y > u)\, du + e^{vK_1} \mathbb{P}(Y \geq K_1) - (e^{vK_2} - 1)\mathbb{P}(Y > K_2)$$

to show $\mathbb{E}[e^{c\ell_N x_N^{-1} X \mathbb{1}_{X \le r x_N}}] = 1 + \mathcal{O}(N^{-\varepsilon})$ for some $\varepsilon > 0$. $\square$.

Random walk with heavy tailed jump distribution

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Use with $x_N = \text{const} \cdot a_N$.

Recall $a_N = h^{-1}(2N\ell_N)$, so a_N "$\sim$" $(2N\ell_N)^{1/\alpha}$.

Fix $e \in (0,1)$ small. A jump larger than $e a_N$ is a "big jump".

For $c \gg e$, it is very unlikely that there is a time-t particle $> c a_N$ away from its time-$(t - O(\ell_N))$ ancestor unless an ancestor made a big jump.

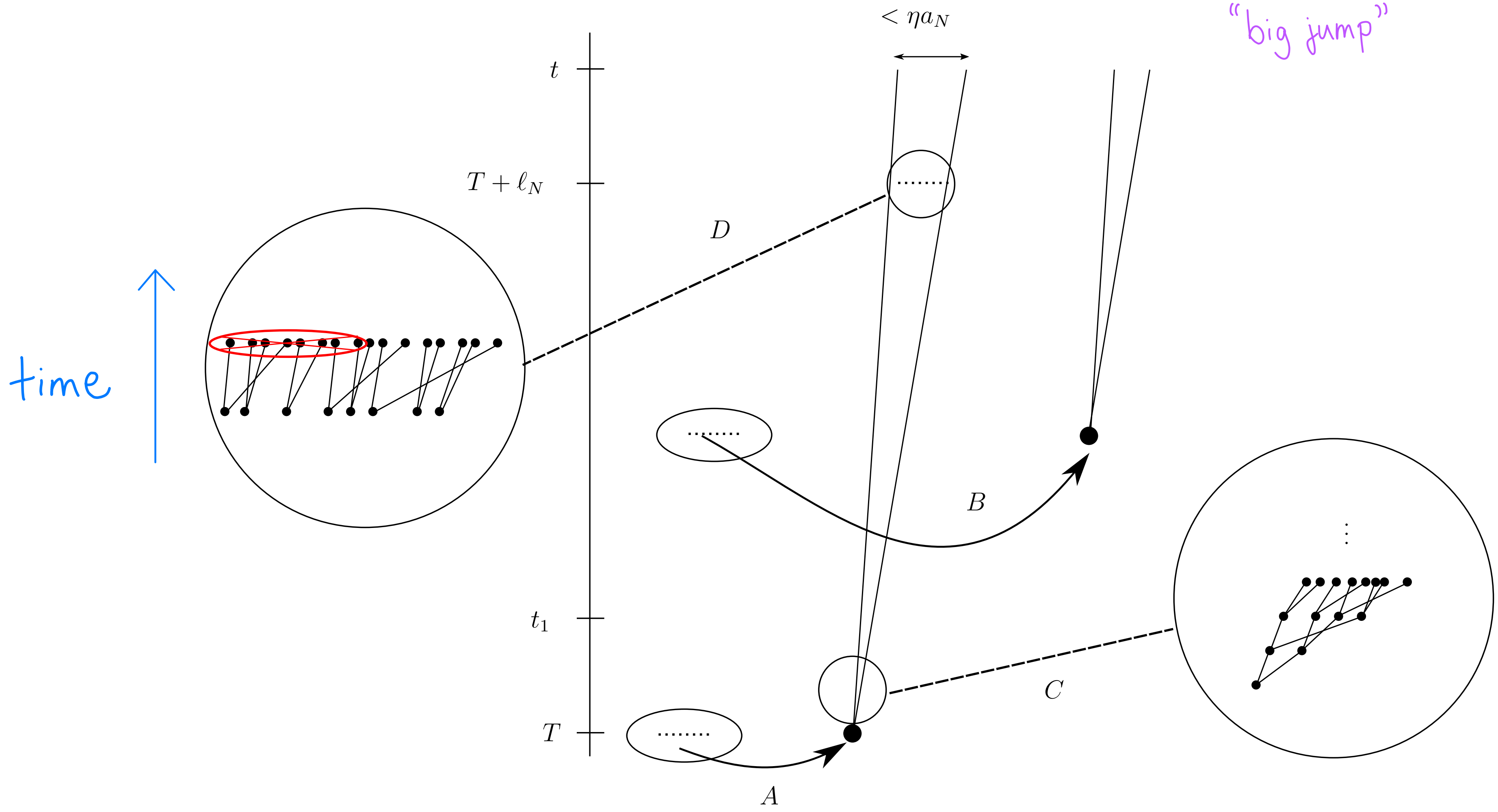
A jump is big w.p. $h(e a_N)^{-1} \sim e^{-\alpha}(2N\ell_N)^{-1}$,

so we see big jumps at rate $\Theta(\ell_N^{-1})$.

Proof heuristics $\quad t_1 := t - \ell_N.$

Let T = last time before time t_1 when a particle makes a jump $\geq \varrho a_N$ and takes the lead.

"big jump"



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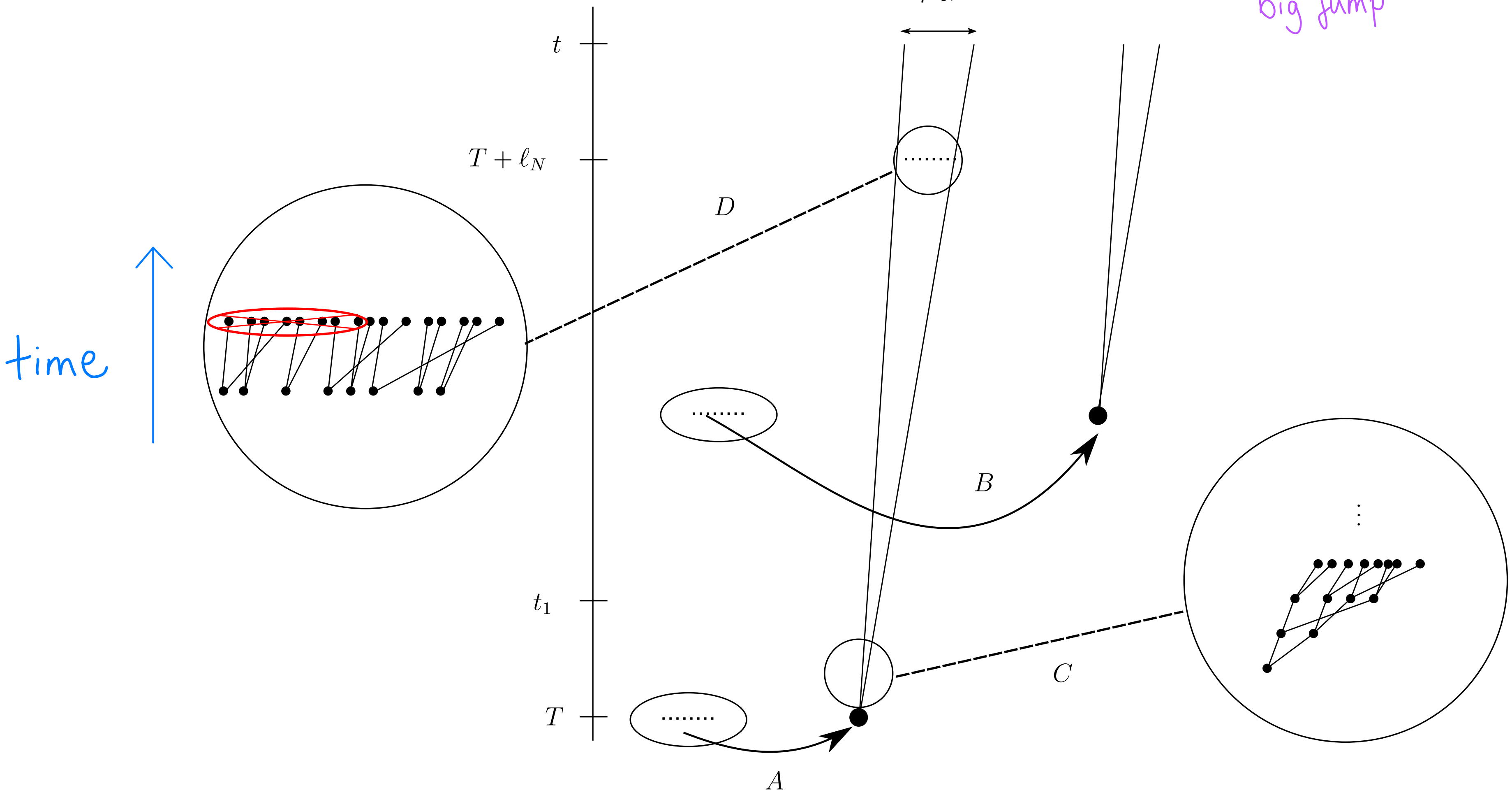


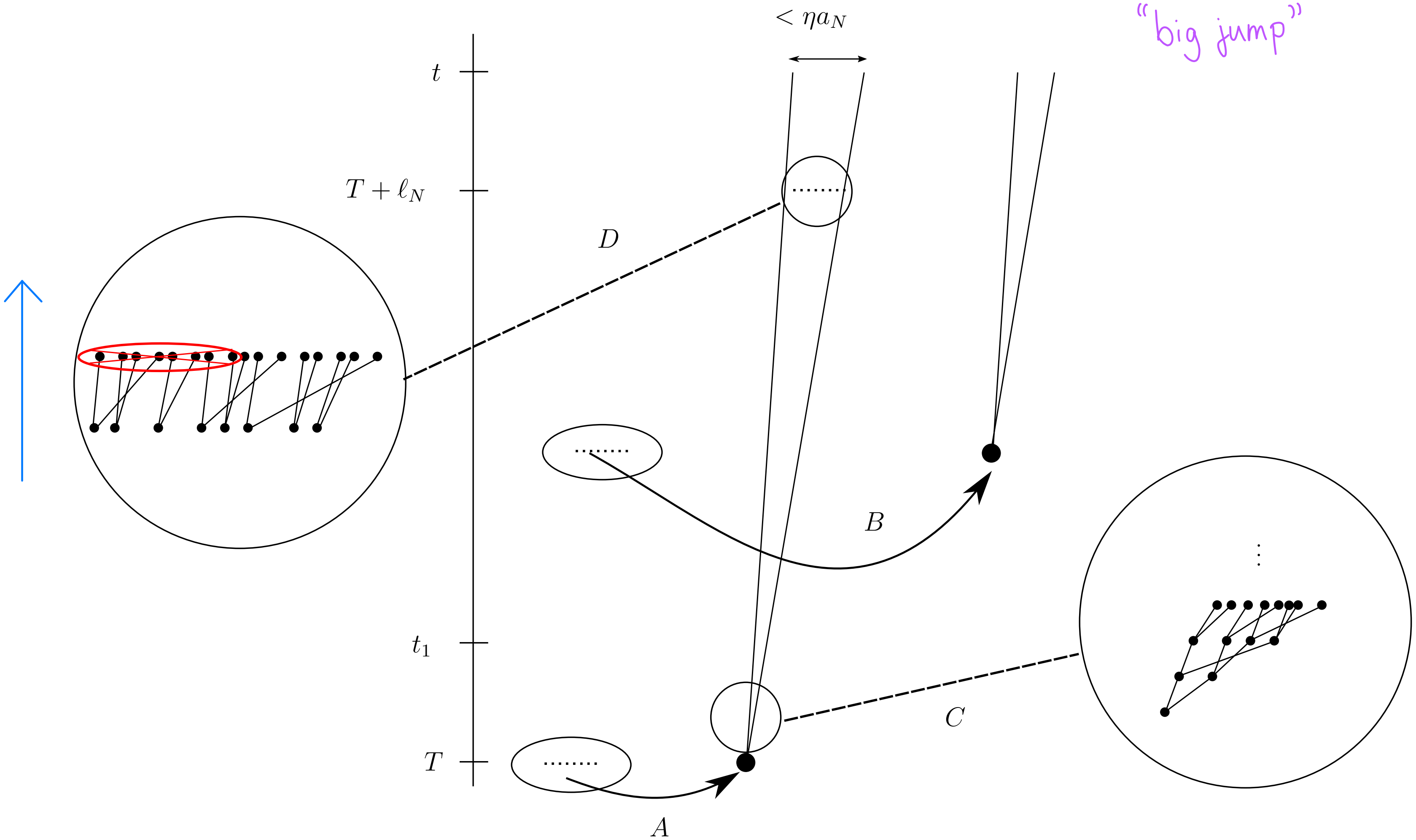
A: A particle makes a big jump at time T and takes the lead (by $\Theta(a_N)$). Its descendants stay in the lead until time t_1 (other particles can't take the lead with a big jump, and can't move far without a big jump).

Proof heuristics $\qquad t_1 := t - \ell_N$.

Let T = last time before time t_1 when a particle makes a jump $\geq \varrho a_N$ and takes the lead.

"big jump"

time



B: There are $O(1)$ big jumps in time interval $[t_1, t]$, each with $o(N)$ descendants at time t.

Proof heuristics

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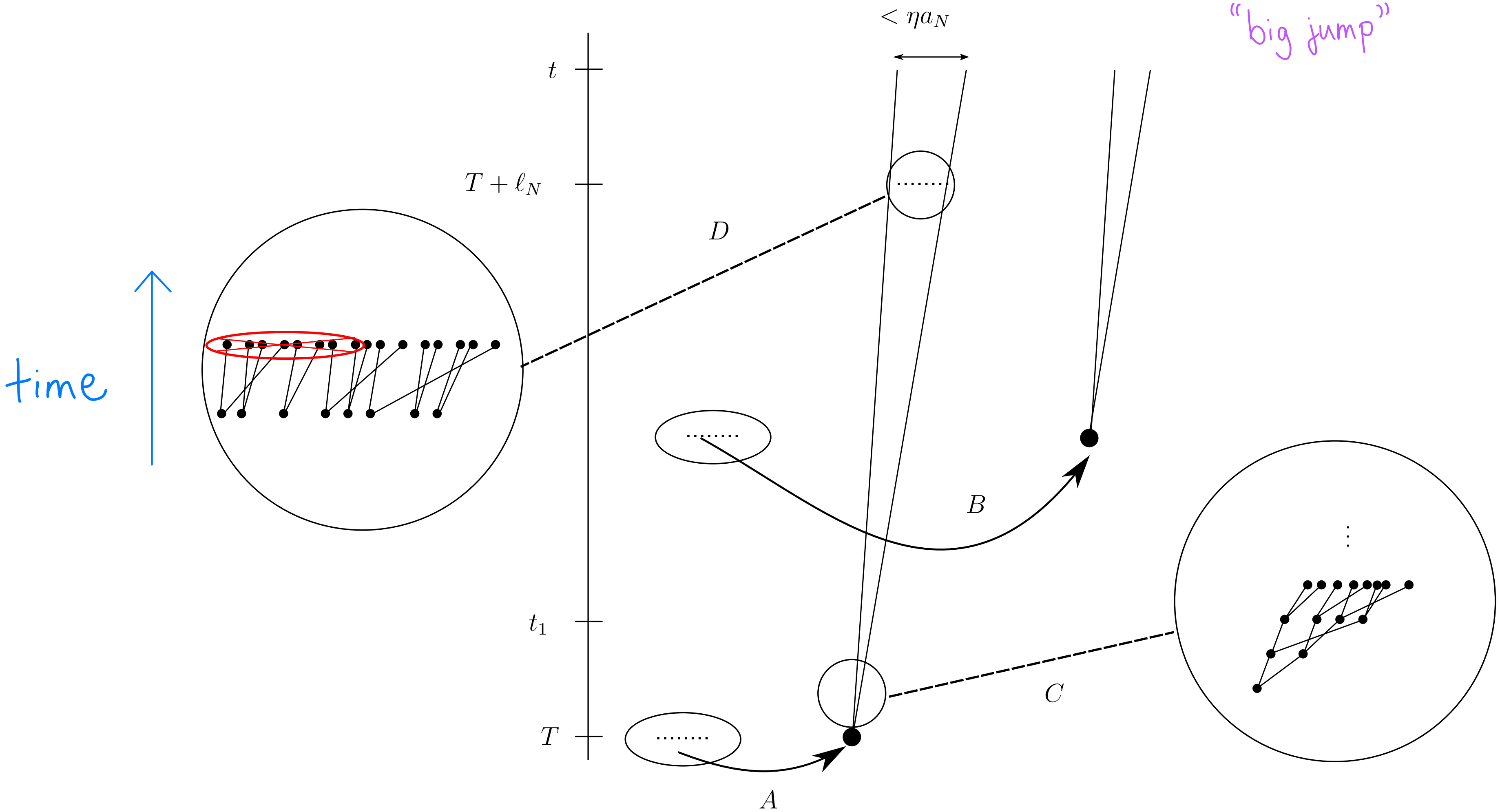
Let T = last time before time t_1 when a particle makes a jump $\geqslant \varrho a_N$ and takes the lead.

"big jump"

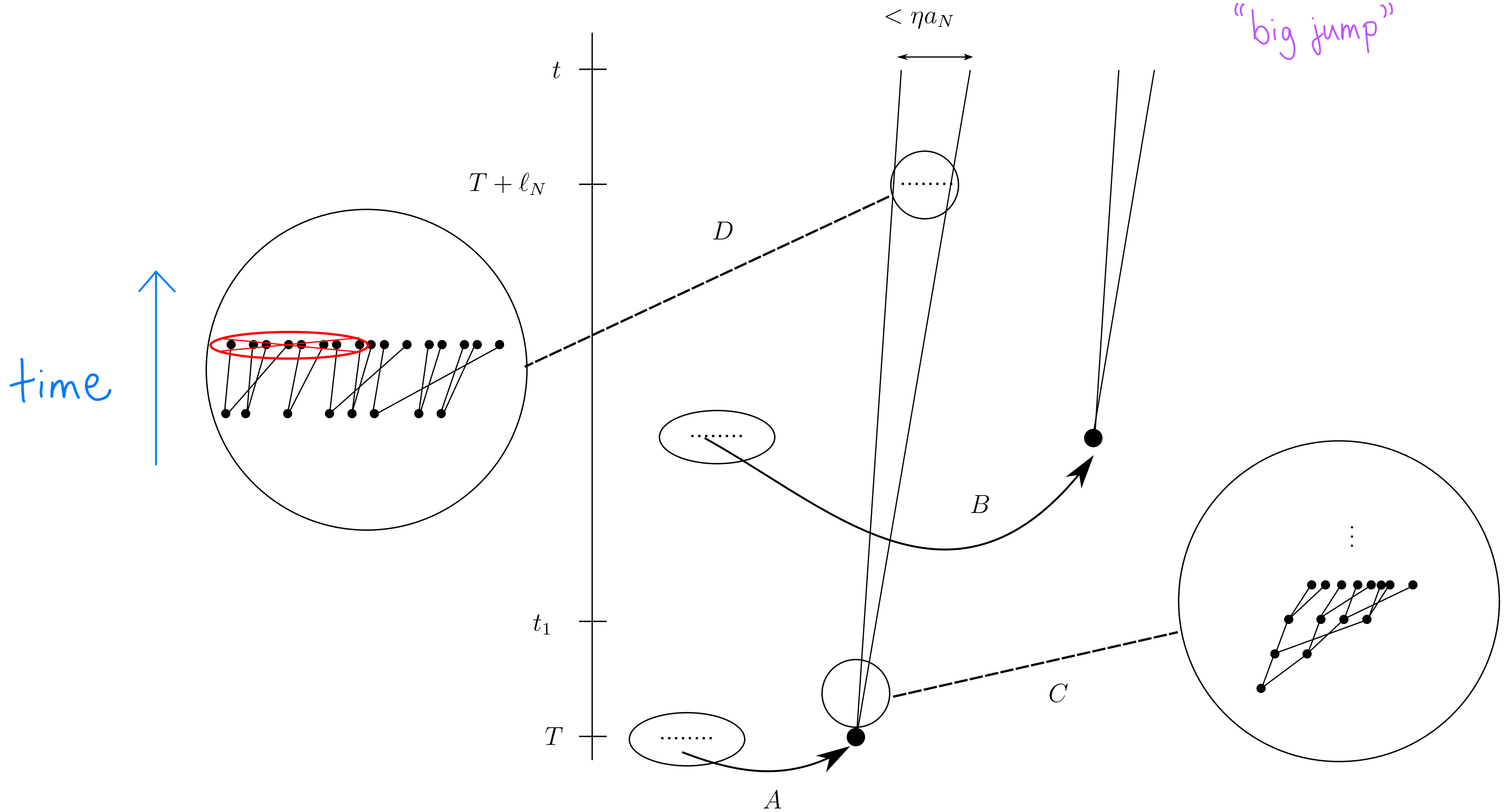


C: The tribe descended from the time-T leader doubles in size at each timestep until almost time $T + \ell_N$.

Proof heuristics $t_1 := t - \ell_N$.

Let T = last time before time t_1 when a particle makes a jump $\geq \varrho a_N$ and takes the lead.

"big jump"



D: On the time interval $[T+\ell_N, t]$, the time-T leader's tribe has size $N - o(N)$.

Proof heuristics

$t_1 := t - \ell_N$.

Let T = last time before time t_1 when a particle makes a jump $\geq \varrho a_N$ and takes the lead.

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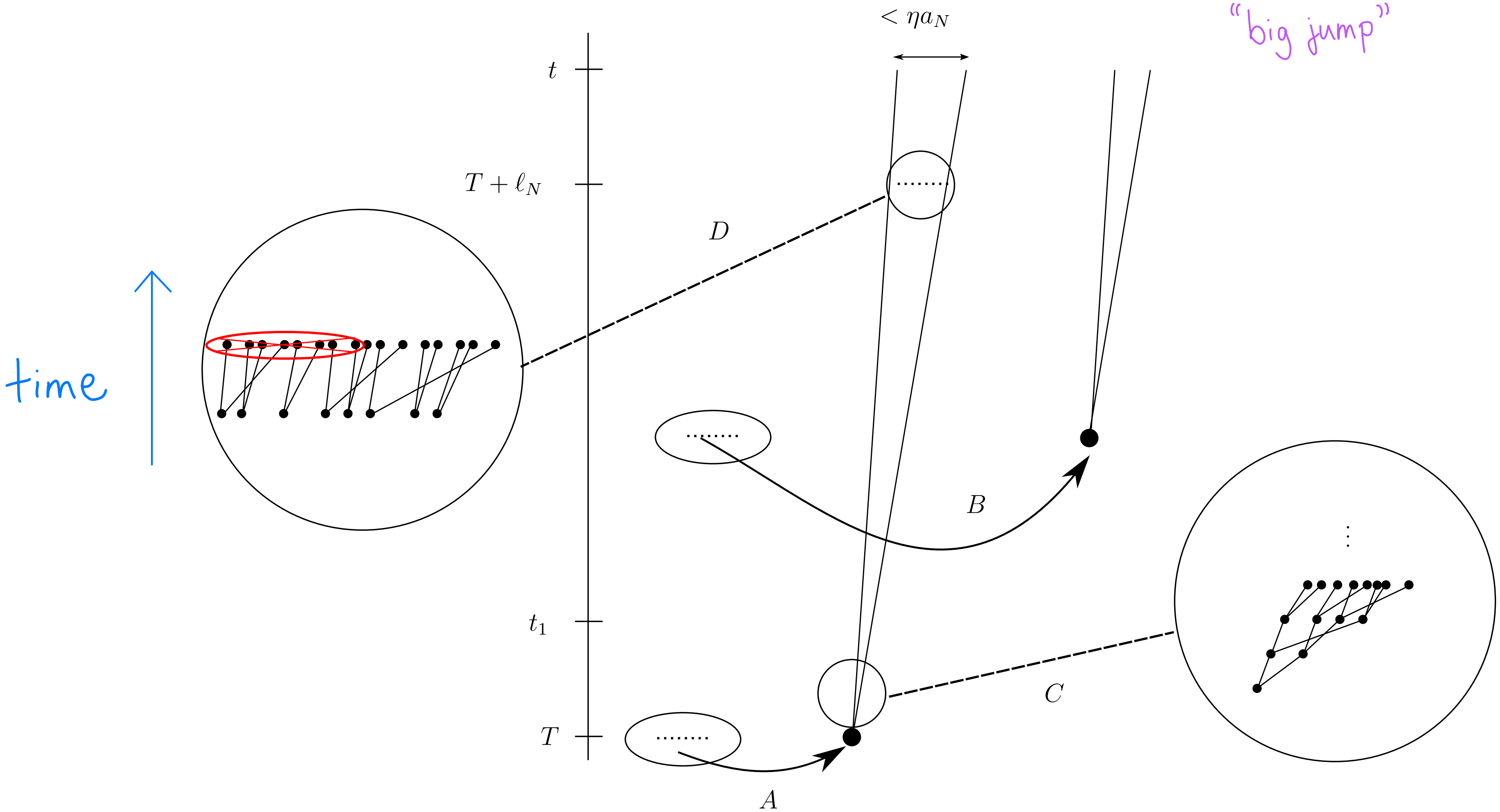

$T \in [t_1 - \ell_N, t_1]$ w.h.p. :

If no particle takes the lead with a big jump during $[s, s+\ell_N]$, then $X_1^{(N)}(s+\ell_N) \geq X_N^{(N)}(s)$ and so $\frac{1}{a_N}\left(X_N^{(N)}(s+\ell_N) - X_1^{(N)}(s+\ell_N)\right)$ is small w.h.p.

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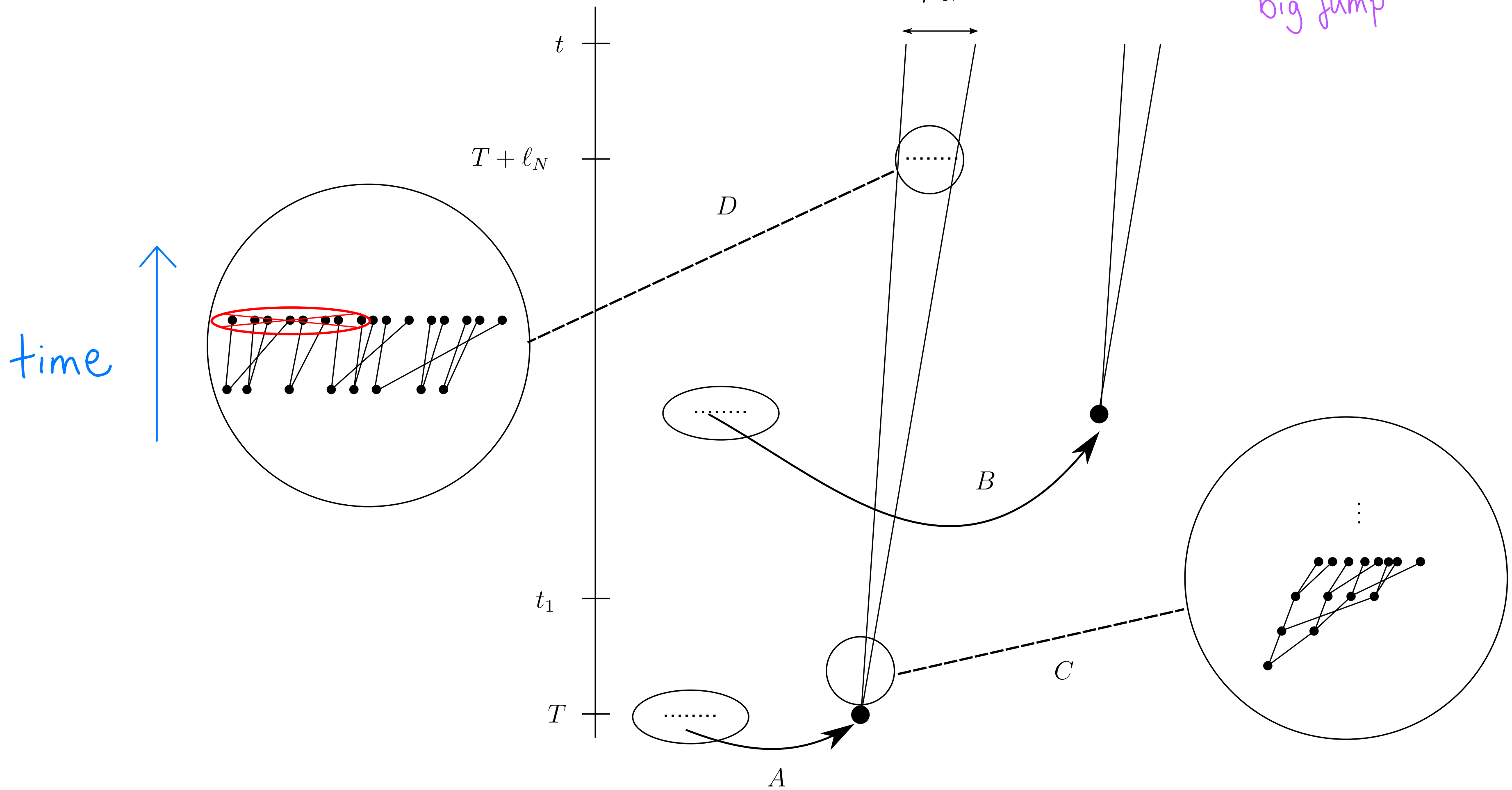

N−o(N) particles are close to leftmost at time t (on a_N space scale)

No big jumps in the time-T leader's tribe up to time $T+\ell_N > t_1$.

O(1) big jumps in the tribe during $[T+\ell_N, t]$, each with o(N) descendants.

Proof heuristics $\quad t_1 := t - \ell_N$.

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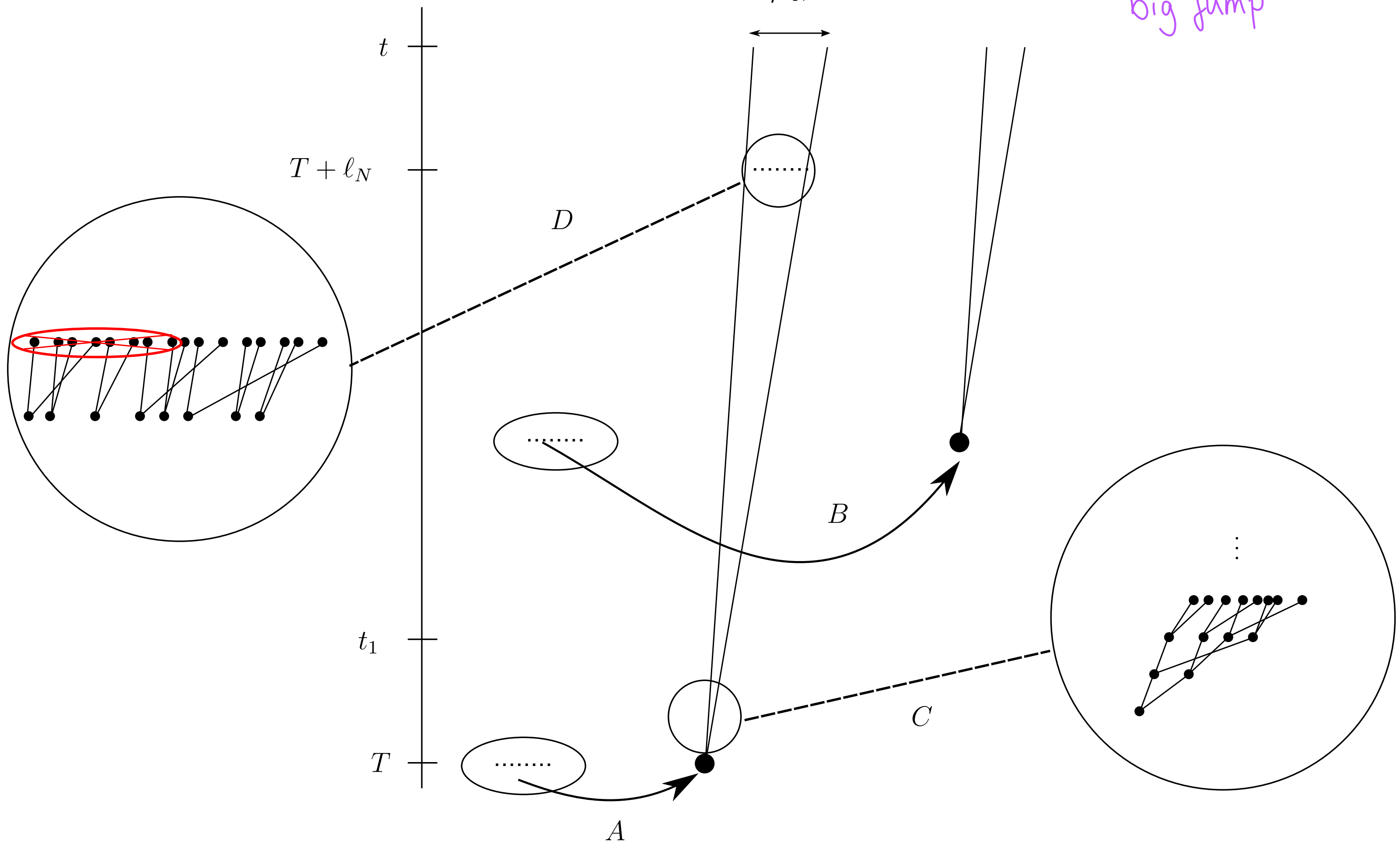
"big jump"

time



Star-shaped genealogy $\quad$ No time-$(T+\varepsilon_N \ell_N)$ particles have $\Theta(N)$ time-t descendants.

None of the particles in the time-T leader's tribe have moved far by time $T+\varepsilon_N \ell_N$, so each has $\Theta(N 2^{-\varepsilon_N \ell_N}) = o(N)$ descendants at time t.