Environment seen from infinite geodesics in Liouville quantum gravity

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A digression: some other models of random growth

First passage percolation on $\mathbb{Z}^d$

- IID positive weights on the edges.
- Consider the random metric obtained by considering the minimum path weight between two vertices.

Exponential last passage percolation on $\mathbb{Z}^2$

- IID exponential weights on the vertices.
- Consider the maximum (oriented) path weight (last passage time between ordered vertices).

- Planar FPP/LPP models are expected to belong to the KPZ universality class, rigorously known only for exactly solvable models such as exponential LPP.
- Directed landscape is believed to be the universal scaling limit.

Hoffman's question

- Consider geodesics in FPP/LPP.
- Geodesics are expected to pass through lower (higher) weight regions.
- It is of interest to understand the environment on/around the geodesics.

Hoffman's question

Question (Chris Hoffman, AIM 2015)

- Consider FPP in $\mathbb{Z}^d$ with a nive passage time distribution.
- Let Γ_n denote the geodesic between $\mathbf{0}$ and ne_1.
- Let ω_v denote the environment rooted at v for $v \in \mathbb{Z}^d$.
- Consider the empirical measure on environments

$$\mu_n = \frac{1}{|\Gamma_n|} \sum_{v \in \Gamma_n} \omega_v.$$

- Does μ_n converge almost surely as $n \to \infty$?

Results for "generic" FPP

- A variant of Hoffman's question was considered by Erik Bates.
- Bates (2019) showed that the empirical distribution of the weight on the geodesic almost surely converges for a "dense" class of passage time distributions for various notions of denseness.
- Argument hinges on abstract convex analysis.
- Janjigian, Lam and Shen (2020) showed that the limit law must be absolutely continuous w.r.t. the passage time distribution.

Results in the integrable set-up: exponential LPP

- Martin, Sly and Zhang (2021) answered Hoffman's question in the context of planar exponential LPP.
- They showed that the empirical environment along the geodesic from $(0,0)$ to (n,n) converges almost surely as $n \to \infty$.
- Using connections to TASEP and and the correspondence between infinite geodesics and second class particles, they could do some explicit computations on the limiting environment.

Results in the integrable set-up: directed landscape

- Directed landscape is the putative universal scaling limit of planar LPP models under appropriate space time scaling.
- The geodesic from $(0, 0)$ to (n, n) after scaling converges to the geodesic from $(0, 0)$ to $(0, 1)$.
- Dauvergne, Sarkar and Virag (2020) calculated the local limit of the environment around a point on the geodesic.
- The limit is explicitly described as a directed landscape with Brownian-Bessel boundary data.

Formulating the question in the LQG set-up

- Our goal is to formulate and investigate the same question in the LQG set-up.
- Fix the almost surely unique infinite geodesic Γ started at 0 parametrized by the LQG length.
- We want to choose a point randomly on an initial segment of this geodesic and consider its appropriately scaled environment together with induced metric and investigate whether that the environment/metric converges as the endpoint of the segment goes off to ∞.
- To this end, we shall choose a ball around a point on the geodesic and scale it to the unit ball $\mathbb{D}$. We shall use appropriate scaling to obtain a random distribution and a random induced metric on this ball.

Scale dependent dilation

- The primary distinction between the LQG and the earlier FPP/LPP set-up is that the underlying noise here is "IID modulo rescaling".

$$h(r\cdot) - \mathbf{Av}(h, \mathbb{T}_r) \stackrel{d}{=} h(\cdot).$$

$$D_{h(r\cdot)}(rz, rw) = r^{\xi Q} D_{h(\cdot)}(z, w)$$

- Recall also the scale invariance of the geodesic

$$(h, \Gamma.) = (h(r\cdot) - \mathbf{Av}(h, \mathbb{T}_r), r^{-1}\Gamma_{r^{\xi Q} e^{\xi \mathbf{Av}(h, \mathbb{T}_r)}.})$$

Local environment

- This suggests the following definition:

Local environment

Fix $\delta \in (0,1)$. For $x \in \mathbb{C}$, define a field $\underline{h}_x$ on $\mathbb{D}$ by suitably dilating and normalizing/centering the field h in a ball of size $\delta|x|$ around x

$$\underline{h}_x = h \circ \Psi_{x,\delta} - c_x(h),$$

where $\Psi_{x,\delta}$ takes the ball of radius $\delta|x|$ around x to the unit ball and the generalized function $h \circ \Psi_{x,\delta}$ on $\mathbb{D}$ naturally defined by the composition of the field with the map Ψ; the quantity $c_x(h)$ is chosen so that $\mathbf{Av}(\underline{h}_x, \mathbb{T}) = 0$.

Metric on the local environment

- Analogously, we define the following metric on $\mathbb{D}$.

$$\underline{D}_{h,x}(u,v) = |\delta x|^{-\xi Q} e^{-\xi \mathbf{Av}(h, \mathbb{T}_{\delta|x|}(x))} D_h(x + \delta|x|u, x + \delta|x|v; \mathbb{D}_{\delta|x|}(x)).$$

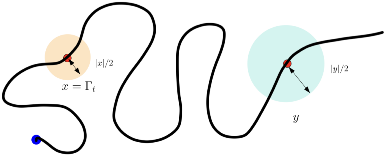
- This are not the only choices for the definition of local environment and metric, but are natural ones.
- The parameter δ is somewhat artificial and is the artefact of our proof.

Log parametrization of the geodesic

- We also want to ensure that the empirical distribution of the environment gets equal contribution from each scale.
- Thus for $t > 0$, and $T \sim Unif[0, t]$, we wish to consider the random point on the geodesic whose LQG distance from the origin is e^T.
- So we are looking at the random environment

$$Field_t = \underline{h}_{\Gamma_{e^T}}.$$

- Let $Metric_t$ denote the corresponding metric.



The main result

Theorem (B., Bhatia, Ganguly (2021))

Fix $\delta \in (0, 1)$.

- *There is a random distribution Field on* $\mathbb{D}$ *such that such that almost surely,*

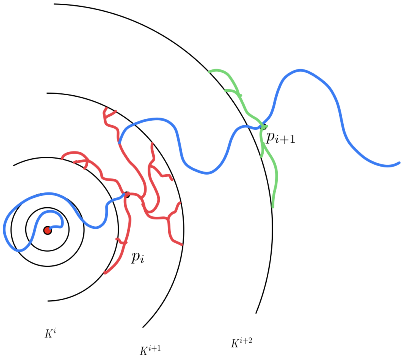
$$Field_t \xrightarrow{d} Field$$

as $t \to \infty$.

- *A similar result holds for* $Metric_t$.

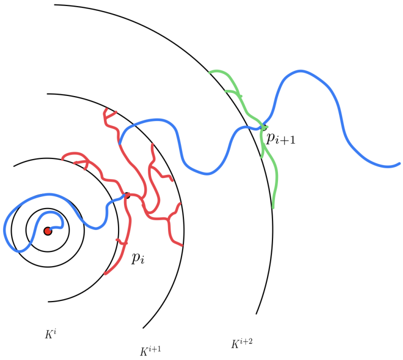
Partitioning the geodesic using coalescence points

- Fix $K > 0$ sufficiently large.



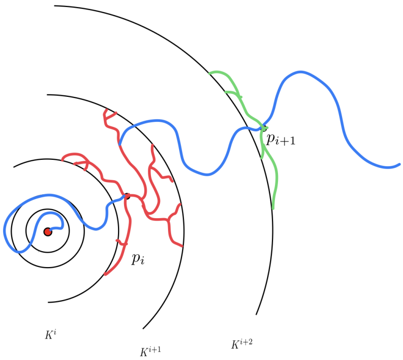
Partitioning the geodesic using coalescence points

- Using the coalescence result one can show that there exists a sequence $\{p_j\}_{j\in\mathbb{Z}}$ with $p_j \to 0$ as $j \to -\infty$ and $p_j \to \infty$ as $j \to \infty$ of coalescence points as in the figure.



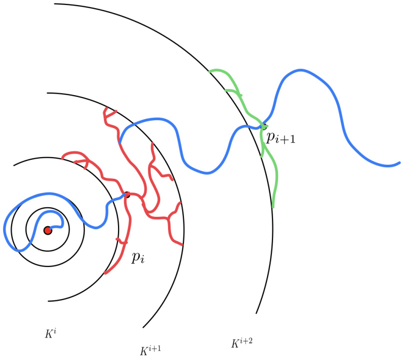
Partitioning the geodesic using coalescence points

- Partition the geodesic Γ into segments $\Gamma(p_j, p_{j+1})$.



Partitioning the geodesic using coalescence points

- The key idea is to show that the contributions coming from the segments $\Gamma(p_j, p_{j+1})$ form a stationary sequence with fast decay of correlations and use a SLLN.



Some key tools

- Let $\mathcal{H}_i$ denote the field obtained by considering h restricted to $\mathbb{C}_{>K^i}$, centering it by making its average on the boundary circle 0, and rescaling to $\mathbb{C}_{>1}$.
- Basic invariance of GFF implies $\mathcal{H}_i$ is a stationary sequence.
- Use domain Markov property to show that for $j \gg i$, the Radon-Nikodym derivative of conditional law of $\mathcal{H}_j$ given $\mathcal{H}_i$ w.r.t. its unconditional law is close to 1 except on a set of probability exponentially small in $j - i$.
- Using this one can show that "local" observables of $\mathcal{H}_i$ have exponential decay of correlation.

Some key tools

- One still needs to control the dependence across scales coming from the log-parametrization of the geodesic.
- For this we set $L_i = D_h(0, p_i)$.
- We show that $G_i = \log L_{i+1} - \log L_i$ is also a stationary sequence with exponential decay of correlations.
- This requires some Brownian computations using scale invariance of Γ and the fact that the circle average process is a Brownian motion.

Properties of the limiting field

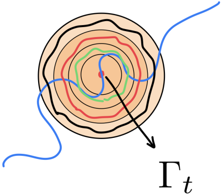
- We do not have any explicit description of the limiting field, however some basic properties can be established.

Theorem (B., Bhatia, Ganguly (2021))

- *The law of Field is mutually singular with respect to the law of h restricted to $\mathbb{D}$.*
- *For each $\delta' \in (0,1)$, the law of Field on the annulus $\mathbb{C}_{(\delta',1)}$ is absolutely continuous with respect to the law of h restricted to $\mathbb{C}_{(\delta',1)}$.*

A sketch of the proof for singularity

- At least heuristically, In *Metric*, there should exist a bigeodesic (i.e. a geodesic between two boundary points of $\mathbb{D}$) passing through 0 with probability 1.
- In $D_h(\cdot, \cdot; \mathbb{D})$, this almost surely does not happen.
- This can be shown by the same argument used to show the non-existence of bigeodesics in LQG metric.
- A more robust version pf this argument shows the mutual singularity of the metrics which then can be transferred to the corresponding fields.



Some interesting directions

- Can we find an explicit description of the limiting field and metric?
- What about local limits near the geodesic?
- Is it possible to characterise the singularity near 0?
- What is the thickness of a typical point on the geodesic?

Thank you

Questions?