Percolation of worms

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Bangalore Probability Seminar, October 4, 2021

Random length worms model

Definition: Random length worms model

Given some $v \in \mathbb{R}^+$ and a probability mass function $m : \mathbb{N} \to \mathbb{R}$ the **random length worms model** is a Poisson point process $\mathcal{X}$ on the space of all the possible $\mathbb{Z}^d$-valued nearest neighbour paths of finite length (the space of worms: W) with intensity measure $v \cdot m(L(w)) \cdot (2d)^{1-L(w)}$, where $L(w)$ is the length of such a path $w \in W$. We denote by $\mathcal{P}_v$ the law of $\mathcal{X}$.

This alternative definition yields that the **random length worms set** $\mathcal{S}^v$ **at level** v can be defined as $\mathcal{S}^v := \mathrm{Tr}(\mathcal{X})$, where $\mathcal{X} \sim \mathcal{P}_v$.

Theorem [Ráth, R.; '21]: Supercritical worm percolation

Let $d \geq 5$. Let $\varepsilon > 0$ and $\ell_0 \geq e^e$. If

$$m(\ell) = c\frac{\ln(\ln(\ell))^{\varepsilon}}{\ell^3 \ln(\ell)} \mathbb{1}[\ell \geq \ell_0], \qquad \ell \in \mathbb{N}$$

then for any $v > 0$ the random length worms model $\mathcal{S}^v$ is supercritical:

$$\mathbb{P}(\,\mathcal{S}^v \text{ percolates}\,) = 1.$$

Equilibrium measure

Given $K \subset\subset \mathbb{Z}^d$ let us denote its **entrance** and **hitting time** by T_K and $\tilde{T}_K$, respectively. The **equilibrium measure** of a set $K \subset\subset \mathbb{Z}^d$ is: $e_K(x) := P_x(\tilde{T}_K = \infty) \cdot \mathbb{1}[x \in K]$.

Definition: Capacity of a set

Given $K \subset\subset \mathbb{Z}^d$, the **capacity** of K is defined by $\mathrm{cap}(K) := \sum_{x \in \mathbb{Z}^d} e_K(x)$.

Lemma: Last exit decomposition

Let $d \geq 3$. For any $x \in \mathbb{Z}^d$ and $K \subset\subset \mathbb{Z}^d$ we have: $P_x(T_K < \infty) = \sum_{y \in K} g(x,y) e_K(y)$, where $g(\cdot,\cdot)$ denotes the Green function. As a consequence:

$$\mathrm{cap}(K) \cdot \min_{y \in K} g(x,y) \leq P_x(T_K < \infty) \leq \mathrm{cap}(K) \cdot \max_{y \in K} g(x,y).$$

Theorem [Jain, Orey; '68]: The capacity of the range of random walk

Let $d \geq 5$. There exists a constant $e_\infty = e_\infty(d) \in (0, +\infty)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \mathrm{cap}\left(\cup_{t=0}^{n-1} \{X(t)\} \right) = e_\infty \qquad P_o - \text{almost surely.}$$

Dirichlet energy

Definition: Dirichlet energy

If μ_1 and μ_2 are measures on $\mathbb{Z}^d$ then their **mutual Dirichlet energy** is

$$\mathcal{E}(\mu_1, \mu_2) := \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} g(x, y)\mu_1(x)\mu_2(y),$$

where $g(\cdot, \cdot)$ is the Green function. When $\mu_1 = \mu_2 = \mu$ then $\mathcal{E}(\mu) := \mathcal{E}(\mu, \mu)$ is called the **Dirichlet energy** of μ.

Theorem [Jain, Orey; '73]: Energy characterization of capacity

The capacity of any set $K \subset\subset \mathbb{Z}^d$

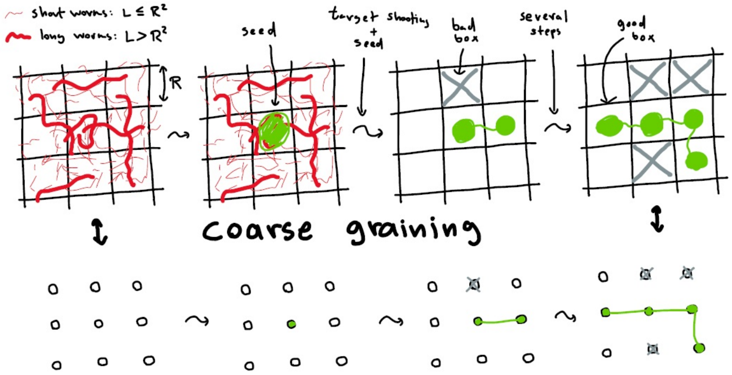
$$\mathrm{cap}(K) = \sup \left\{ \mathcal{E}(\nu)^{-1} \,:\, \nu \text{ is a probability measure supported on } K \right\}.$$

In order to give a lower bound on $\mathrm{cap}(K)$, one just puts a probability measure ν on K and gives an upper bound on $\mathcal{E}(\nu)$.

Example: Capacity of a ball

For any $R > 0$ we have $\mathrm{cap}(\mathcal{B}(R)) \asymp R^{d-2}$, where $\mathcal{B}(R)$ denotes the ball around the origin with radius R by $\mathcal{B}(R)$ (here we considered the sup-norm).

Coarse graining



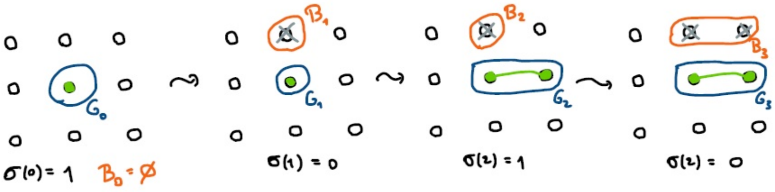
Observation: On the probability of the new box being good

If N denote the number of long worms emanating from the newly considered box, then:

$$\mathbb{E}N \approx \left(v \cdot R^d \cdot \mathbb{P}\left(\text{length} > R^2\right)\right) \cdot R^{2-d} \cdot \mathrm{cap}\left(\text{seed}\right)$$

If we can guarantee that $\mathbb{E}N$ is big enough, then $\mathbb{P}(N \geq 1)$ is also big.

Dynamic renormalization



In this case **dynamic** means that the value of $\sigma(\cdot)$ depends on the previous values.

Lemma [Grimmett, Marstrand; '90]: Sufficient condition for eternal exploration

If there exists a constant $0 < c < 1$ such that $c > p_c^{\text{site}}(\mathbb{Z}^d)$, moreover

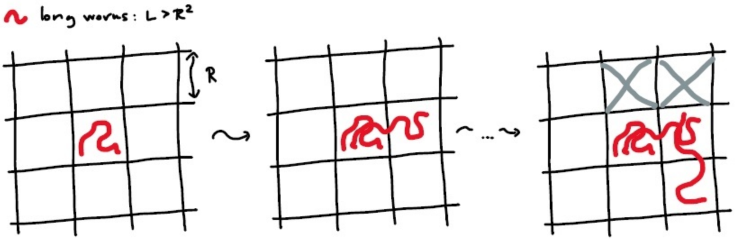
$$\mathbb{P}(\sigma(0) = 1) \geq c \qquad \text{and} \qquad \mathbb{P}(\sigma(t+1) = 1 \mid \sigma(0), \sigma(1), \ldots, \sigma(t)) \geq c$$

hold for all $t \in \mathbb{N}_0$, then we have $\mathbb{P}(|G_\infty| = \infty) > 0$.

Remark: Eternal exploration implies percolation

By induction it is easy to see that $\mathbb{P}(|G_\infty| = \infty) > 0$ implies percolation.

Absence of short worms



Here the seed is a slice of the long worm inside the given box with length comparable to R^2, hence we have $\mathrm{cap}(\text{seed}) \approx R^2$.

Example: Percolation if $m(\ell) \approx \ell^{\varepsilon}/\ell^3$ with $\varepsilon > 0$

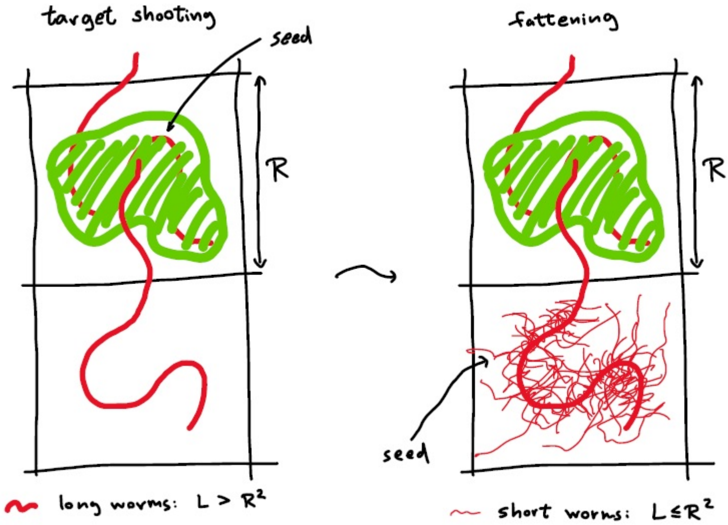
In this case we have $\mathbb{P}\left(\mathcal{L} > R^2\right) \approx R^{2\varepsilon}/R^4$, thus

$$\mathbb{E}N \approx \left(v \cdot R^d \cdot R^{2\varepsilon}/R^4\right) \cdot R^{2-d} \cdot R^2 = v \cdot R^{2\varepsilon}$$

Since v, $\varepsilon > 0$ by choosing R big enough, we can guarantee that $\mathbb{E}N$ is big and hence guarantee that $\mathbb{P}(N \geq 1) \geq 1/2$.

Note that $1/2$ is enough since for site percolation $1/2 > p_c(\mathbb{Z}^d)$ if $d \geq 3$.

Fattening


target shooting
seed
R
fattening
R
seed
long worms: L > R²
short worms: L ≤ R²

Size biasing

Observation: Expected total length of short worms hitting the origin

The expected number of worms of length ℓ that hit o is comparable to $v \cdot m(\ell) \cdot \ell$, thus the expected total length of short worms that hit o is comparable to $v \cdot \mathbb{E}(\mathcal{L}^2 \cdot \mathbb{1}[\mathcal{L} \leq R^2])$.

Using similar reasoning that we will use later, one can show that the probability of the following events are high:

$$\mathrm{cap}(\text{seed}) \approx \begin{array}{c}\text{total length of short worms}\\ \text{hitting the long worm}\end{array} \approx R^2 \cdot v \cdot \mathbb{E}(\mathcal{L}^2 \cdot \mathbb{1}[\mathcal{L} \leq R^2]).$$

Example: Percolation if $m(\ell) \approx \ln(\ell)^\gamma / (\ell^3 \ln(\ell))$ with $\gamma > 1/2$

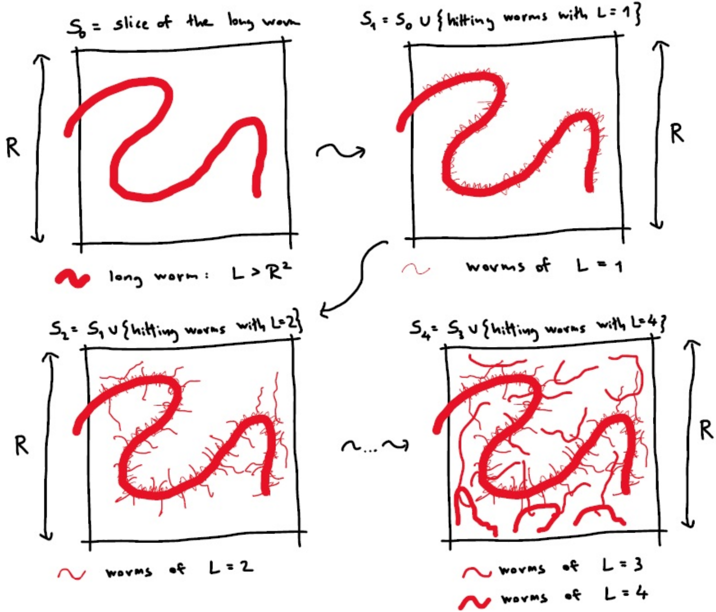
In this case we have $\mathbb{E}(\mathcal{L}^2 \cdot \mathbb{1}[\mathcal{L} \leq R^2]) \approx \ln(R)^\gamma$ and $\mathbb{P}(\mathcal{L} > R^2) \approx \ln(R)^{\gamma-1}/R^4$, thus

$$\mathbb{E}N \approx (v \cdot R^d \cdot \ln(R)^{\gamma-1}/R^4) \cdot (R^{2-d} \cdot R^d \cdot v \cdot \ln(R)^\gamma) = v^2 \cdot \ln(R)^{2\gamma-1}.$$

Since $\gamma > 1/2$, by choosing R big enough we can guarantee that $\mathbb{P}(N \geq 1) \geq 1/2$ holds.

Note that the $\gamma = 1$ case gives $m(\ell) \approx \ell^{-3}$, thus $\mathbb{P}(\mathcal{L} > \ell) \approx \ell^{-2}$, so the behaviour of the planar ellipses percolation (which exhibits percolation phase transition) differs from the behaviour of our high dimensional worm percolation model.

The snowball effect



Dummy calculation

Observation: Snowball effect

Heuristically, for all $\ell \geq 1$ we have $\mathrm{cap}(S_\ell) \approx \mathrm{cap}(S_{\ell-1}) \cdot (1 + v \cdot m(\ell) \cdot \ell^2)$, thus

$$\mathrm{cap}(S_{R^2}) \approx R^2 \cdot \prod_{\ell=1}^{R^2} \left(1 + v \cdot m(\ell) \cdot \ell^2\right) \approx R^2 \cdot \exp\left(v \cdot \sum_{\ell=1}^{R^2} m(\ell) \cdot \ell^2 \right).$$

In the case when $m(\ell) \approx \ln(\ln(\ell))^\varepsilon / (\ell^3 \ln(\ell))$ with $\varepsilon > 0$ we have $\mathbb{E}(\mathcal{L}^2 \cdot \mathbb{1}[\mathcal{L} \leq R^2]) \approx \ln(\ln(R))^{1+\varepsilon}$ and $\mathbb{P}(\mathcal{L} > R^2) \approx \ln(\ln(R))^\varepsilon / (\ln(R) \cdot R^4)$, thus

$$\mathbb{E}N \approx \left(v \cdot R^d \cdot \frac{\ln(\ln(R))^\varepsilon}{\ln(R) \cdot R^4} \right) \cdot R^{2-d} \cdot R^2 \cdot \exp\left\{ v \cdot \ln(\ln(R))^{1+\varepsilon} \right\} \geq v \cdot \frac{\exp\left\{ v \cdot \ln(\ln(R))^{1+\varepsilon} \right\}}{\ln(R)}.$$

Since $\varepsilon > 0$, by choosing R big enough we can guarantee that $\mathbb{P}(N \geq 1) \geq 1/2$.

Remark: Grouping the lengths

Instead of adding the packages of worms of length $\ell = 1, \ldots, R^2$ one by one, we will define a sequence of scales $(R_n)_{n=0}^{N+1}$ (where $R_{N+1} = R$) and we will fatten with all of the worms of length between R_n^2 and R_{n+1}^2 in one round.

Some further comments

Summary: Our method

The infinite component of $\mathcal{S}^v$ will be built up using a dynamic renormalization scheme where for every examined vertex in the coarse grained lattice, there is (i) target shooting and (ii) recursive capacity doubling using a sequence of rapidly growing scales.

In order to do so, we will also subdivide the PPP of worms into disjoint, hence independent, **packages**, where a package contains worms of a certain **length scale** that emanate from a box of a certain **spatial scale**.

Policy: Packages of worms only used once

For the purpose of target shooting or fattening, we will use every package only once.

Remark: Concentration estimates

(a) In order to obtain the requested supercriticality, we need to show that connection probabilities are bounded away from zero.

(b) This will be guaranteed using concentration estimates in a form: *the capacity of the inductively fattened worm cluster is big enough with high enough probability*.

(c) To prove such concentration estimates, we need control the correlation between the amounts of fat produced at distant parts of the set being fattened.

Good sequence of scales

Definition: Good sequence of scales

Let $R_0^*, \gamma_0, \underline{\Delta}, \overline{\Delta}, \underline{\alpha}, \psi, s, \Lambda, v \in (0, +\infty)$. We say that an increasing sequence $(R_n)_{n=0}^{N+1}$ of positive integers is a $(R_0^*, \gamma_0, \underline{\Delta}, \overline{\Delta}, \underline{\alpha}, \psi, s, \Lambda, v)$-good sequence of scales for a probability measure m on $\mathbb{N}$ if

$$R_0 \geq R_0^*, \tag{Initializing}$$

$$\forall\, 0 \leq n \leq N: \quad \sum_{\ell=\underline{\Delta}\cdot R_n^2}^{\overline{\Delta}\cdot R_{n+1}^2} \ell^2 \cdot m(\ell) \geq \underline{\alpha}\,/v \tag{Fattening}$$

$$\forall\, 0 \leq n \leq N: \quad 2^n \cdot \gamma_0 \leq \psi \cdot R_n^{d-4}, \tag{Fattening*}$$

$$\left(2^{N+1} \cdot \gamma_0 \cdot s\right) \cdot v \cdot R_{N+1}^4 \cdot \sum_{\ell=\Lambda\cdot R_{N+1}^2}^{\infty} m(\ell) \geq 2. \tag{Target shooting}$$

If $\mathbb{E}[\mathcal{L}^2] = +\infty$ then we can define $(R_n)_{n=0}^{\infty}$ so that $R_n < +\infty$ and (Fattening) is satisfied for each n, but $\mathbb{E}[\mathcal{L}^2] = +\infty$ is not enough for (Target shooting) to hold for some N.

General version of the main theorem

Theorem [Ráth, R.; '21]: Good sequence of scales implies supercritical percolation

Let $d \geq 5$. There exist constants $R_0^*, \gamma_0, \underline{\Delta}, \overline{\Delta}, \underline{\alpha}, \psi, s, \Lambda \in (0, +\infty)$ such that for any $v \in (0, +\infty)$ and any probability measure m on $\mathbb{N}$: if there exists an $(R_0^*, \gamma_0, \underline{\Delta}, \overline{\Delta}, \underline{\alpha}, \psi, s, \Lambda, v)$-good sequence $(R_n)_{n=0}^{N+1}$ of scales for m then

$$\mathbb{P}(\, \mathcal{S}^v \text{ percolates}\,) = 1.$$

Note that if the length distribution is

$$m(\ell) = c\frac{\ln(\ln(\ell))^{\varepsilon}}{\ell^3 \ln(\ell)} \mathbb{1}\left[\ell \geq \ell_0\right], \qquad \ell \in \mathbb{N},$$

where $\varepsilon > 0$ and $\ell_0 \geq e^e$, then for any choice of the above parameters and $\delta \in (0, \varepsilon)$ the sequence

$$R_n := \exp\left(\exp\left((n + n_0)^{1/(1+\delta)}\right)\right), \qquad n \in \mathbb{N}_0$$

is a good sequence of scales, if n_0 is large enough.

Also note, if we consider the same probability mass function, but with $\varepsilon \in (-1, 0)$, then $\mathbb{E}[\mathcal{L}^2] = +\infty$ still holds, but it can be proved that for any choice of the parameters, there exists ℓ_0 big enough such that there is no good sequence of scales.

End

Thank you for your attention!

Some questions

Questions

(i) Given $d \geq 2$, is there a function $f : \mathbb{N} \to \mathbb{R}_+$ satisfying $\lim_{n\to\infty} f(n)/n^2 = 0$ such that for any choice of ν the condition $\sum_{H\in\mathcal{H}} f(|H|)\cdot\nu(H) < +\infty$ implies $v_c(\nu) > 0$?

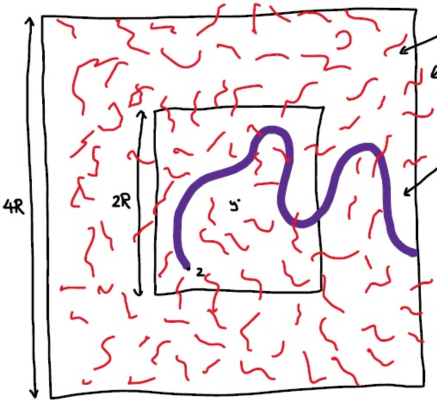
(ii) Is it possible to conclude $v_c = 0$ in the random length worms model if we only assume $m_2 = +\infty$?

(iii) It is not that hard to prove the m_1 and m_2 lemmas are also true for any Cayley graph: how can we characterize v_c there?

(iv) Conjectures in lower dimensional random length worms model: see Balázs' talk.

(v) Other properties of the percolation set?

Input packages



$\mathcal{X}$: PPP of $\sim$ (short) worms:

- $L \le R^2$
- starts at $B(y, 2R)$

Z : independent random walk:

- $Z(0) = z \in B(y, R)$
- running up to:

  $T = \min\{R^2, T_{B(y,2R)^c}\}$

the joint law is denoted:

$(\mathcal{X}, Z) \sim \mathcal{P}_{y,R,z,v}$

Observation: Translation invariant law

The law $\mathcal{P}_{y,R,z,v}$ is invariant under the translations of $\mathbb{Z}^d$ in the sense that if $(\mathcal{X}, Z) \sim \mathcal{P}_{y,R,z,v}$ then $(\mathcal{X} + x, Z + x) \sim \mathcal{P}_{y+x,R,z+x,v}$ for any $x \in \mathbb{Z}^d$.

Good input packages

Definition: Good input package

Let $y \in \mathbb{Z}^d$, $R \in \mathbb{N}$, $z \in \mathcal{B}(y, R)$ and $\gamma \in \mathbb{R}_+$. We say that an input package $(\mathcal{X}, Z)$ is (y, R, z, γ)-**good** if there is a set $H \subset \mathbb{Z}^d$ which satisfies the following properties:

(i) $R^2 \cdot \gamma \leq \mathrm{cap}(H) \leq R^2 \cdot \gamma + 1$, (ii) $z \in H$,
(iii) $H \subseteq \mathcal{B}(y, 3R) \cap (\mathrm{Tr}(\mathcal{X}) \cup \mathrm{Tr}(Z))$, (iv) H is connected.

We call such a set H an (y, R, z, γ)-**good set** for $(\mathcal{X}, Z)$. We call the parameter γ the **capacity-to-length ratio** (since R^2 is the maximal length of Z).

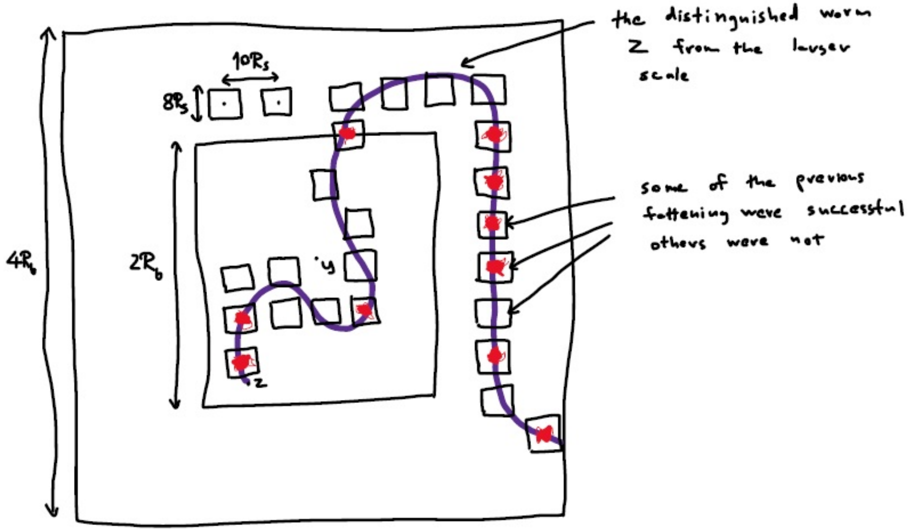
Note that if we choose $H := \mathrm{Tr}(Z)$, then it satisfies (ii), (iii) and (iv). Moreover, by the arrangement of the the input package and the properties of the capacity we can find such γ_0 and R_0^* such that for all $R \geq R_0^*$ that for the chosen set: $\mathbb{P}(\mathrm{cap}(H) \geq R^2 \cdot \gamma_0^2) \geq 3/4$.

Lemma: Initializing the γ of good input packages

There exist dimension-dependent positive finite constants R_0^* and γ_0 such that for any $R \geq R_0^*$ and any $v \geq 0$ we have

$$\min_{z \in \mathcal{B}(R)} \mathcal{P}_{o,R,z,v}\left((\mathcal{X}, Z) \text{ is } (o, R, z, \gamma_0)\text{-good} \right) \geq 3/4.$$

A long necklace of beads



Doubling the capacity-to-length ratio

Lemma: Doubling the γ of good input packages by fattening

There exist dimension-dependent positive finite parameters $\psi < 1$, $\overline{\Delta} < 1$, $\underline{\Delta} > 1$, $\underline{\alpha} > 1$, such that for any $R_s \leq R_b \in \mathbb{N}$, any $\gamma \in \mathbb{R}_+$ satisfying

$$\gamma_0 \leq \gamma \leq \psi \cdot R_s^{d-4}, \tag{Fattening*}$$

any choice of $v \in \mathbb{R}_+$ and any choice of the probability mass function m satisfying the inequality

$$v \cdot \sum_{\ell=\underline{\Delta}\cdot R_s^2}^{\overline{\Delta}\cdot R_b^2} \ell^2 \cdot m(\ell) \geq \underline{\alpha}, \tag{Fattening}$$

the following implication holds: if

$$\min_{z \in \mathcal{B}(R_s)} \mathcal{P}_{o,R_s,z,v}\left((\mathcal{X}, Z) \text{ is } (o, R_s, z, \gamma)\text{-good} \right) \geq 3/4, \tag{Hypothesis}$$

then

$$\min_{z \in \mathcal{B}(R_b)} \mathcal{P}_{o,R_b,z,v}\left((\mathcal{X}, Z) \text{ is } (o, R_b, z, 2\gamma)\text{-good} \right) \geq 3/4. \tag{Doubling}$$

Going up the ladder

Let us choose constants from the:

Initializing lemma: $R_0^*, \gamma_0 \in (0, +\infty)$, and

γ**-doubling lemma:** $\underline{\Delta}, \overline{\Delta}, \underline{\alpha}, \psi \in (0, +\infty)$.

Given $v \in (0, +\infty)$ and probability mass function m, if a sequence $(R_n)_{n=0}^{N+1}$ of scales satisfies the properties

(Initializing), (Fattening), (Fattening*)

from the definition of good sequence of scales, then we can use induction on $0 \le n \le N+1$ with

(i) $n = 0$: $R_0 \ge R_0^*$, and

(ii) $0 < n \le N$: $R_s = R_n$, $R_b = R_{n+1}$ and $\gamma = 2^n \cdot \gamma_0$

to prove the following lemma.

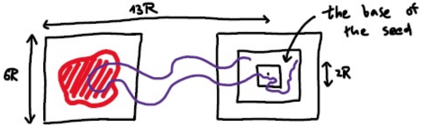
Lemma: Going up the ladder

Under the previous assumptions

$$\min_{z \in \mathcal{B}(R_n)} \mathcal{P}_{o,R_n,z,v}\left((\mathcal{X}, Z) \text{ is } (o, R_n, z, 2^n \cdot \gamma_0)\text{-good} \right) \ge 3/4$$

holds for all $0 \le n \le N+1$.

Joining the boxes



Lemma: Target shooting with a boomerang

There exist $s > 0$ and $\Lambda \in \mathbb{N}$ such that for every $R \in \mathbb{N}$, $y \in \mathbb{Z}^d$ and every $H \subseteq \mathcal{B}(y, 13R)$ satisfying

$$\mathrm{cap}(H) \cdot s \cdot v \cdot R^2 \cdot \mathbb{P}\left(\mathcal{L} \geq \Lambda R^2\right) \geq 2,$$

we have $\mathcal{P}_v$ (the above event) $\geq 3/4$.

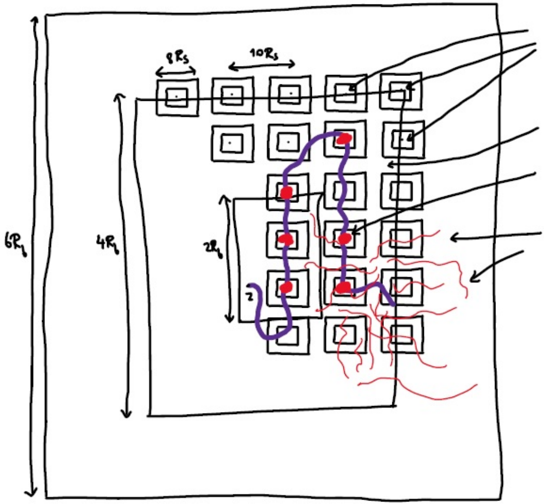
Conclusion: Percolation of worms

If we fix our constants as they were in the previous lemmas and do the coarse graining using radius R_{N+1} then for the dynamic renormalization scheme we obtain:

$$\mathbb{P}(\sigma(0) = 1) \geq 1/2, \quad \text{and} \quad \mathbb{P}(\sigma(t+1) = 1 \mid \sigma(0), \sigma(1), \ldots, \sigma(t)) \geq 1/2 \quad (t \in \mathbb{N}_0).$$

As a consequence, by the Grimmett-Marstrand lemma our result follows.

Inside an input package



$\mathcal{D} := \{ y \in 10R_s\mathbb{Z}^d : B(y,R_s) \cap B(2R_b) \neq \emptyset \}$

$\mathcal{G} := B(2R_b) \setminus \bigcup_{y \in \mathcal{D}} B(y, 4R_s)$

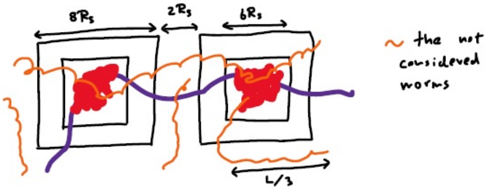
good set of the smaller scale R_s: H^y $(y \in \mathcal{D})$

worms for the next round:

- $\underline{\Delta} R_s^2 \le L \le \bar{\Delta} R_b^2$
- starts at $\mathcal{G}$
- stays in $B(3R_b)$

$\mathcal{F}$: the sigma field generated by Z and worms of $L \le R_s^2$

The candidate set



$$H := \mathrm{Tr}(Z) \cup \bigcup_{y \in \mathcal{D}} H^y \cup \bigcup_{y \in \mathcal{D}} \mathrm{Tr}(\text{the previously mentioned worms starting at } \mathcal{G})$$

The way we defined the set H it follows immediately that:

(ii) $z \in H$; (iii) $H \subseteq \mathcal{B}(3R_b) \cap (\mathrm{Tr}(\mathcal{X}) \cup \mathrm{Tr}(Z))$; (iv) H is connected.

Observation: A lower bound is enough

We only need to show that

$$2 \cdot \gamma \cdot R_b^2 \leq \mathrm{cap}(H)$$

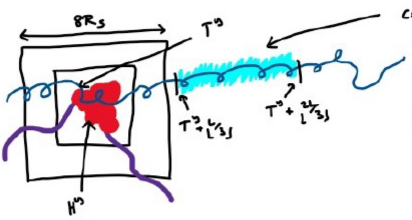
since by throwing away point from H we can also achieve the required upper bound.

Dirichlet energy bound

Observation: Small energy and big total measure

If μ is a measure on H and Σ denotes its total measure, then for any $K > 0$ we have

$$\mathbb{P}\left(\mathrm{cap}(H) < 2 \cdot \gamma \cdot R_b^2\right) \leq \mathbb{P}\left(\frac{\Sigma_\mu^2}{\mathcal{E}(\mu)} < 2 \cdot \gamma \cdot R_b^2\right) \leq \mathbb{P}\left(\mathcal{E}(\mu) \geq \frac{K}{2 \cdot \gamma \cdot R_b^2}\right) + \mathbb{P}\left(\Sigma_\mu \leq \sqrt{K}\right).$$



counting measure for these slices for a given $y \in \mathcal{D}$ where $H^y \neq \emptyset$: μ^y

$\mu := \sum_{y \in \mathcal{D}} \mu^y$ $\quad \Sigma_\mu[y] := \sum_{x \in \mathbb{Z}^d} \mu^y(x)$

$\Sigma_\mu := \sum_{x \in \mathbb{Z}^d} \mu(x)$

Introducing $\alpha := v \cdot \sum_{\ell = \underline{\Delta} \cdot R_s^2}^{\overline{\Delta} \cdot R_b^2} \ell^2 \cdot m(\ell)$, later we find $\Sigma_\mu \approx R_b^2 \cdot \gamma \cdot \alpha$.
Note that if Σ the total length of all worms that only visited one of the sets H^y $(y \in \mathcal{D})$ then we have $\Sigma/3 \leq \Sigma_\mu$.

Small energy: proof ideas

Lemma: Small energy

There exists $C < \infty$ such that if $\underline{\Delta} \geq 16$, then for any $K > 0$

$$\mathbb{P}\big(\mathcal{E}(\mu) \geq K\big) \leq \frac{C \cdot \big(1 + \psi \cdot \alpha\big) \cdot (R_b^2 \cdot \gamma \cdot \alpha)}{K}.$$

Due to the Markov inequality and the bilinearity of the Dirichlet energy we have

$$\mathbb{E}\left[\mathcal{E}(\mu)\right] = \mathbb{E}\left[\sum_{y,y' \in \mathcal{D}} \mathbb{E}\left[\mathcal{E}(\mu^y, \mu^{y'}) \,\middle|\, \mathcal{F}\right]\right].$$

For example, if $y = y'$ then on the event $H^y \neq \emptyset$ we have

$$\mathbb{E}\left[\mathcal{E}\left(\mu^y\right) \,\middle|\, \mathcal{F}\right] \leq v \cdot \sum_{\ell = \underline{\Delta} R_s^2}^{\overline{\Delta} R_b^2} m(\ell) \cdot \sum_{x \in \mathcal{G}} P_x\left(T_{H^y} \leq \lfloor \ell/3 \rfloor \mid \mathcal{F}\right) \cdot E_o\left[\sum_{t,t' = \lfloor \ell/3 \rfloor}^{2\lfloor \ell/3 \rfloor} g\left(X(t), X'(t')\right)\right]$$

$$\leq C \cdot v \cdot \sum_{\ell = \underline{\Delta} R_s^2}^{\overline{\Delta} R_b^2} m(\ell) \cdot (\ell \cdot \mathrm{cap}(H^y)) \cdot \ell \leq C \cdot \alpha \cdot (C' \cdot \gamma \cdot R_s^2)$$

and since the expected number of visited boxes of side length R_s inside a box of side length R_b is at most $C \cdot (R_b/R_s)^2$ this gives the upper bound for in this case.

Big total local time: the main obstacles

Lemma: Big total local time

There exist $c > 0$ and $C < \infty$ such that if $\overline{\Delta}$ small enough and $\underline{\Delta}$ is big enough, then for any K satisfying $c \cdot (R_b^2 \cdot \gamma \cdot \alpha) - 3K/2 > 0$ we have

$$\mathbb{P}\big(\Sigma \leq K\big) \leq \frac{C \cdot (R_b^2 \cdot \gamma \cdot \alpha) \cdot \psi}{K} + 0.02 + \frac{C \cdot R_b^2 \cdot (R_b^2 \cdot \gamma \cdot \alpha)}{\big(c \cdot (R_b^2 \cdot \gamma \cdot \alpha) - 3K/2\big)^2}$$

Remark: Some caution needed

(i) To guarantee positivity we need to show that the number of visited and good boxes of the smaller scale is big with high enough probability.

(ii) Since the considered worms can only hit one of the good sets, we need to guarantee that visited sets on the smaller scale is well spread-out.

(iii) Altough the summands $\Sigma[y]$ are independent given $\mathcal{F}$, the conditioning on this sigma-field might put us in a situation where H^y is surrounded with other sets of form $H^{\tilde{y}}$. The effect of such a "crowd" makes it hard for us to give a lower bound on the total length of worms that only hit H^y.

Big total local time: proof ideas

If Σ' and $\overline{\Sigma}$ denote the total local time of such worms that hit *at least one* good set and *multiple* good sets, respectively, then we have

$$\Sigma = \Sigma' - \overline{\Sigma}$$

form which it follows that for any $K > 0$:

$$\mathbb{P}\left(\Sigma \le K\right) \le \mathbb{P}\left(\overline{\Sigma} \ge K/2\right) + \mathbb{P}\left(\Sigma' \le 3K/2\right).$$

Here, to deal with the first term we can use Markov inequality and an upper bound on a quantity closely related to the Dirichlet energy of the counting measure on the centers of the sub-boxes visited by the random walk. This produces the first term of the lemma. Meanwhile the remaining two terms in the upper bound follows from using conditional Chebysev inequality and the following lemma.

Lemma: Lower bound on the number of good boxes visited by Z

There exists constant $c > 0$ such that if $\underline{\Delta}$ is big enough, then we have

$$\mathbb{P}\left(\text{ number of good boxes visited by } Z > c \cdot R_b^2/R_s^2\right) \ge 0.98.$$

It's proof builds on simple ideas:

(i) a lower bound on the number of smaller scale boxes visited by Z;

(ii) by the induction hypothesis of the γ-doubling lemma at least half of these boxes are good.