# Annealed random walk conditioned on survival among Bernoulli obstacles

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Joint works with Jian Ding, Rongfeng Sun and Changji Xu.

# Model

- $(S := (S_n)_{n\geq 0}, \mathbf{P}_x)$: SRW on $\mathbb{Z}^d$ starting at $x \in \mathbb{Z}^d$ $(d \geq 2)$;
- $(\omega = (\omega_x)_{x\in\mathbb{Z}^d}, \mathbb{P})$: Independent Ber(p) random variables.

Let $\mathcal{O} = \{x \in \mathbb{Z}^d \colon \omega_x = 0\}$. The random walk is killed upon hitting $\mathcal{O}$:

$$\tau_{\mathcal{O}} := \inf\{n \geq 0 : S_n \in \mathcal{O}\}.$$

The question is how $S$ (and $\mathcal{O}$) behaves conditioned on $\{\tau_{\mathcal{O}} > N\}$, i.e., under the measure

$$\mu_N((S,\mathcal{O}) \in \cdot) := \mathbb{P} \otimes \mathbf{P}((S,\mathcal{O}) \in \cdot \mid \tau_{\mathcal{O}} > N).$$

This is called the *annealed law* since the average is taken over the environment. (Called *quenched* if environment is fixed.)

# In other contexts

1. The killed random walk is generated by the random Schrödinger operator $-\frac{1}{2d}\Delta + \infty \cdot 1_{\mathcal{O}}$. Asymptotics of $\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N)$ is related to the Anderson localization through the so-called *Lifshiz tail*. The localization for 2D/3D discrete Bernoulli Anderson model (with finite coupling strength) has very recently proved by Ding–Smart/Li–Zhang.

2. One can integrate out the $\mathcal{O}$-marginal since $\tau_{\mathcal{O}} > N$ is equivalent to $\mathcal{O} \cap S_{[0,N]} = \emptyset$. Then $S$-marginal is

$$\mu_N(S \in \cdot) = \frac{\mathbf{E}\Big[p^{|S_{[0,N]}|} \colon S \in \cdot\Big]}{\mathbf{E}\Big[p^{|S_{[0,N]}|}\Big]}.$$

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$$\mu_N(S \in \cdot) = \frac{\mathbf{E}\left[e^{-\nu |S_{[0,N]}|} : S \in \cdot\right]}{\mathbf{E}\left[e^{-\nu |S_{[0,N]}|}\right]}.$$

This can be regarded as a *self-attractive polymer* model.

# Part 1: Geometry of the range

# Earlier works 1: partition function

Recall:
$$\mu_N((S,\mathcal{O})\in\cdot) := \mathbb{P}\otimes\mathbf{P}((S,\mathcal{O})\in\cdot \mid \tau_{\mathcal{O}}>N) = \frac{\mathbb{P}\otimes\mathbf{P}((S,\mathcal{O})\in\cdot, \tau_{\mathcal{O}}>N)}{\mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}}>N)}.$$

Theorem (Donsker–Varadhan (1979))

*For $d\geq 2$,*

$$\mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}}>N) = \exp\Big\{-c_{DV}N^{\frac{d}{d+2}}(1+o(1))\Big\},$$
$$\textit{with } c_{DV} = \inf_U\{|U|\log(1/p)+\lambda(U)\},$$

*where $\lambda(U)$ is the principal Dirichlet eigenvalue of $-\frac{1}{2d}\Delta$ in $U$.*

Remark

Due to the Faber–Krahn isoperimetric inequality, the above infimum is achieved by a ball $B(x;\varrho_1)$ for some $\varrho_1(d,p)>0$.

# Earlier works 1: partition function

The proof roughly goes as follows:

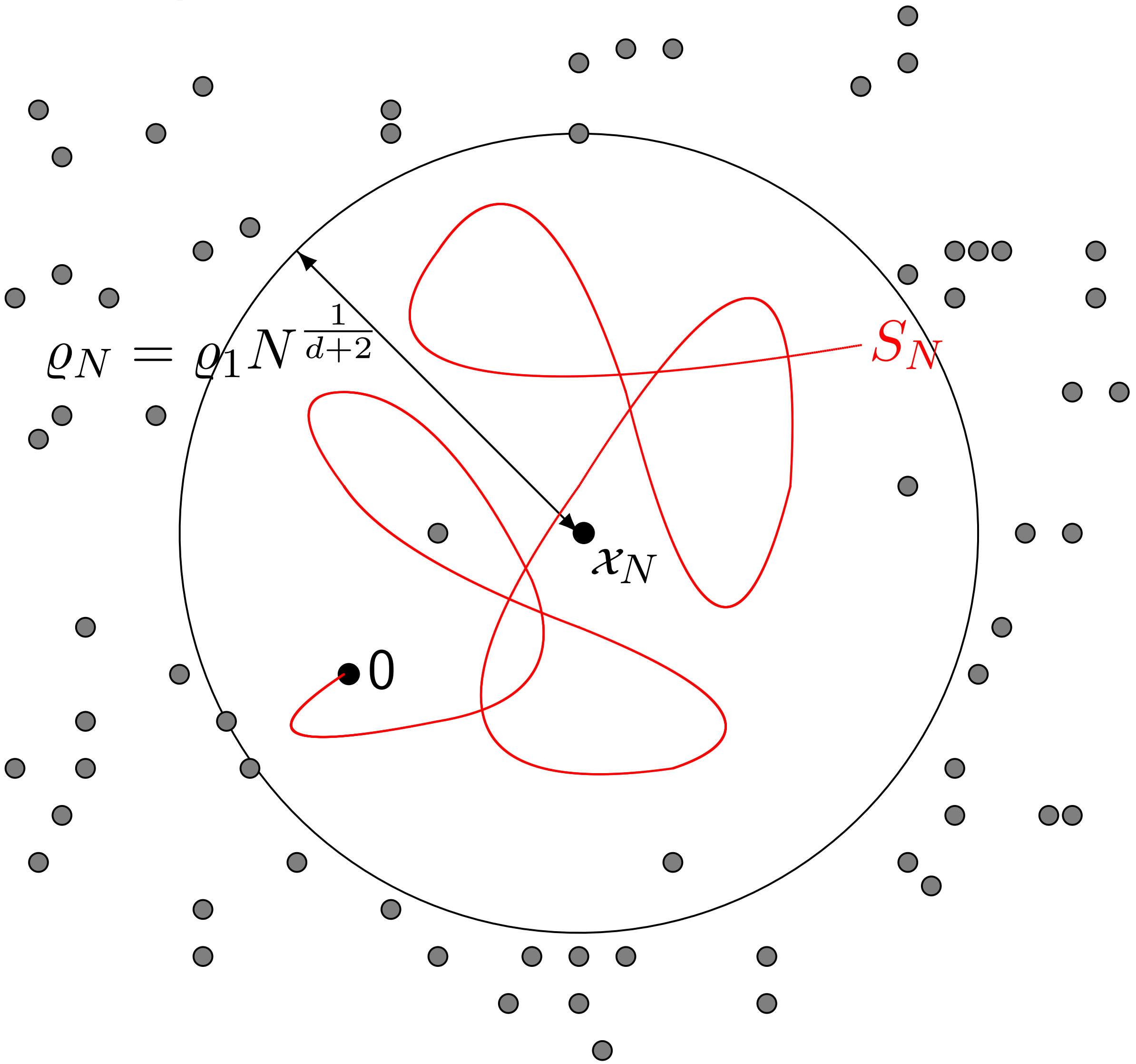
$$\begin{aligned}\mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}}>N)&=\sum_U\mathbb{P}(\mathcal{O}\cap U=\emptyset)\mathbf{P}(S_{[0,N]}=U)\\&\approx\max_U p^{|U|}\exp\{-N\lambda(U)\}\\&=\exp\left\{-N^{\frac{d}{d+2}}\inf_U\{|U|\log(1/p)+\lambda(U)\}\right\}.\end{aligned}$$

The second line is a kind of Laplace principle.

- Donsker–Varadhan proved it by the large deviation principle,
- Antal (1995) gave another proof by Sznitman's "method of enlargement of obstacles".

Anyway, this "indicates" that the best strategy —to stay in a ball of radius $\varrho_N=\varrho_1 N^{\frac{1}{d+2}}$— dominates others.

# Earlier works 1: partition function




This picture is a bit misleading since almost all the sites should be visited $N/N^{\frac{d}{d+2}} = N^{\frac{2}{d+2}}$ times.

# Earlier works 2: confinement property

This "indication" has been made rigorous by Sznitman (1991, $d = 2$), Bolthausen (1994, $d = 2$) and Povel (1999, $d \geq 3$) in the following stronger form:

## Theorem (Confinement property)

*For any $d \geq 2$, there exists $x_N = x_N(\mathcal{O}) \in B(0; \varrho_N)$ such that for any $\varepsilon > 0$,*

$$\lim_{N\to\infty} \mu_N\big(S_{[0,N]} \subset B(x_N; (1+\varepsilon)\varrho_N)\big) = 1.$$

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Why "stronger"? Because the large deviation principle only tells us that the random walk spends most of the time in a ball $B(x; \varrho_N)$.

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### Remark

Why "stronger"? Because the large deviation principle only tells us that the random walk spends most of the time in a ball $B(x; \varrho_N)$. See Bolthausen's LNM 1781 for more detail, where he wrote "it is not clear if one really should believe in this confinement (and had in fact been doubted by experts in the beginning)."

## Earlier works 2.5: clearing/covering ball for $d = 2$

In dimension two, a bit more have been known.

**Proposition (Ball clearing: Sznitman (1991))**

*Let $d = 2$. Then for any $\varepsilon > 0$,*

$$\lim_{N\to\infty} \mu_N(\mathcal{O} \cap B(x_N; (1-\varepsilon)\varrho_N) = \emptyset) = 1.$$

**Proposition (Ball covering: Bolthausen (1994))**

*Let $d = 2$. Then for any $\varepsilon > 0$,*

$$\lim_{N\to\infty} \mu_N\big(B(x_N; (1-\varepsilon)\varrho_N) \subset S_{[0,N]}\big) = 1.$$

Bolthausen used this in his proof of the confinement property and he conjectured that this remains true for $d \geq 3$.

# Main result 1: ball covering in $d \geq 3$

Theorem (Ball covering: Ding–F.–Sun–Xu (2020))

*Let $d \geq 2$. Then for any $\varepsilon > 0$,*

$$\lim_{N \to \infty} \mu_N \big( B(x_N; (1-\varepsilon)\varrho_N) \subset S_{[0,N]} \big) = 1.$$

Remark

This confirms Bolthausen's conjecture in 1994. However, our proof relies on the confinement property and hence does not give a way to extend Bolthausen's proof of confinement to $d \geq 3$. Recently, Berestycki–Cerf announced a proof of the ball covering without assuming the confinement.

# Main result 2: boundary size

The confinement property and the ball covering theorem together imply

$$\partial S_{[0,N]} \subset B(x_N; (1+\varepsilon)\varrho_N) \setminus B(x_N; (1-\varepsilon)\varrho_N).$$

The following theorem is a step toward understanding the surface fluctuation:

Theorem (Boundary size: Ding–F.–Sun–Xu (2020))

*Let $d \geq 2$. Then there exists $\alpha > 0$ such that*

$$\lim_{N\to\infty} \mu_N\big(|\partial S_{[0,N]}| \leq \varrho_N^{d-1}(\log \varrho_N)^{\alpha}\big) = 1.$$

# Part 2: A model with bias

# Earlier works 3: Ballisticity transition

Consider a model with bias $h \in \mathbb{R}^d$:

$$\mu_N^h((S,\mathcal{O}) \in \cdot) := \frac{\mathbb{E} \otimes \mathbf{E}\big[e^{\langle h, S_N \rangle} : \tau_{\mathcal{O}} > N, (S,\mathcal{O}) \in \cdot\big]}{\mathbb{E} \otimes \mathbf{E}\big[e^{\langle h, S_N \rangle} : \tau_{\mathcal{O}} > N\big]}.$$

Grassberger–Procaccia (1982) predicted that this model undergoes a ballisticity transition. Later Eisele–Lang (1987) proved a phase transition at the level of partition function.

## Theorem (Sznitman (1995))

*Let $d \geq 2$. There exists a compact set $K$ such that*

$$\lim_{N \to \infty} \mu_N^h(|S_N| = o(N)) = \begin{cases} 1 & \text{if } h \in K^\circ, \\ 0 & \text{if } h \notin K. \end{cases}$$

*When $h \in K^\circ$, in fact $o(N)$ can be improved to $o(N^{\frac{d}{d+2}})$.*

# Earlier works 3: Ballisticity transition

To describe the criticality, we need the so-called Lyapunov exponent:

$$\beta(x) := -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > \tau_{nx}).$$

(Existence follows from an easy sub-additivity argument.)
This measures a cost for the RW to make a long crossing among the obstacles.

On the other hand, for the biased model, there is a gain of $e^{\langle h, nx\rangle}$ from the above long crossing. Therefore

- $\langle h, x\rangle < \beta(x)$ for all $x$ (cost beats gain) $\Rightarrow$ sub-ballistic,
- $\langle h, x\rangle > \beta(x)$ for some $x$ (gain beats cost) $\Rightarrow$ ballistic.

# Earlier works 4: Ballistic phase

Recently, the understanding of the ballistic phase has been improved a lot.

**Theorem (Ioffe–Velenik (2013))**

*The random walk is ballistic under $\mu_N^h$ for $h \in \partial K$. Moreover, in the whole ballistic phase,*

1. *the random walk has an asymptorics speed,*
2. *the transversal fluctuation converges to a Gaussian law.*

These are based on an intricate renormalization analysis in the spirit of "regeneration time" technique for RWRE. But it is harder here since at criticality (cost=gain), there is no a priori condition that create an effective bias.

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Sad news: On October 1st, 2020, Dmitry Ioffe passed away...

# Main result 3: Sub-ballistic phase

The last result is about the sub-ballistic phase.

Theorem (Ding–F.–Sun–Xu (2020))

*Let $h \in K^\circ$. Then for any $\varepsilon > 0$, with $\mu_N^h$-high probability:*

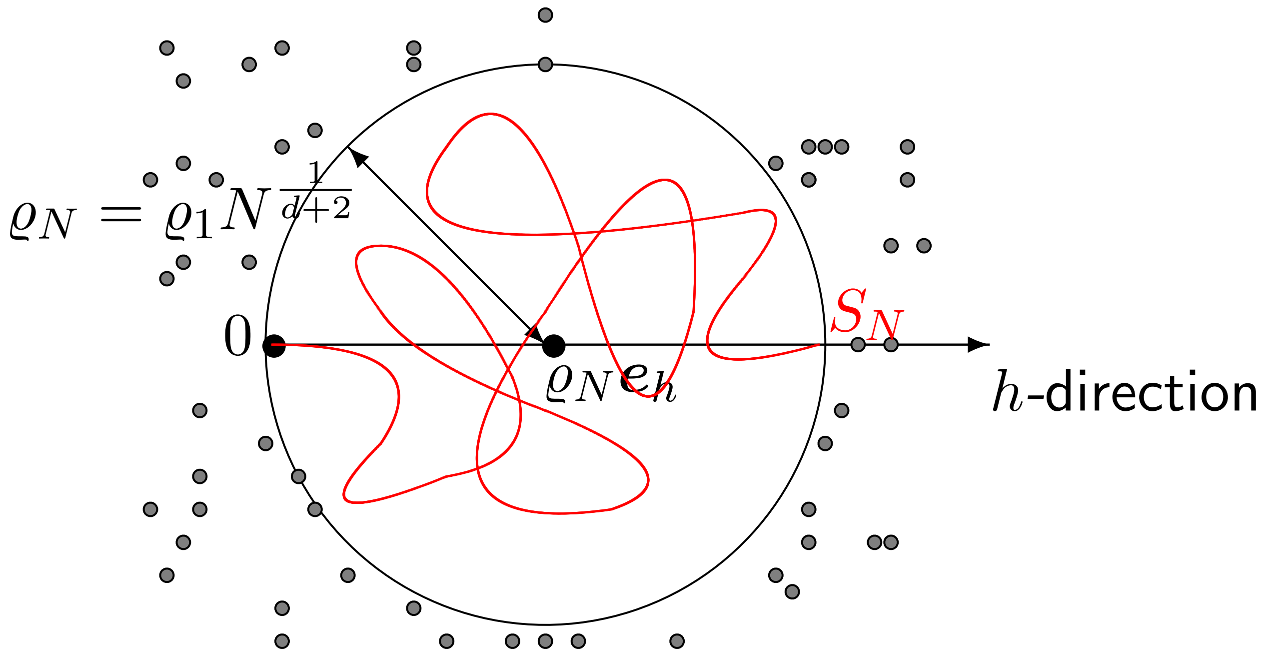
$$B(\varrho_N \boldsymbol{e}_h; (1-\varepsilon)\varrho_N) \subset S_{[0,N]} \subset B(\varrho_N \boldsymbol{e}_h; (1+\varepsilon)\varrho_N),$$
$$S_N \in B(2\varrho_N \boldsymbol{e}_h; \varepsilon\varrho_N),$$

*where $\boldsymbol{e}_h := h/|h|$.*

# Schematic figure in the sub-ballistic phase

## Theorem (Ding–F.–Sun–Xu (2020))

*Let $h \in K^\circ$. Then, with $\mu_N^h$-high probability:*




The center $x_N$ and the endpoint $S_N$ are shifted toward $h$ to maximize the gain $e^{\langle h, S_N \rangle}$, but otherwise the same as unbiased. This improves Sznitman's $o(N^{\frac{d}{d+2}})$ result to $O(N^{\frac{1}{d+2}})$. In the proof, a "geometry of the range" result plays a key role.

# The end of first talk

# Contents of second talk

In this second talk, I explain some of the proof elements. There are three contents:

1. Confinement: Explain a common difficulty.
2. Ball covering: Focus on the "comparison" techniques.
3. Confinement with bias: how to use ball covering.

I will not tell you the proof outline of the "boundary size" result.
It requires a bit complicated notation and good familiarity with the "comparison" techniques.

# Proof Outline for Confinement

# Difficulty

Recall the "confinement property" ($\varrho_N \asymp N^{\frac{1}{d+2}}$):

$$\lim_{N\to\infty} \mu_N\big(S_{[0,N]} \subset B(x_N; (1+\varepsilon)\varrho_N)\big) = 1.$$

According to Bolthausen, "it is not clear if one really should believe in this confinement...". Why?

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According to Bolthausen, “it is not clear if one really should believe in this confinement...”. Why?

The problem is that there are two notions of “system size”.

1. Volume: the partition function asymptotics is

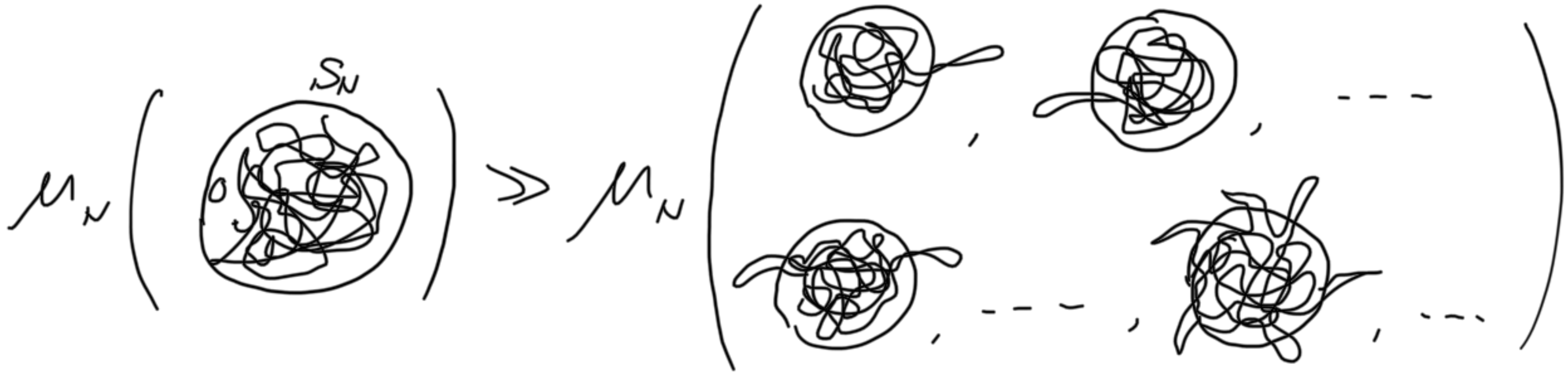
$$\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N) = \exp\Big\{-c_{\mathsf{DV}} N^{\frac{d}{d+2}} + o(N^{\frac{d}{d+2}})\Big\}.$$

2. Length: RW can make an excursion of length $N^{\frac{1}{d+2}}$ with probability $\approx e^{-N^{\frac{1}{d+2}}}$. (Make an obstacle free corridor and follow it.)

# Difficulty

It is certainly clear that


but it is not clear if one really should believe in




The error $o(N^{\frac{d}{d+2}})$ in Donsker–Varadhan's result is not small in the "length scale" $(d \neq 1)$. Need a new idea for the confinement.

# Idea of Sznitman and Povel's proof

Roughly speaking, Sznitman and Povel's proof relies on analytical properties of Schrödinger semigroup and strong Markov.

Key elements are

1. find a sharp but non-explicit (spectral-probabilistic) lower bound for $\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N)$,
2. use "method of enlargement of obstacles" to show that one cannot expect more than a single vacant ball of size $\varrho_N$.
   $\longrightarrow$ Outside the ball is dangerous (in a spec. th. sense).
3. use strong Markov to extract the cost of the first excursion away from the ball.

The special probabilistic form of the lower bound in 1 allows them to compare it with the upper bound in 3 for the probability of having an excursion.

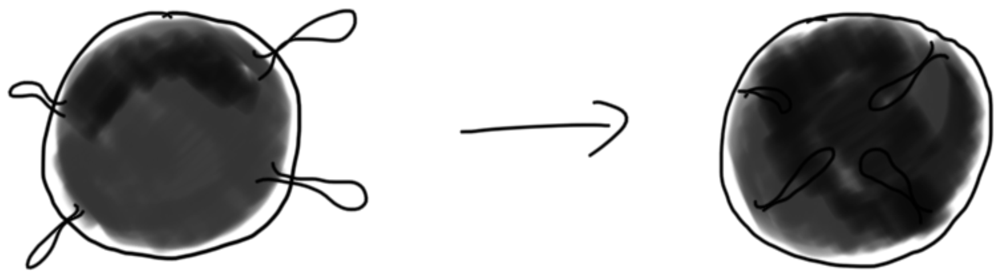
# Idea of Bolthausen's proof

He obeyed *self-attractive polymer* $\mu_N(S \in \cdot) = \frac{\mathbf{E}\left[e^{-\nu|S_{[0,N]}|} : S \in \cdot\right]}{\mathbf{E}\left[e^{-\nu|S_{[0,N]}|}\right]}$ interpretation. The argument roughly goes as follows:

1. first prove the ball covering:

$$\lim_{N \to \infty} \mu_N\big(B(x_N; (1-\varepsilon)\varrho_N) \subset S_{[0,N]}\big) = 1.$$

2. starting from a path with many outgoing excursions, perform a "folding" operation that maps the path to another "confined" one. This reduces the energy $\nu|S_{[0,N]}|$ a lot.



Of course there is a problem of "entropy loss" but he managed to prove that the gain beats the loss.

# Proof Idea for Ball Covering

# Proof idea for weak version of ball covering

Our proof heavily relies on comparison arguments. The following lemma gives an illustrative example:

## Lemma (clearing implies covering)

*Suppose* $\mu_N(\mathcal{O} \cap B(x_N; (1-\varepsilon)\varrho_N) = \emptyset) = 1 - o(\varrho_N^{-d})$.

*Then,* $\lim_{N\to\infty} \mu_N\big(B(x_N; (1-\varepsilon)\varrho_N) \subset S_{[0,N]}\big) = 1$.

## Proof.

Suppose $\mu_N(\exists x \in B(x_N; (1-\varepsilon)\varrho_N) \setminus S_{[0,N]}) \geq c > 0$. Then there is a point $x$ such that

$$\mu_N(x \in B(x_N; (1-\varepsilon)\varrho_N) \setminus S_{[0,N]}) \geq c\varrho_N^{-d}.$$

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$$\mu_N(x \in B(x_N; (1-\varepsilon)\varrho_N) \setminus S_{[0,N]}) \geq c\varrho_N^{-d}.$$

But the left-hand side is bounded by

$$\frac{1}{1-p}\mu_N(x \in B(x_N; (1-\varepsilon)\varrho_N) \setminus S_{[0,N]} \text{ and } x \in \mathcal{O})$$

and this contradicts the assumption. $\square$

# Proof idea for ball clearing

<u>To show</u>: $\lim_{N\to\infty} \mu_N(\mathcal{O} \cap B(\varkappa_N; (1-\varepsilon)\varrho_N) = \emptyset) = 1.$

Suppose $x \in \mathcal{O} \cap B(\varkappa_N; (1-\varepsilon)\varrho_N)$. Then, either

1. $B(x; \varepsilon\varrho_N/2)$ contains a large density of obstacles or
2. $B(x; \varepsilon\varrho_N/2)$ contains a small density of obstacles.

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1. $B(x; \varepsilon\varrho_N/2)$ contains a large density of obstacles or
2. $B(x; \varepsilon\varrho_N/2)$ contains a small density of obstacles.

- Case 1 is easy to exclude since it makes too hard for the random walk to survive.
- Case 2 is more complicated and split into two sub-cases...
  - 2.1 random walk comes close to $x$ many times;
  - 2.2 random walk comes close to $x$ few times.

We deal with them by using comparison arguments.

# Proof idea for ball clearing

Case 2.1: $B(x;\varepsilon\varrho_N/2)$ contains a small density of obstacles and random walk comes close to $x$ many times.

We remove all the obstacles in $B(x;\varepsilon\varrho_N/2)$. This operation

- imposes a cost in the environment probability;
- brings a gain in the random walk probability.

It turns out that the gain beats the cost:

$$\mathbb{P}\otimes\mathbf{P}(\text{Case 2.1}) \ll \mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}} > N, \mathcal{O}\cap B(x;\varepsilon\varrho_N/2) = \emptyset).$$

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However, it is not straightforward because

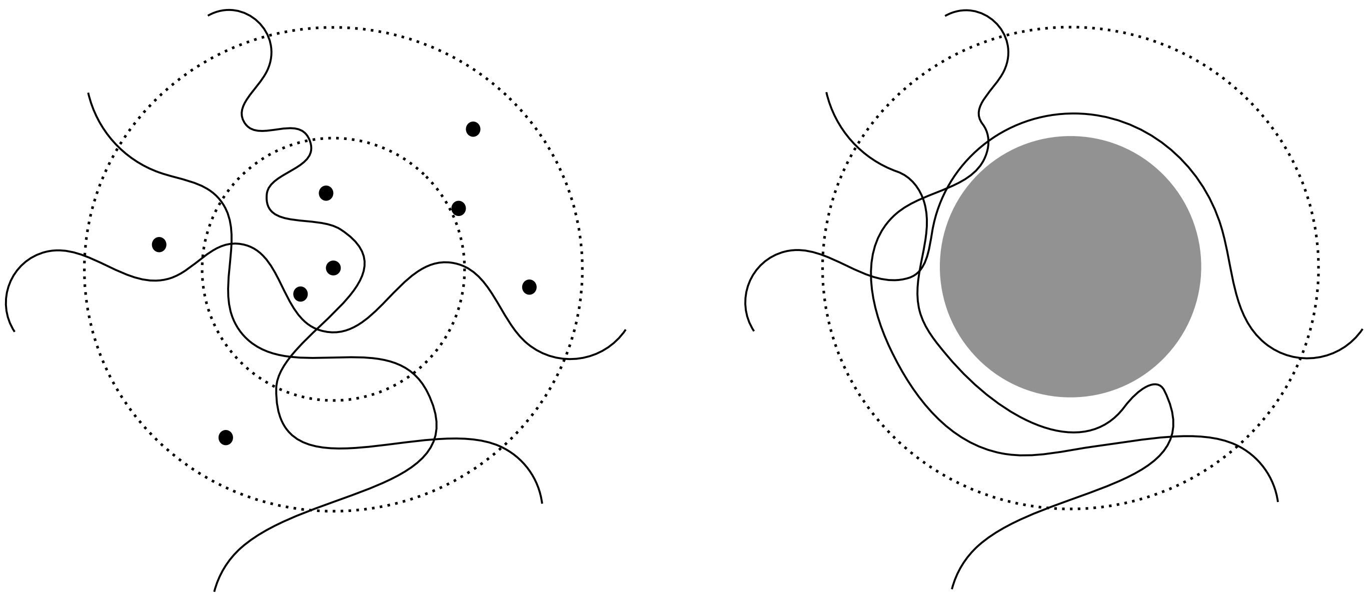
- the cost increases linearly in the number of obstacles in $B(x;\varepsilon\varrho_N/2)$, while
- the gain does NOT increase linearly in the number of obstacles in $B(x;\varepsilon\varrho_N/2)$.

# Proof idea for ball clearing

Case 2.2: $B(x; \varepsilon \varrho_N/2)$ contains a small density of obstacles and random walk comes close to $x$ few times.

We remove all the obstacles in $B(x; \varepsilon \varrho_N/2) \setminus B(x; \varepsilon \varrho_N/4)$, let the random walk avoid $B(x; \varepsilon \varrho_N/4)$, and then change the obstacles configuration in $B(x; \varepsilon \varrho_N/4)$ to typical ones. This operation

- imposes a cost in the random walk probability;
- brings a gain in the environment probability.



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- imposes a cost in the random walk probability;
- brings a gain in the environment probability.

It turns out that the gain beats the cost:

$$\begin{aligned}&\mathbb{P}\otimes\mathbf{P}(\text{Case 2.2})\\ &\qquad\ll \mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}\cup B(x;\varepsilon\varrho_N/4)}>N, \mathcal{O}\cap B(x;\varepsilon\varrho_N/4)\text{ is typical}).\end{aligned}$$

## Remark

This argument looks wasteful since we are comparing the LHS to a tiny fraction of the partition function. But it is more effective than comparing with $\exp\{-c_{\mathsf{DV}}N^{\frac{d}{d+2}}+o(N^{\frac{d}{d+2}})\}$.

# Proof Idea for Confinement with Bias

Recall: $$\mu_N^h(A) = \frac{\mathbb{E} \otimes \mathbf{E}\Big[e^{\langle h, S_N \rangle} :\, A \cap \{\tau_{\mathcal{O}} > N\}\Big]}{\mathbb{E} \otimes \mathbf{E}\Big[e^{\langle h, S_N \rangle} :\, \tau_{\mathcal{O}} > N\Big]}$$

# Partition function

Sznitman (1995) proved that under the annealed measure with a subcritical bias,

$$\mu_N^h\Big(S_N = o(N^{\frac{d}{d+2}})\Big) \to 1.$$

This (morally) means that the extra $e^{\langle h, S_N\rangle}$ term is $e^{o(N^{\frac{d}{d+2}})}$.
As a result, the partition function behaves as

$$\mathbb{E} \otimes \mathbf{E}\Big[e^{\langle h, S_N\rangle} \colon \tau_{\mathcal{O}} > N\Big] = \exp\Big\{-c_{\mathsf{DV}} N^{\frac{d}{d+2}}(1 + o(1))\Big\},$$

exactly like the unbiased case.

So the global picture (in the volume scale) is the same as the unbiased case.

# From volume to length scale

What we can read from the partition function asymptotics is

1. there is a ball of radius $\varrho_N$ almost free of obstacles,
2. RW spends most of the time in that ball.


From this picture, we proceed in two steps:

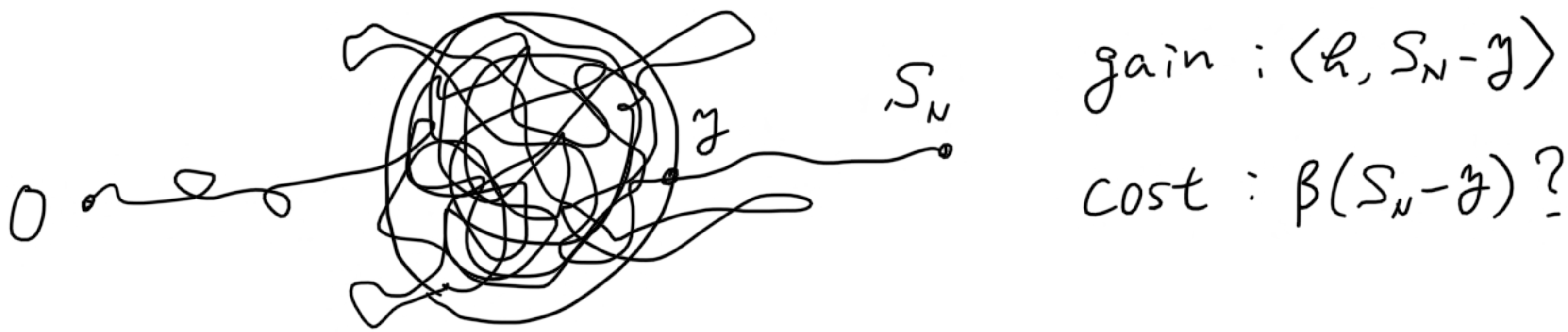
i) RW is confined in the ball from the first visit to last visit,

ii) the first and last pieces also must be in the ball.

# The reason for 2 steps

Technically, the excursions starting and ending in the ball are relatively easy to exclude because they don't contribute to $e^{\langle h, S_N \rangle}$.

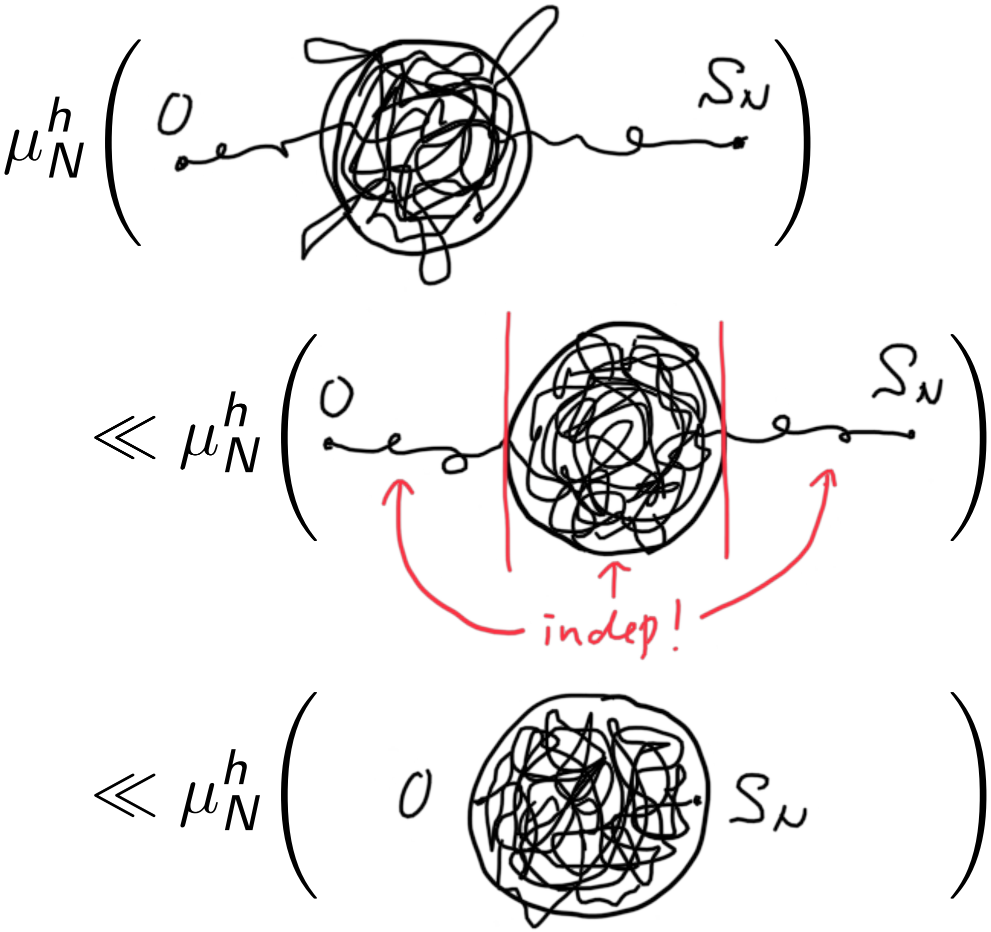
On the other hand, this is a necessary input to exclude the first and last pieces. It is the subcriticality that makes these two pieces stay in the ball but $\beta$ was an "averaged" crossing cost.




It is not clear whether the environment between $y$ and $S_N$ should behave in an "averaged" manner.

# Flow of the argument

So the overall argument goes as follows:




For the proof of first step, we (roughly) follow Bolthausen's proof of the confinement property.

# Confinement of the middle part

Recall Bolthausen's strategy:

1. prove the ball covering first,
2. starting from a path with many outgoing excursions, perform a "folding" operation that maps the path to another "confined" one.

We need to prove the ball covering but now under the biased measure.

**Lemma**

*Let $d \geq 2$ and $B(x_N; \varrho_N)$. For any $\varepsilon > 0$,*

$$\lim_{N\to\infty} \mu_N^h\Big(B(x_N; (1-\varepsilon)\varrho_N) \subset S_{[\tau_B, \tau_B^{\leftarrow}]}\Big) = 1.$$

# Final remark

The proof of the previous lemma is EXACTLY the same as $h = 0$ case. The reason is that our proof for $h = 0$ was essentially combinatorial. I don't know if the large deviation arguments of Bolthausen and Beresticky–Cerf works for $\mu_N^h$ or not.

After having the lemma, we can (roughly) follow Bolthausen's argument to prove the confinement of the middle part.

This completes our program.

Thank you!