Extrema of log-correlated Gaussian fields

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(Based on joint work with Jian Ding & Ofer Zeitouni)
Bengaluru Probability Seminar

Outline

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- tightness of the re-centered maximum and it's convergence in law for log-correlated Gaussian field.

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- connection to cover-times of random walks. (Ding, Lee, Peres ('12))

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- In dimension $d = 3$ it plays an important role in early universe cosmology, where it approximately describes the gravitational potential function of the universe at a fixed time shortly after the big bang.(Dorelson '03)
- The linear combinations of the two models lattice free field and membrane model together are considered as models for semiflexible membranes (or semiflexible polymers if $d = 1$).(Kurt '09)

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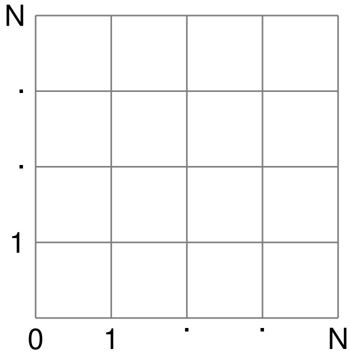
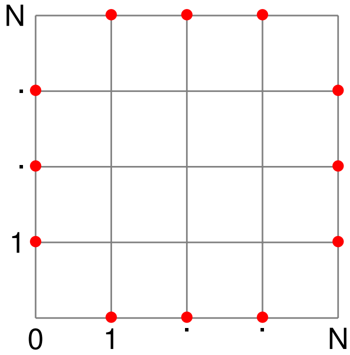
Extension to geometric Brownian (GB) model, which doesn't take into account:

- The volatility fluctuates randomly and follows approximately a lognormal distribution.
- While the returns are rapidly decorrelated, the volatility exhibits long range correlations following a power law
- The returns are heavy tailed.

Lattice(V_N)



Lattice(V_N) with boundary(∂V_N)



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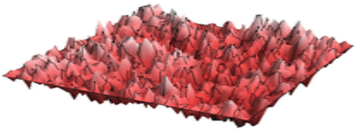
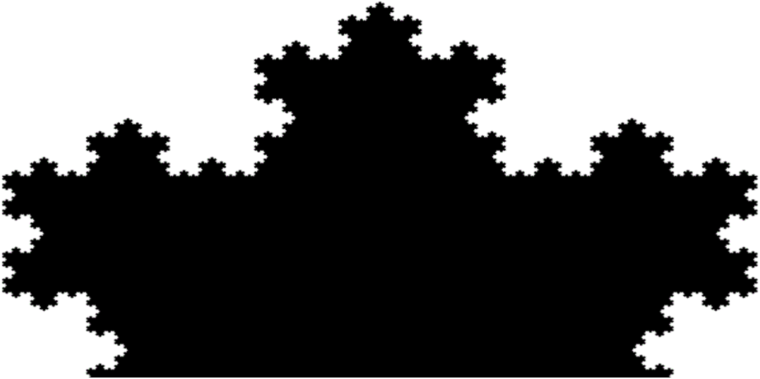


Figure: GFF on 60 X 60 square grid(Watson, S.)

Self-similarity



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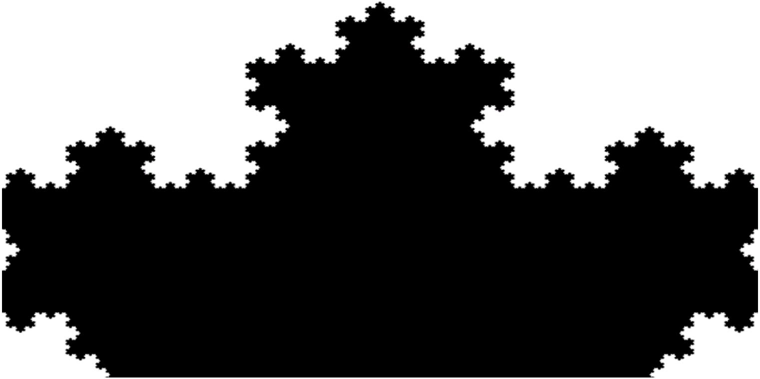

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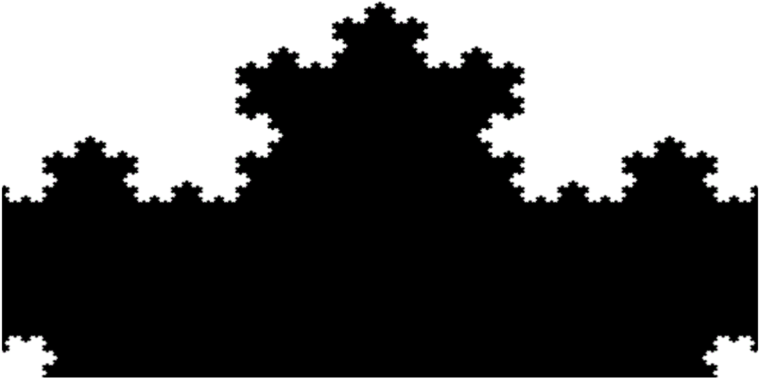

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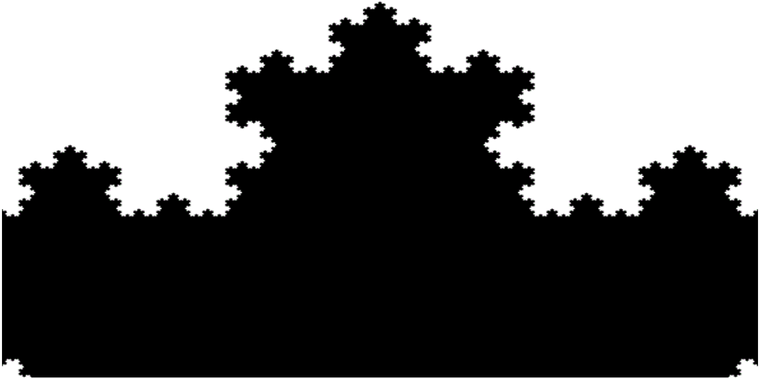

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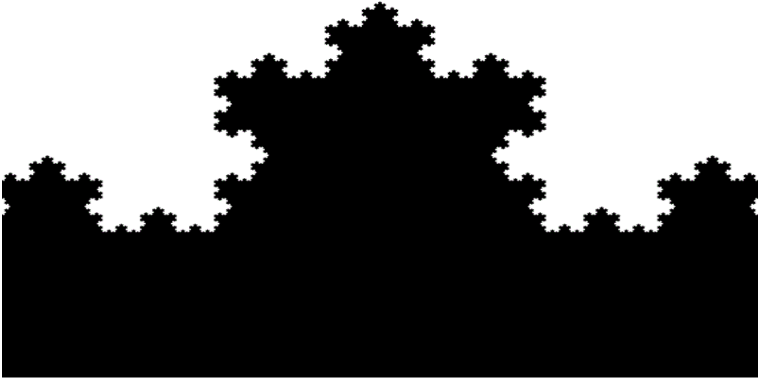

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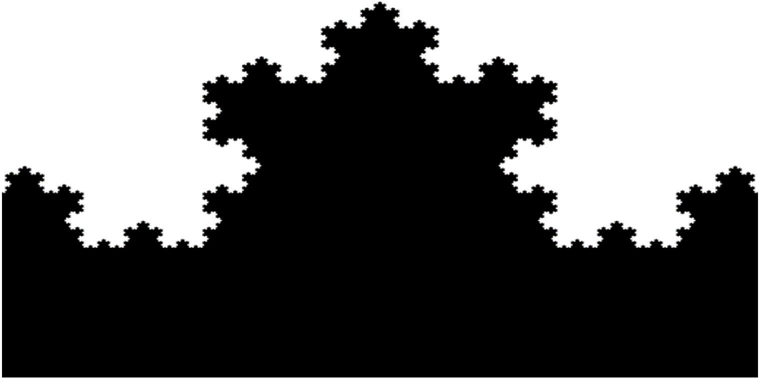

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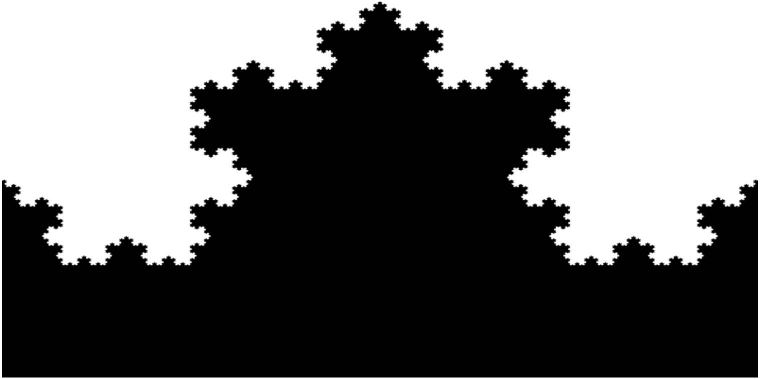

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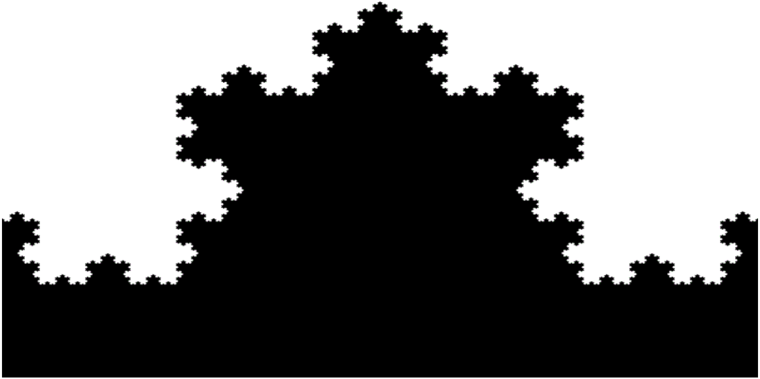

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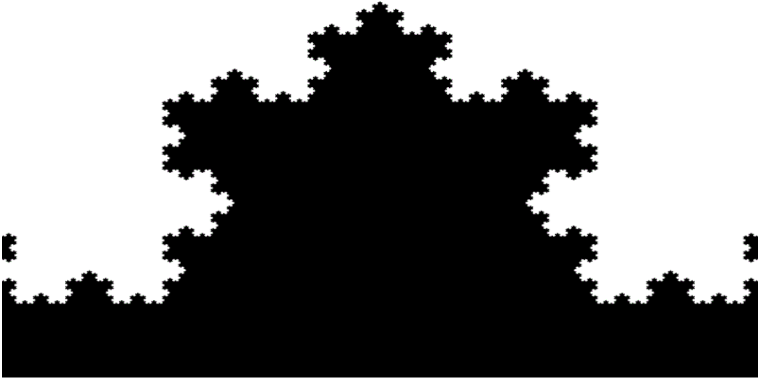

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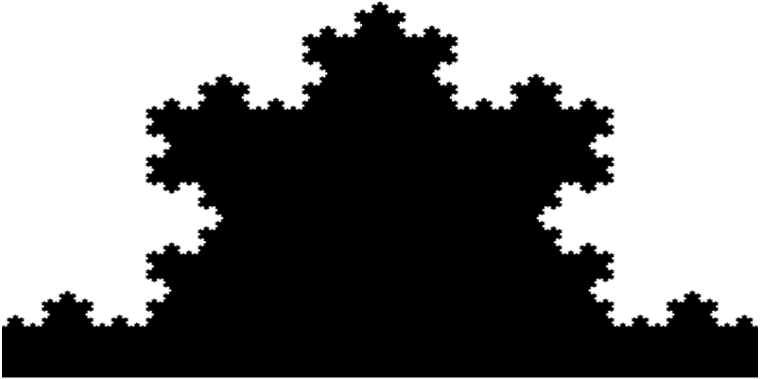

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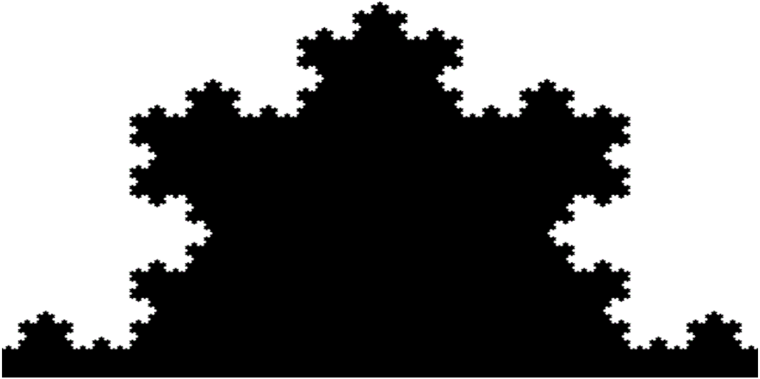

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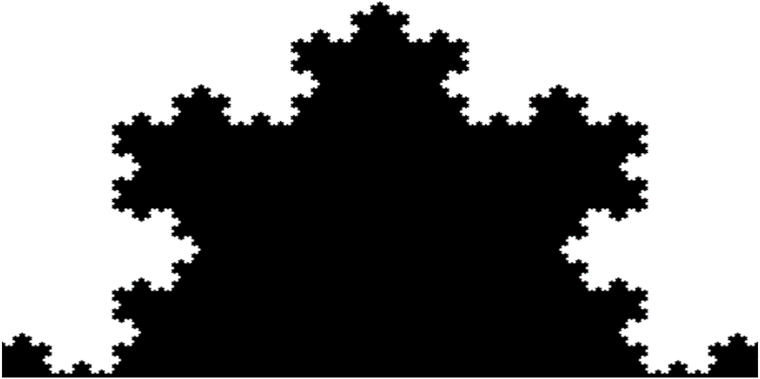

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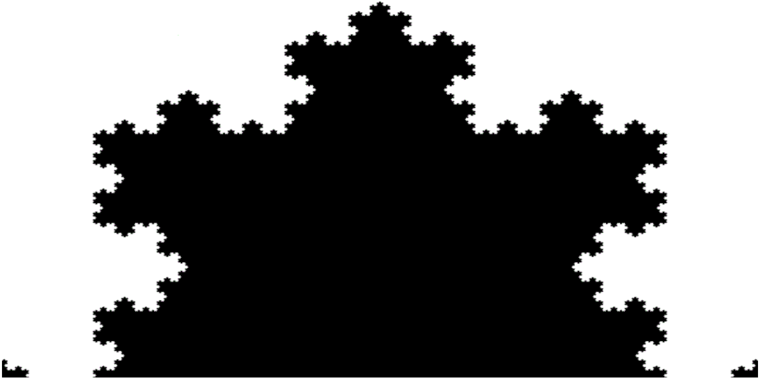

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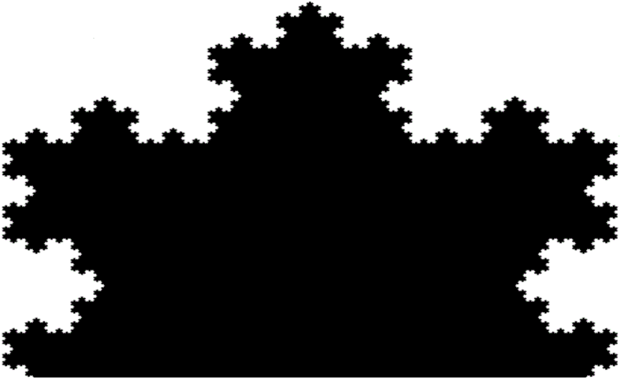

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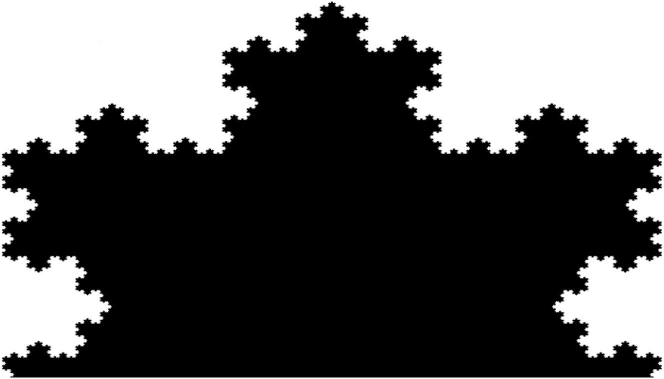

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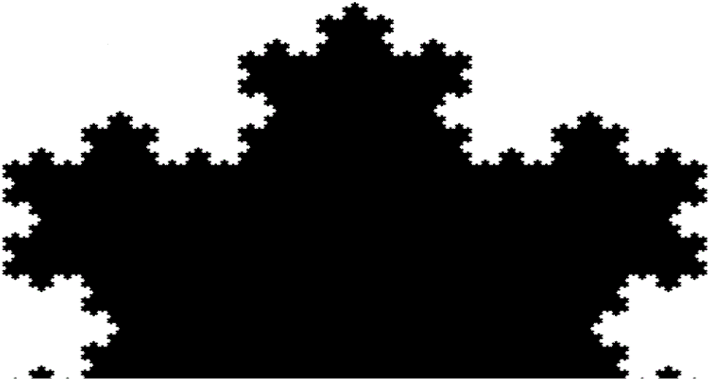

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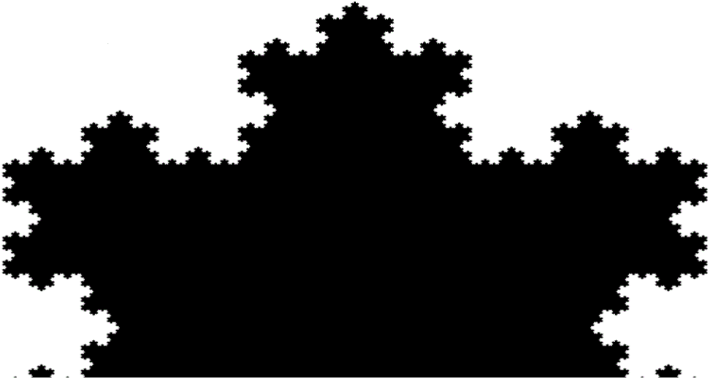

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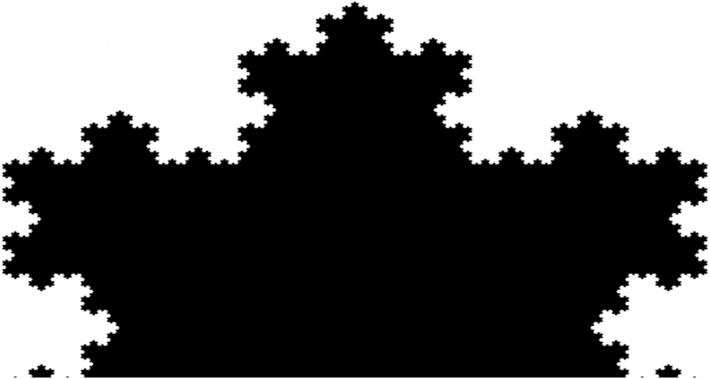

Here we depict a Koch curve, an example of a self similar object.

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- GFF is self-similar

Pictorial presentation

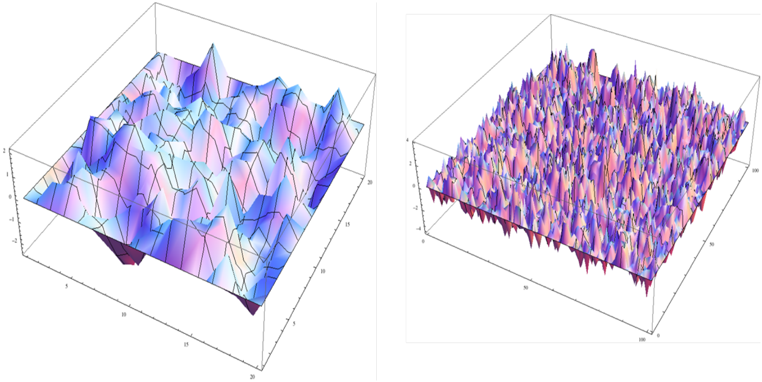

Figure: GFF on a big box(right), with a smaller section(left)(from Sheffield, S.)

Covariance Approximations

The covariance structure is given by the random walk Green's function $G_N(\cdot,\cdot)$ where

$$G_N(u,v) = \mathbb{E}^u\Big(\sum_{n=0}^{\tau_N} 1_{\{S_n=v\}}\Big).$$

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Using approximations from random walk :

$$G_N(u,v) = \frac{2}{\pi}(\log N - \log|u-v|) + O(1).$$

Gaussian membrane model(Kurt ('07,'09))

A Gaussian field $\{\psi_v^N : v \in V_N\}$ where the Hamiltonian is given by

$$\frac{1}{2}\sum_{v\in V_N}(\Delta\psi_v^N)^2.$$

Here Δ is the discrete laplacian operator :

$$\Delta\psi_v^N = \frac{1}{2d}\sum_{i=1}^{d}(\psi_{v+e_i}^N + \psi_{v-e_i}^N - 2\psi_v^N).$$

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For this model the dimension $d = 4$ is critical. It is to be noted that this is a log-correlated gaussian field, again using approximations from random walk.

Log-correlated Gaussian field

A normalized discrete log-correlated Gaussian field $\{\varphi_v^N : v \in V_N\}$ on d-dimensional box of side length N is defined as follows.

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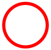
- From (A.0) it follows that assuming (A.1) only for interior points, works.

GFF vs BRW



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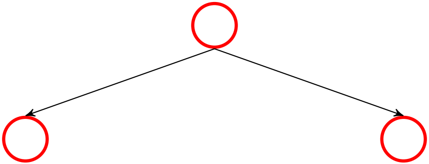

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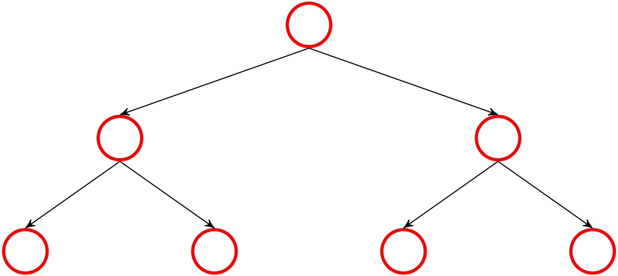

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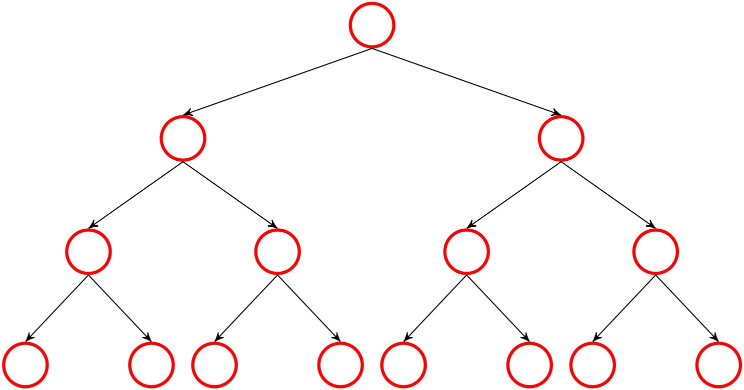

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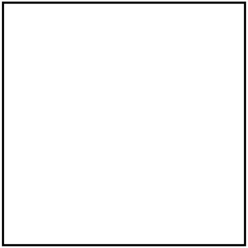
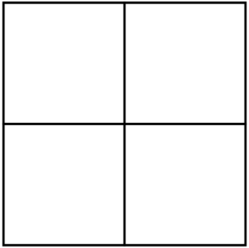


Figure: Tree structure for the lattice

GFF vs BRW



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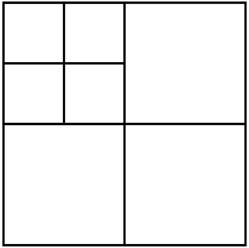

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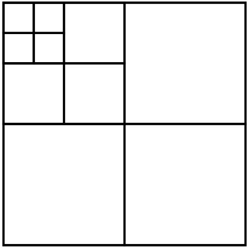

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Branching Random Walk

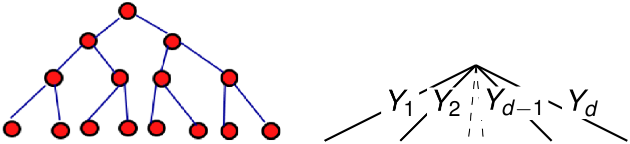

Figure: Branching Random Walk & node of a d-dim BRW

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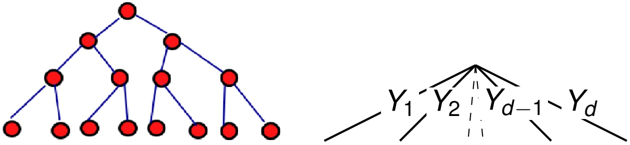
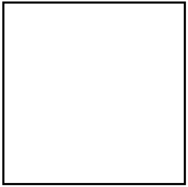


Figure: Branching Random Walk & node of a d-dim BRW

All the edges carry an independent standard Gaussian variable. The process consists of the values at the leaf nodes, obtained by summing over all values on the edges from the root to this node.

MBRW

- Difficulty : Points close by might get separated in trees.



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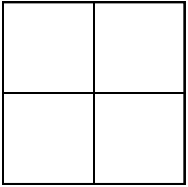

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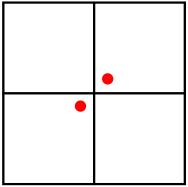

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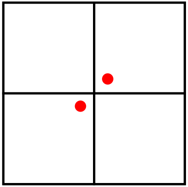

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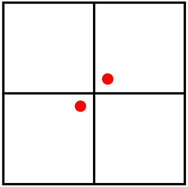

- Solution : *Modified branching random walk.*

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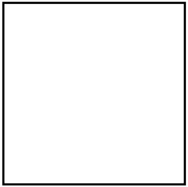

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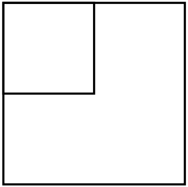

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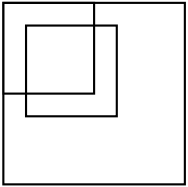

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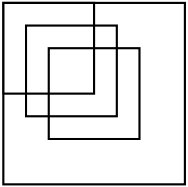

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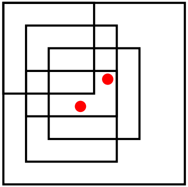

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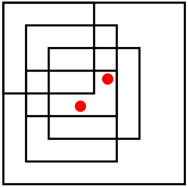

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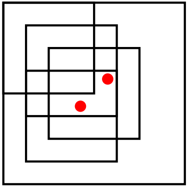

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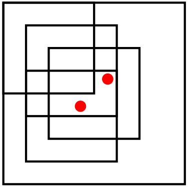

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- Covariance structure of the GFF is similar to that of an MBRW.
- Gives toroid structure to branching random walk.

Tightness

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- This could be extended, in a relatively simple fashion to log-correlated Gaussian fields as long as the covariance can be approximated up to O(1).

Theorem

Under Assumptions (A.0) and (A.1), we have that $\mathbb{E}M_N = m_N + O(1)$ *where the* $O(1)$ *term depends on* α_0 *and* $\alpha^{(1/10)}$*. In addition, the sequence* $M_N - \mathbb{E}M_N$ *is tight.*

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- Goal- to find minimal structural assumptions for limit law to hold.
- Obstacle- Markov Random Field.
- Way out - convergence of covariance at macroscopic and microscopic levels.

Covariance assumptions

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- In the middle level(mesoscopic) we approximate the field by an MBRW.
- The limiting distribution of this new field coincides with the previous.

Picture of new model

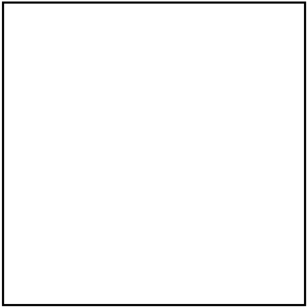

Figure: Hierarchy of construction of the approximating Gaussian field

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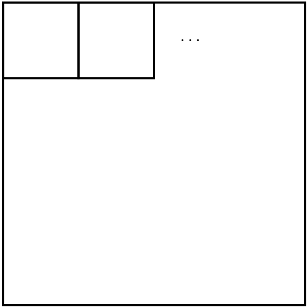

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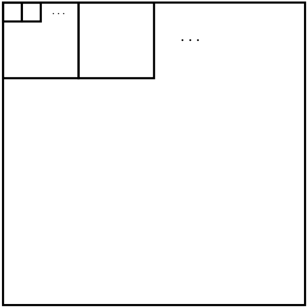

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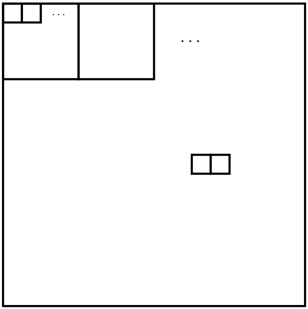

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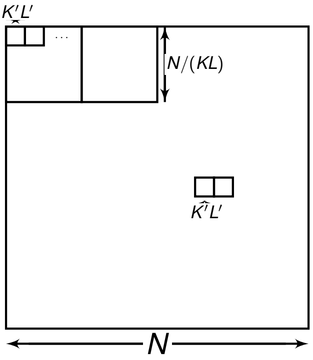

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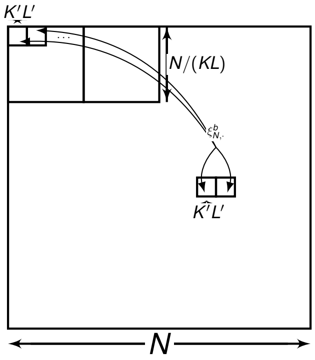

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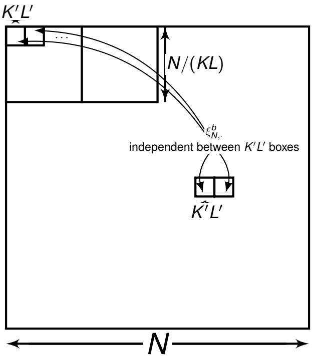

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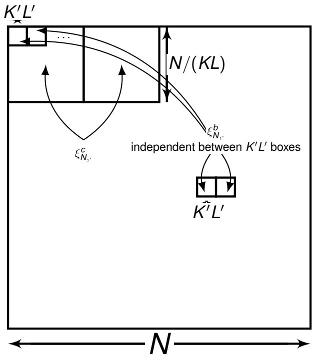

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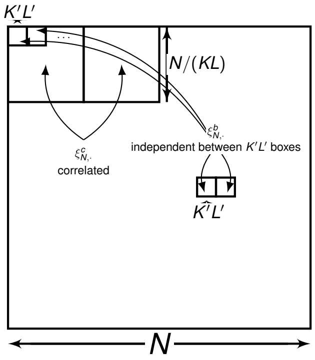

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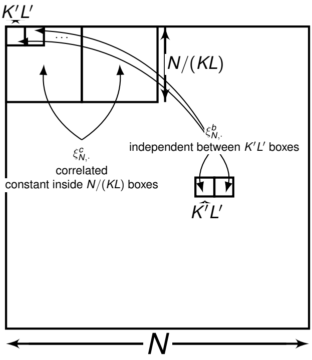

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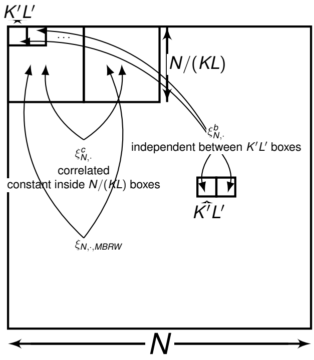

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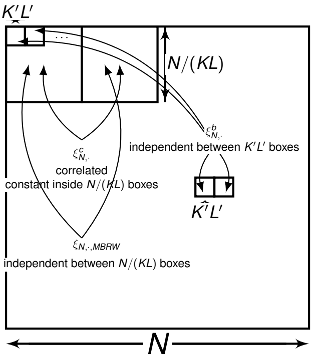

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Convergence

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- Construct fine field comprising the microscopic and mesoscopic approximations.
- Compute the asymptotics of the right tail for the distribution of the maximum of the fine field.
- Combine this with the macroscopic field, to get the result.

Theorem

Under Assumptions (A.0), (A.1), (A.2) and (A.3), the sequence $\{M_N - \mathbb{E}M_N\}_N$ converges in distribution.

Convergence

The limiting law of $(M_N - m_N)$ is characterized as a Gumbel distribution with random shift. $\mathcal{Z}_N$ is defined as

$$\mathcal{Z}_N = \sum_{v \in V_N} (\sqrt{2d} \log N - \varphi_{N,v}) e^{-\sqrt{2d}(\sqrt{2d} \log N - \varphi_{N,v})}.$$

Theorem

Under ssumptions (A.0), (A.1), (A.2) and (A.3) the the derivative martingale $\mathcal{Z}_N$ converges in law to a positive random variable $\mathcal{Z}$. In addition, the limiting law μ_∞ of $M_N - m_N$ can be expressed by $\mu_\infty((-\infty, x]) = \mathbb{E} e^{-\beta^* \mathcal{Z} e^{-\sqrt{2d} x}}$, for all $x \in \mathbb{R}$, where β^* is a positive constant.

Thank You

Markov property

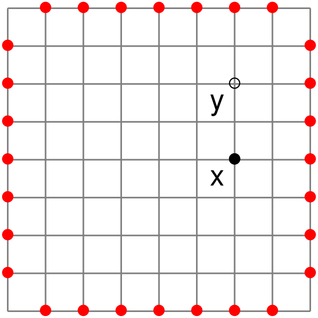
Markov property is a kind of memoryless property. It says that *future, given present, is independent of the past.*

For example, for a sequence of random variables $\{X_1, X_2, \ldots, X_n, \ldots\}$ this means that

$$\begin{aligned} & \mathbb{P}(X_n = x_n \mid X_1 = x_1, X_2 = x_2, \ldots, X_{n-1} = x_{n-1}) \\ = \ & \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}) \end{aligned}$$

Back to GFF.

Covariance of GFF



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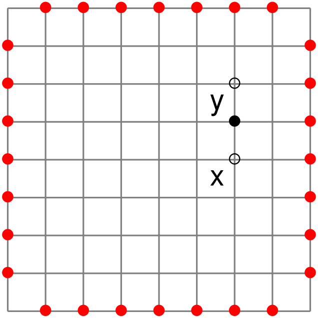

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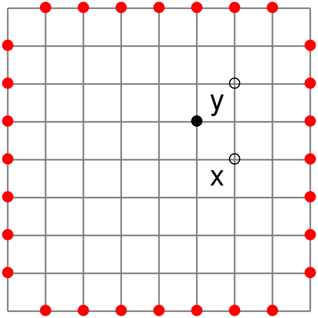

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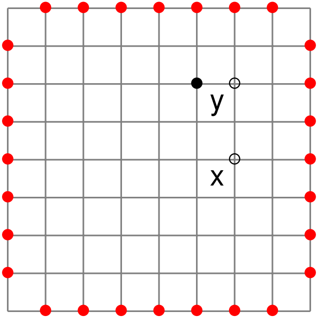

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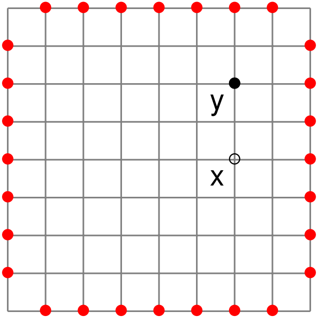

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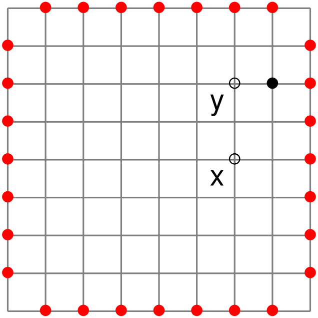

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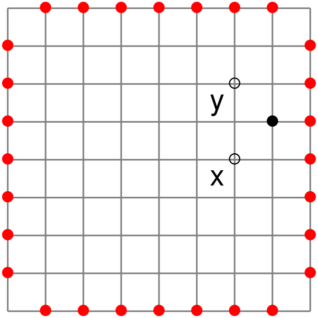

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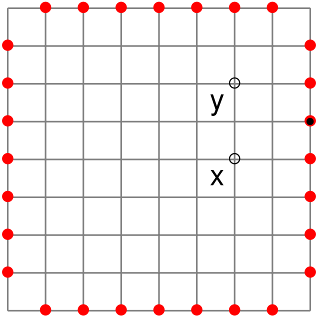


For this realization the random walk hits y once before hitting the boundary.

Back to GFF.

Logarithmically bounded

(A.0)

There exists a constant $\alpha_0 > 0$ such that for all $u, v \in V_N$,
$\operatorname{Var} \varphi_{N,v} \le \log N + \alpha_0$ &
$\mathbb{E}(\varphi_{N,v} - \varphi_{N,u})^2 \le 2 \log_+ |u - v| - |\operatorname{Var} \varphi_{N,v} - \operatorname{Var} \varphi_{N,u}| + 4\alpha_0$

Back to assumptions.

Lattice points

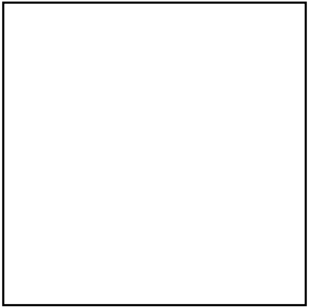
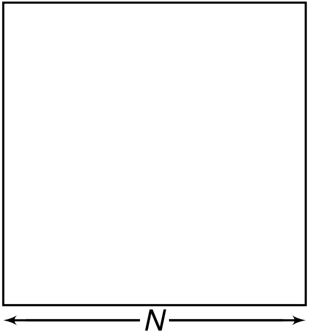


Figure: Full lattice

Lattice points



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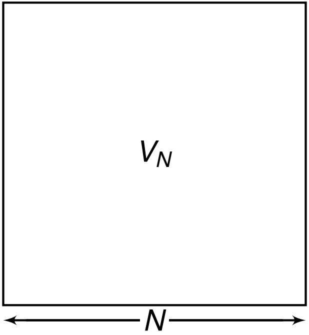

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Interior points

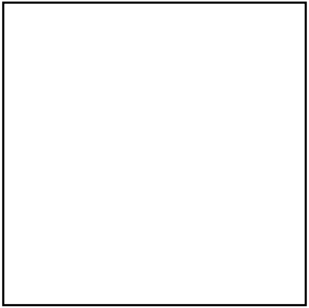

Figure: Interior points of lattice

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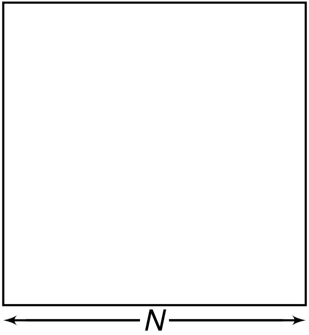

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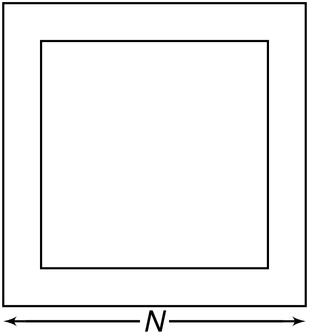

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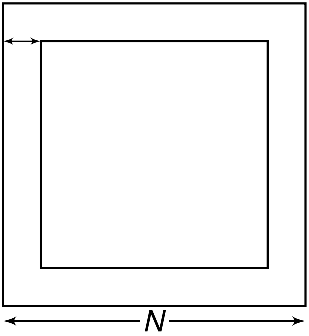

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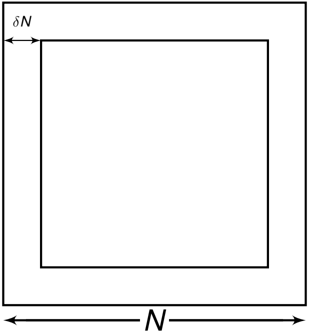

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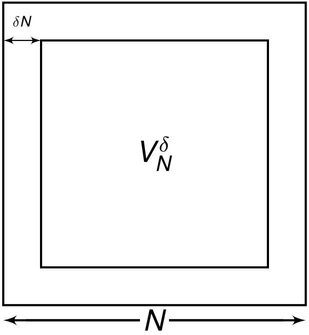


Figure: Interior points of lattice

Logarithmically correlated

(A.1)

For any $\delta > 0$ there exists a constant $\alpha^{(\delta)} > 0$ such that for all $u, v \in V_N^\delta$, $|\mathrm{Cov}(\varphi_{N,v}, \varphi_{N,u}) - (\log N - \log_+ |u - v|)| \leq \alpha^{(\delta)}$.
$(V_N^\delta = \{z \in V_N : d(z, \partial V_N) \geq \delta N\})$

Back to assumptions.

Right tail of maximum

Lemma

Under Assumption (A.1), there exists a constant $C > 0$ *depending only on* $(\alpha_0, \alpha^{(1/10)}, d)$ *such that for all* $\lambda \in [1, \sqrt{\log N}]$,

$$C\lambda e^{-\sqrt{2d}\lambda} \geq \mathbb{P}(M_N > m_N + \lambda) \geq C^{-1}\lambda e^{-\sqrt{2d}\lambda}.$$

Back to tightness.

Expected maximum

The expected value of the maximum of the field is:

$$m_N = \sqrt{2d}\log N - \frac{3}{2\sqrt{2d}}\log\log N\,. \tag{1}$$

Back to tightness.

Tightness

The sequence of random variables $M_N - m_N$ is tight if $\forall \epsilon > 0$ there exists K_ϵ such that for all sufficiently large N:

$$\mathbb{P}(|M_N - m_N| > K_\epsilon) < \epsilon$$

Back to tightness.

Microscopic

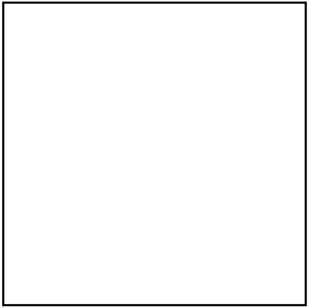
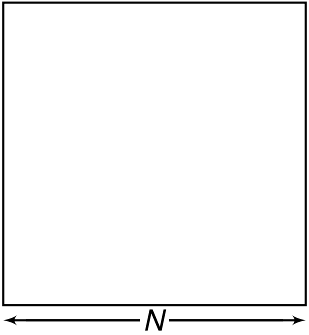


Figure: Near diagonal behavior

Microscopic



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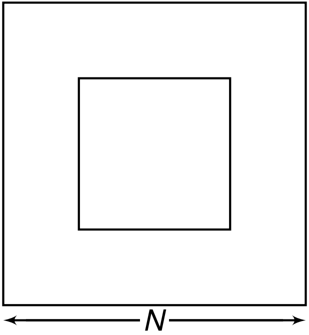

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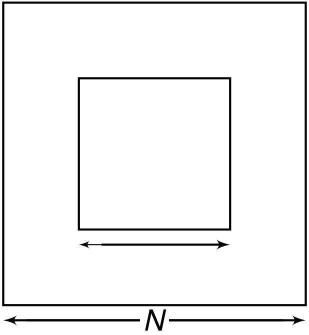

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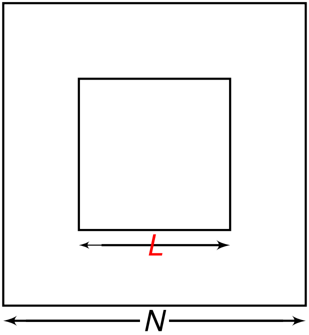

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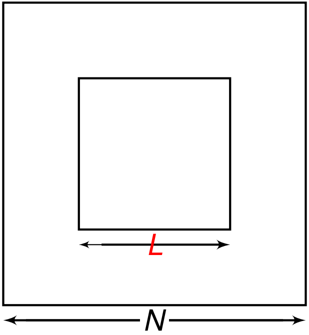

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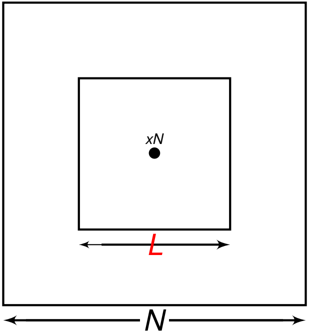

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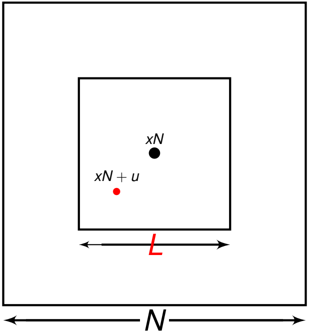

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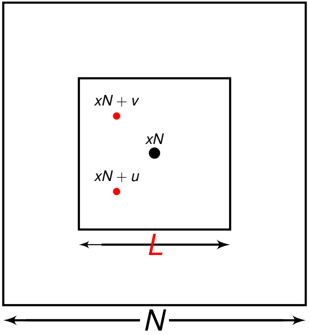


Figure: Near diagonal behavior

Microscopic

(A.2)(Near diagonal behavior)

There exist a continuous function $f : (0,1)^d \mapsto \mathbb{R}$ and a function $g : \mathbb{Z}^d \times \mathbb{Z}^d \mapsto \mathbb{R}$ such that the following holds. For all $L, \epsilon, \delta > 0$, there exists $N_0 = N_0(\epsilon, \delta, L)$ such that for all $x \in V^\delta$, $u, v \in [0, L]^d$ and $N \geq N_0$ we have

$$|\mathrm{Cov}(\varphi_{N,xN+v}, \varphi_{N,xN+u}) - \log N - f(x) - g(u,v)| < \epsilon\,.$$

Back to microscopic.

Macroscopic

Figure: Off diagonal behavior

Macroscopic

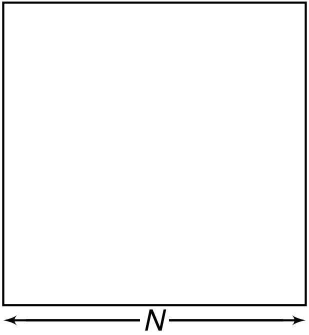


Figure: Off diagonal behavior

Macroscopic

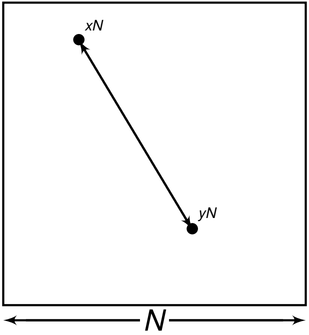


Figure: Off diagonal behavior

Macroscopic

$$\mathcal{D}^d = \{(x,y) : x, y \in (0,1)^d, x \neq y\}$$

(A.3)(Off diagonal behavior)

There exists a continuous function $h : \mathcal{D}^d \mapsto \mathbb{R}$ such that the following holds. For all $L, \epsilon, \delta > 0$, there exists $N_1 = N_1(\epsilon, \delta, L) > 0$ such that for all $x, y \in V^\delta$ with $|x - y| \geq \frac{1}{L}$ and $N \geq N_1$ we have

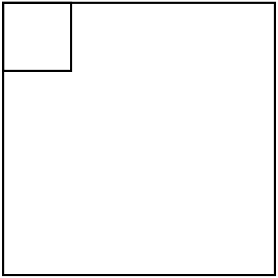
$$|\mathrm{Cov}(\varphi_{N,xN}, \varphi_{N,yN}) - h(x,y)| < \epsilon\,.$$

Back to macroscopic.

Perturbation

Figure: Perturbation levels of the Gaussian field

Perturbation



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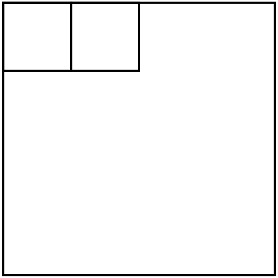

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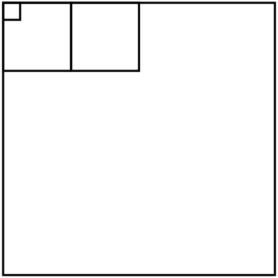

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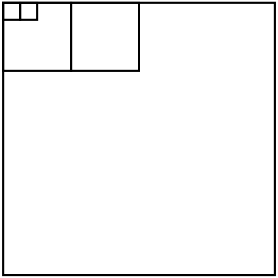

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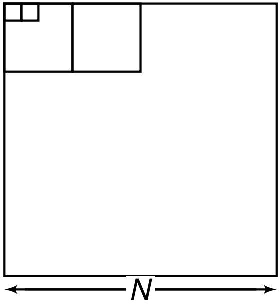
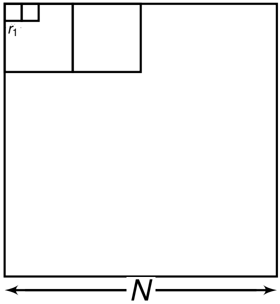


Figure: Perturbation levels of the Gaussian field

Perturbation



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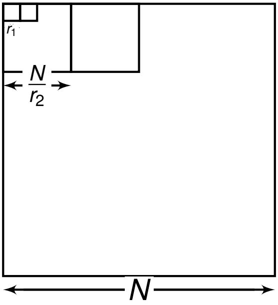

Figure: Perturbation levels of the Gaussian field

Perturbation

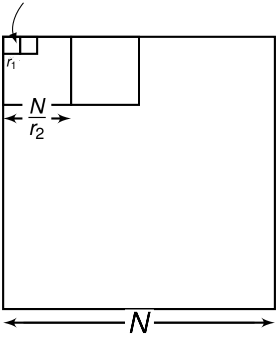

Figure: Perturbation levels of the Gaussian field

Perturbation

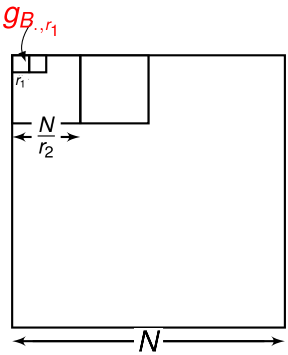

Figure: Perturbation levels of the Gaussian field

Perturbation

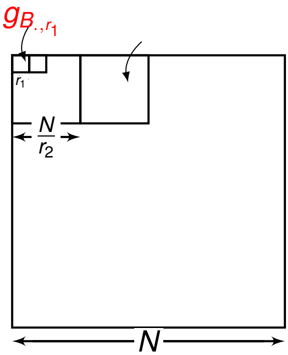

Figure: Perturbation levels of the Gaussian field

Perturbation

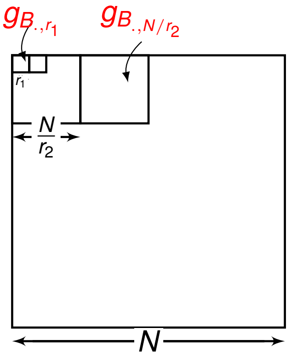

Figure: Perturbation levels of the Gaussian field

Perturbation

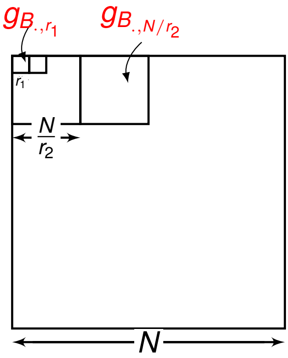

Figure: Perturbation levels of the Gaussian field

Back to Robustness.