Particle Systems, Conservation Laws, Weak solutions and Entropy

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- $$\eta = \{\eta(x) : \ x \in \mathbf{Z}\} \qquad \eta(x) = 0 \text{ or } 1$$

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- $F(t, \eta) = E[F(\eta_t)|\eta_0 = \eta]$ is obtained by solving

$$\frac{d}{dt}E[F(t, \eta)] = (\mathcal{A}F)(t, \eta);\ F(0, \eta) = F(\eta)$$

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  exists.
- One can have x vary over either $\mathbf{Z}$ or $\mathbf{Z}_N$ leading to the range of u being $\mathcal{R}$ or $\mathcal{T}$.

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- Time is Nt because we assume $m = \sum z\pi(z) > 0$.

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- $$\frac{\partial\rho(t,u)}{\partial t} + \frac{\partial[m\rho(t,u)(1-\rho(t,u))]}{\partial u} = 0$$

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- Which is the real solution?

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- Rezakhanlou has proved convergence to such a limit in more general situations.

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- Large deviation rate function.

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- Entropy inequality for h implies entropy inequality for any convex H

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- Connections with growth processes, Random Matrices, Tracy-Widom distribution and many other exactly solvable models are known.
- There are some natural questions one can ask for which the answers are not known.

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- What if they want to go on opposite directions.

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- Is there some kind of phase transition? With exact computation not being possible, what techniques can we use?
- Important issue is the invariant measure or the stationary distribution.
- Does a nontrivial one exist and is it unique?

Thank You