Metastability for the Curie-Weiss model on inhomogeneous random graphs: results and challenges

Saeda Marello

Institute for Applied Mathematics, University of Bonn

Based on ongoing joint work with Anton Bovier and Frank den Hollander

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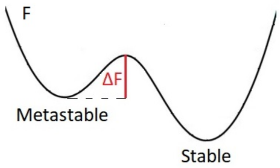
Metastability: quantities of interest

Given F free energy

1. Metastable parameter regime ($\to$ multiple minima of F)
2. Critical sets/points F:
   - local minima (metastable state), global minimum (stable state)
   - local maxima/saddle points
3. **Metastable exit time**: metastable $\to$ (more) stable.
   - Mean
   - Distribution

   In the limit $N \to \infty$ it is usually exponential $\sim \exp(N\Delta F)$.



Ferromagnetic Ising model on a lattice of size N

Spin flip model on a lattice of size N: N particles/spins.
Configurations $\sigma = (\sigma_i)_{1 \le i \le N} \in \{-1,1\}^N = \mathcal{S}_N$.
Energy/Hamiltonian:

$$H_N(\sigma) = -\frac{1}{2N} \sum_{i,j=1}^{N} J(i,j)\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \qquad \sigma \in \{-1,1\}^N$$

$J(i,j) \ge 0$ interaction/coupling coefficients, $h \ge 0$ external magnetic field.

We use **Glauber dynamics** with **Metropolis transition** rates and $\beta > 0$ inverse temperature

$$p_N(\sigma,\sigma') = \begin{cases} \exp(-\beta[H_N(\sigma') - H_N(\sigma)]_+) & \text{if } \sigma \sim \sigma' \text{ single spin flip,} \\ 0 & \text{otherwise.} \end{cases}$$

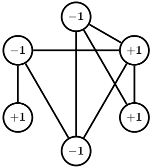
The Gibbs measure $\mu_N(\sigma) \propto \mathrm{e}^{-\beta H_N(\sigma)}$ is the **invariant** and **reversible measure**.

Curie–Weiss on graphs

$$H_N(\sigma) = -\frac{1}{2N}\sum_{i,j=1}^{N} J(i,j)\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \qquad \sigma \in \{-1,1\}^N$$

The **interaction graph** $G = (\mathcal{E}, [N])$: $(i,j) \in \mathcal{E} \iff J(i,j) \neq 0$.

When $J(i,j) \in \{0,1\}$, we can write

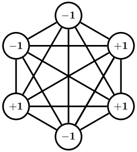


$$H_N(\sigma) = -\frac{1}{2N}\sum_{(i,j)\in\mathcal{E}} \sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i$$

$\implies$ *Curie-Weiss* **on** *the graph* $G = (\mathcal{E}, [N])$.

Models from the previous talk:

- Curie–Weiss model: $J(i,j) \equiv 1$
  $\implies$ complete graph
- Randomly dilute CW: $J(i,j) \sim Be(p)$ i.i.d.
  $\implies$ CW on the Erdős–Rényi random graph



Complete graph

Results for CW on the Erdős–Rényi random graph

From the previous talk

Theorem (Mean metastable exit time for CW on E-R, in [BMP21])

For $\beta > 1$, $h > 0$ small enough and for all $s > 0$,

$$\lim_{N \uparrow \infty} \mathbb{P}_J \left(C_1 \mathrm{e}^{-s} \leq \frac{\boxed{\mathbb{E}^{ER}_{\nu_{m_-,m_+}} \left[\tau_{\mathcal{S}_N[m_+(N)]} \right]}}{\mathbb{E}^{CW}_{m_-(N)} \left[\tau_{m_+(N)} \right]} \leq C_2 \mathrm{e}^{s} \right) \geq 1 - k_1 \mathrm{e}^{-k_2 s^2}.$$

where

$$\mathbb{E}(H_N^{ER}(\sigma)) = p H_N^{CW}(\sigma)$$

Ideas:

- compare $\mathrm{cap}(A,B)$ and $\sum_{\sigma' \in \mathcal{S}_N} \mu_N(\sigma')\, h_{AB}(\sigma')$ of the target model with its "mean" correspondents, using **concentration**.
- use potential-theoretic approach

$$\mathbb{E}_{\nu_{A,B}}[\tau_B] = \frac{1}{\mathrm{cap}(A,B)} \sum_{\sigma' \in \mathcal{S}_N} \mu_N(\sigma')\, h_{AB}(\sigma')$$

Conjecture for CW on inhomogeneous random graphs

Conjecture

In the metastable regime, let $J(i,j)$ be independently distributed Bernoulli, and A, B metastable sets for the "mean" model. Then for all $s > 0$,

$$\lim_{N\uparrow\infty} \mathbb{P}_J\left(C_1 \mathrm{e}^{-s} \leq \frac{\boxed{\mathbb{E}_{\nu_{A,B}}[\tau_B]}}{\mathbb{E}^{\mathbb{E}}_{\nu^{\mathbb{E}}_{A,B}}[\tau_B]} \leq C_2 \mathrm{e}^{s} \right) \geq 1 - k_1 \mathrm{e}^{-k_2 s^2}.$$

where

$$\mathbb{E}(H_N(\sigma)) = H_N^{\mathbb{E}}(\sigma)$$

Ideas:

- compare $\mathrm{cap}(A,B)$ and $\sum_{\sigma' \in \mathcal{S}_N} \mu_N(\sigma') h_{AB}(\sigma')$ of the target model with its "mean" correspondents, using **concentration**;
- use potential-theoretic approach

$$\mathbb{E}_{\nu_{A,B}}[\tau_B] = \frac{1}{\mathrm{cap}(A,B)} \sum_{\sigma' \in \mathcal{S}_N} \mu_N(\sigma') h_{AB}(\sigma').$$

Results for inhomogeneous random graphs

$$\text{Target quantity: } \mathbb{E}_{\nu_{A,B}}[\tau_B] = \frac{1}{\mathsf{cap}_N(A,B)} \sum_{\sigma' \in \mathcal{S}_N} \mu_N(\sigma') h_{AB}(\sigma')$$

where

$$\mathsf{cap}_N(A,B) = \sum_{x \in A} \mu_N(x) \mathbb{P}_x(\tau_B < \tau_A).$$

Theorem (Capacity estimates)

Assume $J(i,j) \sim Be(v_N(i,j))$ independent. Then, for any disjoint A, B and any $s > 0$

$$\lim_{N\to\infty} \mathbb{P}_J \left(c_1 \mathrm{e}^{-s} \leq \boxed{\frac{Z_N \, \mathit{cap}_N(A,B)}{Z_N^{\mathbb{E}} \, \mathit{cap}_N^{\mathbb{E}}(A,B)}} \leq c_2 \mathrm{e}^{s} \right) \geq 1 - k_1 \mathrm{e}^{-k_2 s^2}.$$

where c_1, c_2 are both $O(1)$, explicit and depend on $\beta, h, \sum_{i,j} \mathit{Var} J(i,j)$.

Idea of the **proof**: use Dirichlet and Thomson variational principles, together with **concentration**.

Challenge: estimates on $\sum_{\sigma' \in \mathcal{S}_N} \mu_N(\sigma') h_{AB}(\sigma')$

Concentration

Extension of the concentration results proven in [BMP21]

$$H_N(\sigma) = -\frac{1}{2N}\sum_{i,j=1}^{N} J(i,j)\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \qquad \sigma \in \{-1,1\}^N$$

with $J(i,j) \sim Be(v_N(i,j))$.

Theorem (Concentration)

Assume $(J(i,j))_{ij}$ are independent. Then there exist absolute constants $k_1, k_2 > 0$ such that, for any $g\colon \mathcal{S}_N \to [0,\infty)$ and any $s > 0$

$$\lim_{N\to\infty} \mathbb{P}_J\left(\mathrm{e}^{-s+\kappa(N)} \le \frac{\sum_{\sigma\in\mathcal{S}_N} g(\sigma)\mathrm{e}^{-\beta(H_N(\sigma)-\mathbb{E}[H_N(\sigma)])}}{\sum_{\sigma\in\mathcal{S}_N} g(\sigma)} \le \mathrm{e}^{s+\alpha(N)}\right) \ge 1 - k_1\mathrm{e}^{-k_2 s^2},$$

where $\quad \alpha(N) = \frac{\beta^2}{8N^2}\sum_{i,j=1}^{N} \mathit{Var}(J(i,j)) \quad$ *and* $\quad \kappa(N) = \alpha(N) - \varepsilon(\alpha(N)).$

Ideas of the proof

$$\lim_{N\to\infty} \mathbb{P}\left(\mathrm{e}^{-s+\kappa(N)} \le \frac{\sum_{\sigma\in\mathcal{S}_N} g(\sigma)\mathrm{e}^{-\beta(H_N(\sigma)-\mathbb{E}[H_N(\sigma)])}}{\sum_{\sigma\in\mathcal{S}_N} g(\sigma)} \le \mathrm{e}^{s+\alpha(N)} \right) \ge 1 - k_1\mathrm{e}^{-k_2 s^2},$$

is equivalent to

Target result

$$\mathrm{e}^{\mathcal{Y}_N}\mathrm{e}^{\kappa(N)} \sum_{\sigma\in\mathcal{S}_N} g(\sigma) \le \sum_{\sigma\in\mathcal{S}_N} g(\sigma)\,\mathrm{e}^{-\beta(H_N(\sigma)-\mathbb{E}[H_N(\sigma)])} \le \mathrm{e}^{\mathcal{Y}_N}\mathrm{e}^{\alpha(N)} \sum_{\sigma\in\mathcal{S}_N} g(\sigma)$$

with $\mathcal{Y}_N$ sub-Gaussian r.v. i.e. $\mathbb{P}_J\left(|\mathcal{Y}_N| \ge s\right) \le k_1 \exp\left(-k_2 s^2\right)$, for any $s > 0$.

Ideas of the proof

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$$\sum_{\sigma\in\mathcal{S}_N} g(\sigma)\mathrm{e}^{-\beta[H_N(\sigma)-\mathbb{E}[H_N(\sigma)]]}$$
$$\equiv \exp\left(N\mathcal{F}_N\right) = \exp\left(N(\mathcal{F}_N - \mathbb{E}\mathcal{F}_N)\right)\cdot\exp\left(N\mathbb{E}\mathcal{F}_N\right)$$

Ideas of the proof

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Step 1. Proof stochastic part

Proposition

$N(\mathcal{F}_N - \mathbb{E}\mathcal{F}_N)$ is a sub-Gaussian r.v., i.e. for any $s > 0$

$$\mathbb{P}_J\Big(N|\mathcal{F}_N - \mathbb{E}\mathcal{F}_N| \geq s\Big) \leq k_1 \exp\Big(-\frac{8k}{\beta^2}s^2\Big).$$

Proof: $$N\mathcal{F}_N = \log \sum_{\sigma\in\mathcal{S}_N} g(\sigma)\mathrm{e}^{-\beta[H_N(\sigma)-\mathbb{E}[H_N(\sigma)]]}$$

$$= \log \sum_{\sigma\in\mathcal{S}_N} g(\sigma)\exp\left[\tfrac{\beta}{2N}\sum_{i,j=1}^{N}[J(i,j)-\mathbb{E}J(i,j)]\sigma_i\sigma_j\right]$$

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Proof: $$N\mathcal{F}_N = \log \sum_{\sigma \in \mathcal{S}_N} g(\sigma) \exp\Big[\tfrac{\beta}{2N}\sum_{i,j=1}^{N}[J(i,j) - \mathbb{E}J(i,j)]\sigma_i\sigma_j\Big]$$

Talagrand's concentration inequality

$$\mathbb{P}\Big(|G(g) - \mathbb{E}G(g)| \geq tK\Big) \leq k_1 \exp\big(-k_2 t^2\big),$$

for $G : \mathbb{R}^n \to \mathbb{R}$ convex and 1-Lipschitz, and $g = (g_i)_{i\in[n]}$ **independent** r.v.'s uniformly bounded by $K > 0$.

with $G = \frac{2\sqrt{2}}{\beta}N\mathcal{F}_N$ and the r.v.'s $[J(i,j) - \mathbb{E}J(i,j)]_{ij}$.

Step 2. Proof deterministic part

Proposition

$$e^{\kappa(N)} \sum_{\sigma \in \mathcal{S}_N} g(\sigma)(1+o(1)) \leq \exp\left(N \mathbb{E}_J \mathcal{F}_N\right) \leq e^{\alpha(N)} \sum_{\sigma \in \mathcal{S}_N} g(\sigma)(1+o(1))$$

Lower bound:

- $\boxed{\mathbb{E}_J[\exp(N\,\mathcal{F}_N)]}$
- $\mathbb{E}_J[\exp(2N\,\mathcal{F}_N)] \leq k\mathbb{E}_J^2[\exp(N\,\mathcal{F}_N)]$
- Talagrand's concentration inequality

Upper bound:

- $\boxed{\mathbb{E}_J[\exp(N\,\mathcal{F}_N)]}$
- Jensen's inequality

Step 2. Proof deterministic part: upper bound

Upper bound

$$\exp\left(N\mathbb{E}_J\mathcal{F}_N\right) \leq \boxed{\mathbb{E}_J[\exp(N\,\mathcal{F}_N)] = \mathrm{e}^{\alpha(N)} \sum_{\sigma\in\mathcal{S}_N} g(\sigma)(1+o(1))}$$

Step 2. Proof deterministic part: upper bound

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$$\exp(N\mathcal{F}_N) = \sum_{\sigma\in\mathcal{S}_N} g(\sigma) \prod_{i,j=1}^{N} \exp\left[\tfrac{\beta}{2N}[J(i,j) - \mathbb{E}J(i,j)]\sigma_i\sigma_j\right]$$

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$$\begin{aligned}\mathbb{E}_J \exp\left[\tfrac{\beta}{2N}[J(i,j)-\mathbb{E}J(i,j)]\sigma_i\sigma_j\right] &= 1+\frac{1}{2}\left(\tfrac{\beta}{2N}\sigma_i\sigma_j\right)^2 \mathsf{Var}[J(i,j)] + o\left(\tfrac{1}{N^2}\right) \\ &= \exp\left[\tfrac{\beta^2}{8N^2}\mathsf{Var}[J(i,j)] + o\left(\tfrac{1}{N^2}\right)\right]\end{aligned}$$

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$$\begin{aligned}\Longrightarrow\ \mathbb{E}_J[\exp(N\,\mathcal{F}_N)] &= \sum_{\sigma\in\mathcal{S}_N} g(\sigma)\,\mathrm{e}^{\frac{\beta^2}{8N^2}\sum_{i,j=1}^{N}\mathsf{Var}(J(i,j))}(1+o(1))\\ &= \mathrm{e}^{\alpha(N)} \sum_{\sigma\in\mathcal{S}_N} g(\sigma)\,(1+o(1))\end{aligned}$$

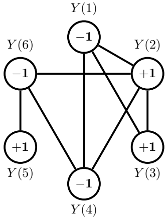
An inhomogeneous random graph: the Chung-Lu r. g.

Chung-Lu random graph

$G = (\mathcal{E}, \{1, 2, \ldots, N\})$

Assign random weights on vertices $Y \colon \{1, 2, \ldots, N\} \to [0, 1]$.

$$\mathbb{P}\left((i, j) \in \mathcal{E}\right) = Y(i)Y(j), \quad i, j \in \{1, 2, \ldots, N\}$$



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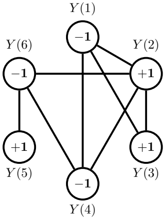

$$\mathbb{P}\left((i,j) \in \mathcal{E}\right) = Y(i)Y(j), \quad i, j \in \{1, 2, \dots, N\}$$

Hamiltonian of **CW on the Chung-Lu random graph**

$$H_N(\sigma) = -\frac{1}{2N} \sum_{i,j=1}^{N} J(i,j)\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \qquad \sigma \in \{-1, 1\}^N$$

with $J(i,j) \sim \mathrm{Be}(Y(i)Y(j))$.

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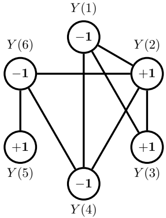

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with $J(i,j) \sim \text{Be}(Y(i)Y(j))$.

The mean w.r.t. J is

$$\mathbb{E}(H_N(\sigma)) = -\frac{1}{2N} \sum_{i,j=1}^{N} Y(i)Y(j)\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \qquad \sigma \in \{-1, 1\}^N$$

It is the Hamiltonian of **CW with disorder**.

Recap: Curie–Weiss model and stochastic modifications

$$H_N(\sigma) = -\frac{1}{2N}\sum_{i,j=1}^{N} J(i,j)\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \qquad \sigma \in \{-1,1\}^N$$

Curie-Weiss model (CW) [1]
$J(i,j) \equiv 1$ complete graph

Randomly Dilute CW (RDCW) [2]
$J(i,j) \sim \text{Be}(p)$ iid Erdős–Rényi random graph

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CW on the Chung–Lu r.g. [3]
$J(i,j) \sim$ Be$(Y(i)Y(j))$, $Y(i) \geq 0$ iid r.v. Chung–Lu random graph

Recap: Curie–Weiss model and stochastic modifications

$$H_N(\sigma) = -\frac{1}{2N}\sum_{i,j=1}^{N} J(i,j)\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \qquad \sigma \in \{-1,1\}^N$$

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Randomly Dilute CW (RDCW) [2]
$J(i,j) \sim$ Be(p) iid — Erdős–Rényi random graph

CW with disorder [2]
$J(i,j) = Y(i)Y(j)$, $Y(i) \geq 0$ iid r.v.

CW on the Chung–Lu r.g. [3]
$J(i,j) \sim$ Be$(Y(i)Y(j))$, $Y(i) \geq 0$ iid r.v. — Chung–Lu random graph

Model reduction in CW-Dis: coarse graining and magnetisation

Coarse-graining in CW-Dis with finite support $\{a_1, \ldots, a_k\}$ of $Y(i)$.
Partition: $A_\ell = \{i \in \{1, \ldots, N\} : Y(i) = a_\ell\}$, $\ell \in \{1, \ldots, k\}$.

Model reduction in CW-Dis: coarse graining and magnetisation

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Partition: $A_\ell = \{i \in \{1, \ldots, N\} : Y(i) = a_\ell\}$, $\ell \in \{1, \ldots, k\}$. Thus $Y(i) = \sum_{\ell=1}^{k} a_\ell \mathbb{1}(i \in A_\ell)$. Then

$$
\begin{aligned}
H_N(\sigma) &= -\frac{N}{2}\left(\frac{1}{N}\sum_{i=1}^{N} Y(i)\sigma_i\right)^2 - h\sum_{i=1}^{N}\sigma_i, \qquad \sigma \in \{-1,1\}^N \\
&= -\frac{N}{2}\left(\frac{1}{N}\sum_{\ell=1}^{k} a_\ell \sum_{i\in A_\ell}\sigma_i\right)^2 - h\sum_{\ell=1}^{k}\sum_{i\in A_\ell}\sigma_i
\end{aligned}
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Magnetisation $m : \{-1, 1\}^N \to M_N \subset [-1, 1]^k$, $m = (m_1, \ldots, m_k)$,

$$\boxed{m_\ell(\sigma) = \frac{1}{|A_\ell|}\sum_{i \in A_\ell} \sigma_i,} \qquad \ell \in \{1, \ldots, k\}, \sigma \in \{-1, 1\}^N.$$

$$H_N(\sigma) = -N\left[\frac{1}{2}\left(\sum_{\ell=1}^{k} a_\ell \frac{|A_\ell|}{N} m_\ell(\sigma)\right)^2 + h \sum_{\ell=1}^{k} \frac{|A_\ell|}{N} m_\ell(\sigma)\right] = N E_N(m(\sigma))$$

Well-known results in CW

$J(i,j) \equiv 1$

Theorem (Metastability)

- **Metastable regime**

$$\beta \in (1,\infty), \qquad h \in [0, \bar{h}_c(\beta)).$$

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$$\mathbb{E}_\sigma \left[\tau_{\mathcal{S}_N[\mathbf{m}_+]} \right] = [1+o(1)] \sqrt{\frac{1-\mathbf{m}^{*2}}{1-\mathbf{m}_-^2}} \frac{\mathrm{e}^{\beta N [F_{\beta,h}(\mathbf{m}^*) - F_{\beta,h}(\mathbf{m}_-)]}}{\sqrt{F''_{\beta,h}(\mathbf{m})[-F''_{\beta,h}(\mathbf{m}^*)]}} \frac{\pi}{\beta(1-\mathbf{m}^*)}.$$

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Results in CW with finite disorder

$J(i,j) = Y(i)Y(j)$, $Y(i) \geq 0$ iid r.v. with **finite support** $\{a_1, \ldots, a_k\}$ and $\omega_\ell = \mathbb{P}(Y(1) = a_\ell)$.

Theorem (Metastable regime and critical points)

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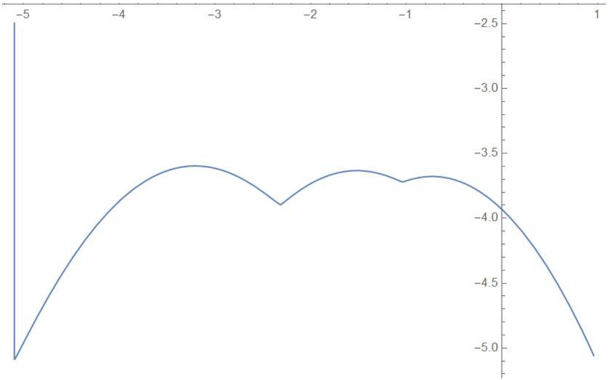
$$m_\ell = \tanh\left(\beta\left[a_\ell\left(\sum_{\ell'=1}^{k} a_{\ell'} \, \omega_{\ell'} \, m_{\ell'}\right) + h\right]\right), \qquad \ell \in \{1, \ldots, k\}.$$

Results in CW with finite disorder: example critical points

Example of free energy with more than three critical points



Energy functional. 1-dim. example: k=4, 7 critical points

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- **Randomness of the exponent** *(in distribution)*

$$N[F_N(\mathbf{m}_N^*) - F_N(\mathbf{m}_N)] = N[F_{\beta,h}(\mathbf{m}^*) - F_{\beta,h}(\mathbf{m})] + Z\sqrt{N} + O(1),$$

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[*A. Bovier, F. den Hollander and S. M., "Metastability for Glauber dynamics on the complete graph with coupling disorder", 2021. arXiv:2107.04543*]

Techniques: Potential-theoretic approach

Lumpable: $H_N(\sigma) = E_N(m(\sigma))$.

Magnetisation: $m = (m_1, \dots, m_k)$, $\quad m_\ell(\sigma) = \frac{1}{|A_\ell|} \sum_{i \in A_\ell} \sigma_i$.

$\forall \sigma \in m^{-1}(\mathbf{m})$,

$$\mathbb{E}_\sigma[\tau_{m^{-1}(\mathcal{B})}] = \boxed{\mathbb{E}^*_{m(\sigma)}[\tau^*_{\mathcal{B}}] = \frac{\tilde{\mu}^*_N(m(\sigma))}{\mathsf{cap}^*(m(\sigma), \mathcal{B})}}$$

Same technique for any lumpable model (e.g. CW, finite RFCW, ...).

Challenge for not lumpable models (infinite RFCW, RDCW, ...): we can use only

$$\mathbb{E}_{\nu_{A,B}}[\tau_B] = \frac{1}{\mathsf{cap}(A, B)} \sum_{\sigma' \in \mathcal{S}_N} \mu_N(\sigma') h_{AB}(\sigma').$$

Summary and outlook

Summary of results:

- Extension of some results for CW on E-R r.g. to general inhomogeneous r.g.
- Focus on CW on the Chung-Lu r.g. and detailed study on its mean, the CW with (finite) disorder

Outlook of next challenges:

- Conclude the proof of our conjecture
- Find other $J(i, j)$ s.t. the mean model is sufficiently solvable (problems with randomness on edges)

Thanks for your attention!

References

- RDCW:
  - A. Bovier and V. Gayrard. *The thermodynamics of the Curie–Weiss model with random couplings*, Journal of Statistical Physics, 72(3-4), 1993 [BG];
  - F. den Hollander and O. Jovanovski. *Glauber dynamics on the Erdős-Rényi random graph*. In and Out of Equilibrium 3, Celebrating Vladas Sidoravicius. Progress in Probability 77, Birkhäuser, (2021), pp. 519–589;
  - A. Bovier, S. M., and E. Pulvirenti. *Metastability for the dilute Curie-Weiss model with Glauber dynamics*, Electronic Journal of Probability, 26:1-38, 2021 [BMP21];
- **CW with disorder**:
  A. Bovier, F. den Hollander and S. M., *Metastability for Glauber dynamics on the complete graph with coupling disorder*, arXiv:2107.04543, 2021 [BdHM];
- RFCW:
  - A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. *Metastability in stochastic dynamics of disordered mean-field models*, Probab. Theory Related Fields 119, 2001 [BEGK];
  - A. Bianchi, A. Bovier, and D. Ioffe. *Sharp asymptotics for metastability in the random field Curie-Weiss model*, Electron. J. Probab. 14 (2009),no 53 [BBI09];
- Monograph:
  A. Bovier and F. den Hollander, *Metastability – A Potential-Theoretic Approach*, Grundlehren der mathematischen Wissenschaften 351, Springer, 2015 [BdH15].