A beginner's introduction to the Liouville quantum gravity metric

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Liouville quantum gravity (LQG)

- Model for a random two-dimensional Riemannian manifold.
- A γ-LQG surface parametrized by $\mathbb{C}$ informally is a random Riemannian manifold with area measure $e^{\gamma h} d^2 z$ and the Riemannian metric tensor $e^{2\xi h}(dx^2 + dy^2)$ for some parameter γ and some $\xi = \xi(\gamma)$ where h is the Gaussian free field on $\mathbb{C}$.
- Considering the scaling exponent relating the the length and area suggests ξ and γ should be related by $\xi = \gamma/d_\gamma$ is the "dimension" of the surface.
- Because of the roughness of h, LQG surfaces cannot be Riemannian manifolds in the usual sense, defining these rigorously has been a major challenge.

Ranges of γ and ξ

- Usual parameter range: $\gamma \in (0, 2)$ (subcritical).
- ξ is an increasing function of γ on $(0, 2]$ and $\xi(2) \approx 0.41$.
- $\xi = \xi(2)$ (critical) and $\xi > \xi(2)$ (supercritical) corresponds to $\gamma = 2$ and $\gamma \in \mathbb{C}$ with $|\gamma| = 2$ respectively.
- We shall only discuss the subcritical case.

Motivation and history-some physics buzzwords

- LQG surfaces were first considered by Polyakov in 1981-motivation came from string theory.
- Also connected to quantum field theory.
- LQG surface parametrized by a domain is "uniform sample from the space of Riemannian metric tensors weighted by $(\det \Delta_g)^{-c_M/2}$" where Δ_g is the Laplace-Beltrami operator and c_M is the matter central charge.
- $c_M = 25 - 6Q^2$ where

$$Q = \left(\frac{2}{\gamma} + \frac{\gamma}{2}\right)$$

- $c_M \in (-\infty, 1]$ corresponds to subcritical and critical cases, whereas $c_M \in (1, 25)$ corresponds to the supercritical regime.

Motivation-random planar maps

- Uniform planar maps, i.e., uniform planar triangulation are believed converge to LQG with $c_M = 0$, i.e., $\gamma = \sqrt{8/3}$.
- For other values of $\gamma \in (0, 2)$, γ-LQG is believed to be the scaling limit of planar maps decorated with statistical physics models.
- For example, a planar map decorated with spanning tree is believed to converge to γ-LQG for $\gamma = \sqrt{2}$.

Gaussian free fields (GFF)

- For a bounded open set U, and a compactly supported smooth function $f \in C_c^\infty(U)$ consider the Dirichlet norm of f given by

$$||f||_\nabla^2 = \int_U |\nabla f|^2 d^2 z.$$

- Let $H_0^1(U)$ denote the Hilbert space closure of $C_c^\infty(U)$ with respect to the Dirichlet norm.
- For an orthonormal basis $f_1, f_2, \ldots,$ of $H_0^1(U)$, zero boundary GFF is formally defined as

$$h = \sum_i X_i f_i$$

where X_i are i.i.d. $N(0,1)$.

Gaussian free fields (GFF)

- Does not make sense as a random function.
- We can make sense of this as a random generalised function (distribution) on U.
- We set

$$(h, \phi)_\nabla = \sum_{i=1}^{\infty} X_i (f_i, \phi)_\nabla$$

which is distributed as $N(0, \|\phi\|_\nabla^2)$.

Zero boundary GFF

- One can then define the action of h on $\phi \in C_c^\infty(U)$ by setting

$$(h, \phi) = -2\pi(h, \Delta^{-1}\phi)_\nabla$$

  where Δ^{-1} is the inverse Laplacian with zero boundary conditions.
- h is thus defined as a random element of the continuous dual of $C_c^\infty(U)$.
- Zero boundary GFF is conformally invariant: if $\Phi : U \to V$ is a conformal map, and h is a zero boundary GFF on V, then $h \circ \Phi$ is a zero boundary GFF on U.

Whole plane GFF

- On $\mathbb{C}$, constant functions have zero Dirichlet energy, it does not induce a norm on $H_0^1(\mathbb{C})$.
- One defines the whole plane GFF h similarly as before as a random element of $(C_c^\infty(\mathbb{C}))'$ modulo constants.
- To make sense of h as an element of the dual itself we shall fix the normalization
$$\mathbf{Av}(h, \mathbb{T}) = 0$$
where $\mathbf{Av}(h, \mathbb{T})$ denotes the GFF integrated against the uniform measure on the unit circle (one can show that this can be defined).
- With this normalization h is no longer conformally invariant. However, using conformal invariance at the level of a generalized function modulo constants, we get
$$h(r\cdot) - \mathbf{Av}(h, \mathbb{T}_r) \stackrel{d}{=} h(\cdot).$$

Key properties of GFF

Domain Markov property

Roughly speaking, on a domain U, one can write

$$h = h_1 + h_2$$

where h_1 is the harmonic extension of of the boundary condition on ∂U and h_2 is an independent GFF on U with zero boundary condition.

Circle average process

$$B(t) = \mathbf{Av}(h, \mathbb{T}_{e^t})$$

is distributed as a standard two-sided Brownian motion.

Mollification of GFF

- To define LQG measure and metric rigorously, one considers a mollification of GFF which is a random function and then takes an appropriate limit for the mollification parameter.
- One particular convenient choice of mollification is to take convolution with the heat kernel.

$$h_\varepsilon(z) = \int_{\mathbb{C}} h(w)\Big(\frac{1}{\pi\varepsilon^2} e^{-|z-w|^2/\varepsilon^2}\Big) d^2w.$$

- It is possible to consider other mollification schemes.

Renormalisation and LQG area measure

- Consider the family of random measures on $\mathbb{C}$:

$$\mu_{h,\varepsilon} = \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} d^2 z.$$

- For $\gamma \in (0,2)$, one can show that $\mu_{h,\varepsilon}$ converges to a random measure μ_h on $\mathbb{C}$ almost surely in the randomness of h.
- This limit is the LQG area measure.
- Duplantier-Sheffield (2011), Rhodes-Vargas (2014).

Renormalisation and LQG metric

- For $\xi > 0$, define the following metric induced by h_ε:

$$D_{h,\varepsilon}(z,w) = \inf_\pi \int_0^1 e^{\xi h_\varepsilon(\pi(t))} |\pi'(t)| dt$$

  where the infimum is taken over all sufficiently nice and properly parametrized paths between z and w.

- It is shown that there exists a family of constants a_ε such that

$$a_\varepsilon = \varepsilon^{1-\xi Q + o(1)}$$

  as $\varepsilon \to 0$ such that for $\xi < \xi(2) = \frac{2}{d_2}$ the metrics

$$\tilde{D}_{h,\varepsilon} = a_\varepsilon^{-1} D_{h,\varepsilon}$$

  converges in probability to a metric D_h.

Renormalisation and LQG metric

- Ding-Gwynne (2018): showed the existence of d_γ via discrete approximation schemes.
- Ding-Dubedat-Dunlop-Falconet (2020): showed the tightness of the renormalised metrics.
- Gwynne-Miller (2021): established uniqueness.

Basic properties of LQG metric

- $\mathbb{C}$, equipped with D_h is a length space.
- d_γ is the Hausdorff dimension of $\mathbb{C}$ equipped with the metric D_h.
- D_h is local, i.e., for any open set U, the internal metric $D_h(\cdot, \cdot; U)$ is determined by the restriction of h to U.
- Induces the Euclidean topology on $\mathbb{C}$ and is bi-Hölder with respect to the Euclidean metric (quantitative estimates available).

Basic properties of LQG metric

- One can similarly define the LQG metric $D_{\mathtt{h}}$ where $\mathtt{h} = h + f$, GFF plus a continuous function.
- Weyl Scaling: $D_{h+f} = e^{\xi f} \cdot D_h$ almost surely.
- Translation invariance: for each $z \in \mathbb{C}$, $D_{h(\cdot+z)} = D_h(\cdot + z, \cdot + z)$ almost surely.
- Coordinate change formula: almost surely for any fixed $r > 0$ and all $z, w \in \mathbb{C}$,
$$D_{h(r\cdot)}(rz, rw) = r^{\xi Q} D_{h(\cdot)}(z, w).$$

Geodesics in LQG

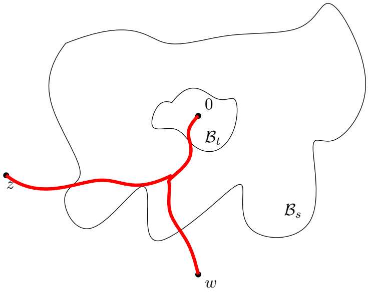
- Almost surely, for any two $z, w \in \mathbb{C}$ there exists a D_h geodesic between z and w. For fixed z and w the geodesic is almost surely unique, we shall denote it by $\Gamma(z,w)$. Geodesics will always be parametrized by the LQG distance.
- For each $z \in \mathbb{C}$, almost surely there exists a unique infinite geodesic started from z.
- Scale invariance of the geodesic: For the infinite geodesic Γ started from the origin we have

$$(h, \Gamma_{\cdot}) = (h(r\cdot) - \mathbf{Av}(h, \mathbb{T}_r), r^{-1}\Gamma_{r^{\xi Q} e^{\xi \mathbf{Av}(h,\mathbb{T}_r)}\cdot})$$

  in distribution.
- Almost surely there are no bigeodesics.
- Gwynne-Pfeffer-Sheffield (2020).

Coalescence of geodesics

- Fix $z \in \mathbb{C}$. For any $s > 0$, there is $t > 0$ such that all geodesics from z to points at LQG distance larger than s coincide within the LQG metric ball centred at z with radius t.

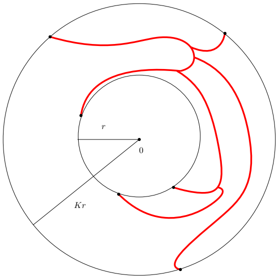


Coalescence of geodesics

- Let $\mathbf{Coal}_{r,K}$ denote the event

$$\mathbf{Coal}_{r,K} = \left\{ \cap_{z \in \mathbb{T}_r, w \in \mathbb{T}_{Kr}} \Gamma(z,w) \neq \emptyset \right\}.$$

- For K sufficiently large, $\mathbb{P}(\mathbf{Coal}_{r,K})$ is bounded away from 0 uniformly in r.



Critical and supercritical LQG metrics

- Many similar results have very recently been proved for critical and supercritical LQG metrics as well.
- The critical LQG metric still induces the Euclidean topology but is no longer bi-Holder with respect to the Euclidean metric.
- Supercritical LQG metric has Hausdorff dimension ∞.

References

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Thank you

Questions?