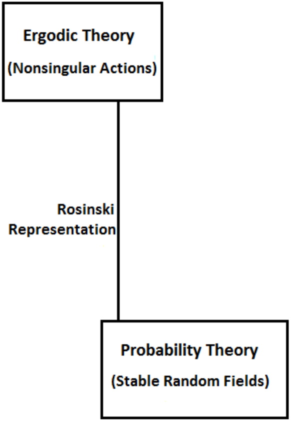
Group measure space construction, ergodicity and W^*-rigidity for stable random fields

Parthanil Roy, Indian Statistical Institute

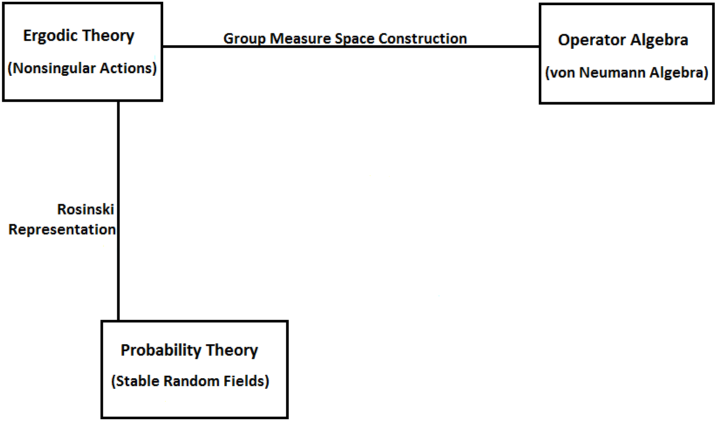
arXiv:2007.14821

Dedicated to the memory of Prof. Jayanta Kumar Ghosh

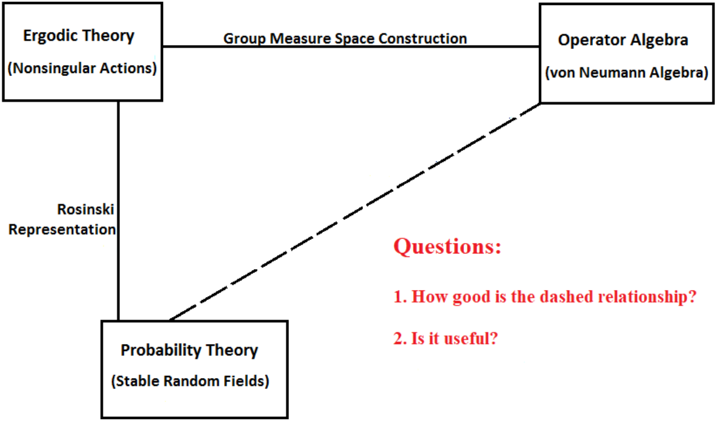
What is this work about?



What is this work about?



What is this work about?



Questions:

1. How good is the dashed relationship?

2. Is it useful?

A Crash Course on Stable Random Fields

Symmetric α-stable distribution

Symmetric α-stable distribution

All random objects (e.g., random variables, random vectors, random processes, random measures, etc.) discussed here will be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Corresponding expectation operator will be denoted by $\mathbb{E}(\cdot)$.

Symmetric α-stable distribution

All random objects (e.g., random variables, random vectors, random processes, random measures, etc.) discussed here will be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Corresponding expectation operator will be denoted by $\mathbb{E}(\cdot)$.

A random variable $X : \Omega \to \mathbb{R}$ (measurable) is said to follow $S\alpha S$ distribution $(0 < \alpha \leq 2)$ with scale parameter $\sigma > 0$ $\big($denoted by $X \sim S\alpha S(\sigma)$ $\big)$ if

$$\mathbb{E}(e^{i\theta X}) := \int_\Omega e^{i\theta X(\omega)} \mathbb{P}(d\omega) = e^{-\sigma^\alpha |\theta|^\alpha}, \quad \theta \in \mathbb{R}.$$

Symmetric α-stable distribution

All random objects (e.g., random variables, random vectors, random processes, random measures, etc.) discussed here will be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Corresponding expectation operator will be denoted by $\mathbb{E}(\cdot)$.

A random variable $X : \Omega \to \mathbb{R}$ (measurable) is said to follow $S\alpha S$ distribution $(0 < \alpha \leq 2)$ with scale parameter $\sigma > 0$ $\big($denoted by $X \sim S\alpha S(\sigma)$ $\big)$ if

$$\mathbb{E}(e^{i\theta X}) := \int_\Omega e^{i\theta X(\omega)} \mathbb{P}(d\omega) = e^{-\sigma^\alpha |\theta|^\alpha}, \qquad \theta \in \mathbb{R}.$$

- See, e.g., Samorodnitsky and Taqqu (1994).

Symmetric α-stable distribution

All random objects (e.g., random variables, random vectors, random processes, random measures, etc.) discussed here will be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Corresponding expectation operator will be denoted by $\mathbb{E}(\cdot)$.

A random variable $X : \Omega \to \mathbb{R}$ (measurable) is said to follow $S\alpha S$ distribution $(0 < \alpha \leq 2)$ with scale parameter $\sigma > 0$ $\big($denoted by $X \sim S\alpha S(\sigma)$ $\big)$ if

$$\mathbb{E}(e^{i\theta X}) := \int_\Omega e^{i\theta X(\omega)} \mathbb{P}(d\omega) = e^{-\sigma^\alpha |\theta|^\alpha}, \qquad \theta \in \mathbb{R}.$$

- See, e.g., Samorodnitsky and Taqqu (1994).
- $\alpha = 2 \;\Rightarrow\; X \sim$ Normal. $\qquad \alpha = 1 \;\Rightarrow X \;\sim$ Cauchy.

Symmetric α-stable distribution

All random objects (e.g., random variables, random vectors, random processes, random measures, etc.) discussed here will be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Corresponding expectation operator will be denoted by $\mathbb{E}(\cdot)$.

A random variable $X : \Omega \to \mathbb{R}$ (measurable) is said to follow $S\alpha S$ distribution $(0 < \alpha \leq 2)$ with scale parameter $\sigma > 0$ $\big($denoted by $X \sim S\alpha S(\sigma)$ $\big)$ if

$$\mathbb{E}(e^{i\theta X}) := \int_\Omega e^{i\theta X(\omega)} \mathbb{P}(d\omega) = e^{-\sigma^\alpha |\theta|^\alpha}, \qquad \theta \in \mathbb{R}.$$

- See, e.g., Samorodnitsky and Taqqu (1994).
- $\alpha = 2 \;\Rightarrow\; X \sim$ Normal. $\qquad \alpha = 1 \;\Rightarrow X \;\sim$ Cauchy.
- **Assume:** $0 < \alpha < 2$

Symmetric α-stable distribution

All random objects (e.g., random variables, random vectors, random processes, random measures, etc.) discussed here will be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Corresponding expectation operator will be denoted by $\mathbb{E}(\cdot)$.

A random variable $X : \Omega \to \mathbb{R}$ (measurable) is said to follow $S\alpha S$ distribution $(0 < \alpha \leq 2)$ with scale parameter $\sigma > 0$ $\big($denoted by $X \sim S\alpha S(\sigma)$ $\big)$ if

$$\mathbb{E}(e^{i\theta X}) := \int_\Omega e^{i\theta X(\omega)} \mathbb{P}(d\omega) = e^{-\sigma^\alpha |\theta|^\alpha}, \qquad \theta \in \mathbb{R}.$$

- See, e.g., Samorodnitsky and Taqqu (1994).
- $\alpha = 2 \;\Rightarrow\; X \sim$ Normal. $\qquad \alpha = 1 \;\Rightarrow X \;\sim$ Cauchy.
- **Assume:** $0 < \alpha < 2 \;\Rightarrow\; \mathbb{P}(|X| > x) \sim C_\alpha \, x^{-\alpha}$ as $x \to \infty$.

Symmetric α-stable distribution

All random objects (e.g., random variables, random vectors, random processes, random measures, etc.) discussed here will be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Corresponding expectation operator will be denoted by $\mathbb{E}(\cdot)$.

A random variable $X : \Omega \to \mathbb{R}$ (measurable) is said to follow $S\alpha S$ distribution $(0 < \alpha \le 2)$ with scale parameter $\sigma > 0$ $\big($denoted by $X \sim S\alpha S(\sigma)$ $\big)$ if

$$\mathbb{E}(e^{i\theta X}) := \int_\Omega e^{i\theta X(\omega)} \mathbb{P}(d\omega) = e^{-\sigma^\alpha |\theta|^\alpha}, \qquad \theta \in \mathbb{R}.$$

- See, e.g., Samorodnitsky and Taqqu (1994).
- $\alpha = 2 \;\Rightarrow\; X \sim$ Normal. $\qquad \alpha = 1 \;\Rightarrow X \;\sim$ Cauchy.
- **Assume:** $0 < \alpha < 2 \;\Rightarrow\; \mathbb{P}(|X| > x) \sim C_\alpha\, x^{-\alpha}$ as $x \to \infty$.
- In particular, $\mathbb{E}(|X|^p) < \infty$ if and only if $p < \alpha$.

Stationary $S\alpha S$ random fields

Stationary $S\alpha S$ random fields

Let $(G, .)$ be a countable (possibly noncommutative) group with identity element e.

Stationary $S\alpha S$ random fields

Let $(G, .)$ be a countable (possibly noncommutative) group with identity element e.

A collection $\{X_t\}_{t \in G}$ of random variables (all defined on $(\Omega, \mathcal{A}, \mathbb{P})$) is called an $S\alpha S$ random field if for all $k \geq 1$, for all $t_1, t_2, \ldots, t_k \in G$ and for all $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^{k} c_i X_{t_i} \sim S\alpha S.$$

Stationary $S\alpha S$ random fields

Let $(G, .)$ be a countable (possibly noncommutative) group with identity element e.

A collection $\{X_t\}_{t \in G}$ of random variables (all defined on $(\Omega, \mathcal{A}, \mathbb{P})$) is called an $S\alpha S$ random field if for all $k \geq 1$, for all $t_1, t_2, \ldots, t_k \in G$ and for all $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^{k} c_i X_{t_i} \sim S\alpha S.$$

An $S\alpha S$ random field $\{X_t\}_{t \in G}$ is (left) stationary if for all $s \in G$,

$$\{X_{s.t}\}_{t \in G} \stackrel{\mathcal{L}}{=} \{X_t\}_{t \in G}.$$

Stationary $S\alpha S$ random fields

Let $(G, .)$ be a countable (possibly noncommutative) group with identity element e.

A collection $\{X_t\}_{t \in G}$ of random variables (all defined on $(\Omega, \mathcal{A}, \mathbb{P})$) is called an $S\alpha S$ random field if for all $k \geq 1$, for all $t_1, t_2, \ldots, t_k \in G$ and for all $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^{k} c_i X_{t_i} \sim S\alpha S.$$

An $S\alpha S$ random field $\{X_t\}_{t \in G}$ is (left) stationary if for all $s \in G$,

$$\{X_{s.t}\}_{t \in G} \stackrel{\mathcal{L}}{=} \{X_t\}_{t \in G}.$$

Important special cases: $G = \mathbb{Z}$, $G = \mathbb{Z}^d$, $G = \mathbb{F}_d$ $(d > 1)$, discrete Heisenberg groups, discrete hyperbolic groups, lamplighter groups, etc.

Nonsingular G-action

Nonsingular G-action

Let $(G, \cdot)$ be a countable group with identity element e. $\{\phi_t\}_{t \in G}$ is called a nonsingular (also known as quasi-invariant) G-action on a σ-finite standard measure space $(S, \mathcal{S}, \mu)$ if

Nonsingular G-action

Let $(G, \cdot)$ be a countable group with identity element e. $\{\phi_t\}_{t \in G}$ is called a nonsingular (also known as quasi-invariant) G-action on a σ-finite standard measure space $(S, \mathcal{S}, \mu)$ if

- $\phi_t : S \to S$ is a measurable map for each $t \in G$,
- $\phi_e(s) = s$ for all $s \in S$,
- $\phi_{t_1 . t_2} = \phi_{t_2} \circ \phi_{t_1}$ for all $t_1, t_2 \in G$

Nonsingular G-action

Let $(G, \cdot)$ be a countable group with identity element e. $\{\phi_t\}_{t \in G}$ is called a nonsingular (also known as quasi-invariant) G-action on a σ-finite standard measure space $(S, \mathcal{S}, \mu)$ if

- $\phi_t : S \to S$ is a measurable map for each $t \in G$,
- $\phi_e(s) = s$ for all $s \in S$,
- $\phi_{t_1 . t_2} = \phi_{t_2} \circ \phi_{t_1}$ for all $t_1, t_2 \in G$ (i.e., $\phi_t : s \mapsto t^{-1}.s$),

Nonsingular G-action

Let $(G, \cdot)$ be a countable group with identity element e. $\{\phi_t\}_{t \in G}$ is called a nonsingular (also known as quasi-invariant) G-action on a σ-finite standard measure space $(S, \mathcal{S}, \mu)$ if

- $\phi_t : S \to S$ is a measurable map for each $t \in G$,
- $\phi_e(s) = s$ for all $s \in S$,
- $\phi_{t_1.t_2} = \phi_{t_2} \circ \phi_{t_1}$ for all $t_1, t_2 \in G$ (i.e., $\phi_t : s \mapsto t^{-1}.s$),
- $\mu \circ \phi_t \sim \mu$ for all $t \in G$

Nonsingular G-action

Let $(G, \cdot)$ be a countable group with identity element e. $\{\phi_t\}_{t \in G}$ is called a nonsingular (also known as quasi-invariant) G-action on a σ-finite standard measure space $(S, \mathcal{S}, \mu)$ if

- $\phi_t : S \to S$ is a measurable map for each $t \in G$,
- $\phi_e(s) = s$ for all $s \in S$,
- $\phi_{t_1 . t_2} = \phi_{t_2} \circ \phi_{t_1}$ for all $t_1, t_2 \in G$ (i.e., $\phi_t : s \mapsto t^{-1}.s$),
- $\mu \circ \phi_t \sim \mu$ for all $t \in G$ (important special case: $\mu \circ \phi_t = \mu$ for all t).

Nonsingular G-action

Let $(G, \cdot)$ be a countable group with identity element e. $\{\phi_t\}_{t \in G}$ is called a nonsingular (also known as quasi-invariant) G-action on a σ-finite standard measure space $(S, \mathcal{S}, \mu)$ if

- $\phi_t : S \to S$ is a measurable map for each $t \in G$,
- $\phi_e(s) = s$ for all $s \in S$,
- $\phi_{t_1.t_2} = \phi_{t_2} \circ \phi_{t_1}$ for all $t_1, t_2 \in G$ (i.e., $\phi_t : s \mapsto t^{-1}.s$),
- $\mu \circ \phi_t \sim \mu$ for all $t \in G$ (important special case: $\mu \circ \phi_t = \mu$ for all t).

See, for instance, Varadarajan (1970), Zimmer (1984), Krengel (1985), Aaronson (1997).

Rosinski (1995) representation of a stationary $S\alpha S$ field

Given a stationary $S\alpha S$ $(0 < \alpha < 2)$ random field $\{X_t\}_{t \in G}$, there exist

Rosinski (1995) representation of a stationary $S\alpha S$ field

Given a stationary $S\alpha S$ $(0 < \alpha < 2)$ random field $\{X_t\}_{t \in G}$, there exist

(i) a σ-finite standard measure space $(S, \mathcal{S}, \mu)$,

Rosinski (1995) representation of a stationary $S\alpha S$ field

Given a stationary $S\alpha S$ $(0 < \alpha < 2)$ random field $\{X_t\}_{t \in G}$, there exist

(i) a σ-finite standard measure space $(S, \mathcal{S}, \mu)$,

(ii) a function $f : S \to \mathbb{R}$ such that $\|f\|_\alpha := \left(\int |f|^\alpha d\mu \right)^{1/\alpha} < \infty$,

Rosinski (1995) representation of a stationary $S\alpha S$ field

Given a stationary $S\alpha S$ $(0 < \alpha < 2)$ random field $\{X_t\}_{t \in G}$, there exist

(i) a σ-finite standard measure space $(S, \mathcal{S}, \mu)$,

(ii) a function $f : S \to \mathbb{R}$ such that $\|f\|_\alpha := \left(\int |f|^\alpha d\mu\right)^{1/\alpha} < \infty$, and

(iii) a nonsingular G-action $\{\phi_t\}_{t \in G}$ on $(S, \mathcal{S}, \mu)$

Rosinski (1995) representation of a stationary $S\alpha S$ field

Given a stationary $S\alpha S$ $(0 < \alpha < 2)$ random field $\{X_t\}_{t \in G}$, there exist

(i) a σ-finite standard measure space $(S, \mathcal{S}, \mu)$,

(ii) a function $f : S \to \mathbb{R}$ such that $\|f\|_\alpha := \left(\int |f|^\alpha d\mu \right)^{1/\alpha} < \infty$, and

(iii) a nonsingular G-action $\{\phi_t\}_{t \in G}$ on $(S, \mathcal{S}, \mu)$

such that each real linear combination

$$\sum_{i=1}^{k} c_i X_{t_i} \sim S\alpha S\left(\left\| \sum_{i=1}^{k} c_i f_{t_i} \right\|_\alpha \right), \tag{1}$$

Rosinski (1995) representation of a stationary $S\alpha S$ field

Given a stationary $S\alpha S$ $(0 < \alpha < 2)$ random field $\{X_t\}_{t \in G}$, there exist

(i) a σ-finite standard measure space $(S, \mathcal{S}, \mu)$,

(ii) a function $f : S \to \mathbb{R}$ such that $\|f\|_\alpha := \left(\int |f|^\alpha d\mu \right)^{1/\alpha} < \infty$, and

(iii) a nonsingular G-action $\{\phi_t\}_{t \in G}$ on $(S, \mathcal{S}, \mu)$

such that each real linear combination

$$\sum_{i=1}^{k} c_i X_{t_i} \sim S\alpha S\left(\left\| \sum_{i=1}^{k} c_i f_{t_i} \right\|_\alpha \right), \tag{1}$$

where

$$f_t = \pm \left(\frac{d\mu \circ \phi_t}{d\mu} \right)^{1/\alpha} f \circ \phi_t\,, \quad t \in G.$$

Rosinski (1995) representation of a stationary $S\alpha S$ field

Given a stationary $S\alpha S$ $(0 < \alpha < 2)$ random field $\{X_t\}_{t \in G}$, there exist

(i) a σ-finite standard measure space $(S, \mathcal{S}, \mu)$,

(ii) a function $f : S \to \mathbb{R}$ such that $\|f\|_\alpha := \left(\int |f|^\alpha d\mu \right)^{1/\alpha} < \infty$, and

(iii) a nonsingular G-action $\{\phi_t\}_{t \in G}$ on $(S, \mathcal{S}, \mu)$

such that each real linear combination

$$\sum_{i=1}^{k} c_i X_{t_i} \sim S\alpha S \left(\left\| \sum_{i=1}^{k} c_i f_{t_i} \right\|_\alpha \right), \tag{1}$$

where

$$f_t = \pm \left(\frac{d\mu \circ \phi_t}{d\mu} \right)^{1/\alpha} f \circ \phi_t \,, \quad t \in G.$$

Converse also holds:

Rosinski (1995) representation of a stationary $S\alpha S$ field

Given a stationary $S\alpha S$ $(0 < \alpha < 2)$ random field $\{X_t\}_{t \in G}$, there exist

(i) a σ-finite standard measure space $(S, \mathcal{S}, \mu)$,

(ii) a function $f : S \to \mathbb{R}$ such that $\|f\|_\alpha := \left(\int |f|^\alpha d\mu\right)^{1/\alpha} < \infty$, and

(iii) a nonsingular G-action $\{\phi_t\}_{t \in G}$ on $(S, \mathcal{S}, \mu)$

such that each real linear combination

$$\sum_{i=1}^{k} c_i X_{t_i} \sim S\alpha S\left(\left\| \sum_{i=1}^{k} c_i f_{t_i} \right\|_\alpha\right), \tag{1}$$

where

$$f_t = \pm\left(\frac{d\mu \circ \phi_t}{d\mu}\right)^{1/\alpha} f \circ \phi_t\,, \quad t \in G.$$

Converse also holds: given (i), (ii) and (iii), there exists a stationary $S\alpha S$ random field $\{X_t\}_{t \in G}$ satisfying (1).

Rosinski (1995) representation of a stationary $S\alpha S$ field

Given a stationary $S\alpha S$ $(0<\alpha<2)$ random field $\{X_t\}_{t\in G}$, there exist

(i) a σ-finite standard measure space $(S,\mathcal{S},\mu)$,

(ii) a function $f:S\to\mathbb{R}$ such that $\|f\|_\alpha := \left(\int |f|^\alpha d\mu\right)^{1/\alpha} < \infty$, and

(iii) a nonsingular G-action $\{\phi_t\}_{t\in G}$ on $(S,\mathcal{S},\mu)$

such that each real linear combination

$$\sum_{i=1}^k c_i X_{t_i} \sim S\alpha S\left(\left\|\sum_{i=1}^k c_i f_{t_i}\right\|_\alpha\right), \tag{1}$$

where

$$f_t = \pm\left(\frac{d\mu\circ\phi_t}{d\mu}\right)^{1/\alpha} f\circ\phi_t\,, \quad t\in G.$$

Converse also holds: given (i), (ii) and (iii), there exists a stationary $S\alpha S$ random field $\{X_t\}_{t\in G}$ satisfying (1).

$\{f_t\}_{t\in G}$ = **a** Rosinski representation of $\{X_t\}$.

Properties of stationary stable fields via the action

Various probabilistic facets of a stable random field have been connected to the ergodic theoretic properties of the underlying nonsingular action:

Properties of stationary stable fields via the action

Various probabilistic facets of a stable random field have been connected to the ergodic theoretic properties of the underlying nonsingular action:

- Mixing features: Rosinski and Samorodnitsky (1996), Samorodnitsky (2005), Roy (2007, 2012), Wang, R. and Stoev (2013)
- Large deviations issues: Mikosch and Samorodnitsky (2000), Fasen and R. (2016)
- Growth of maxima: Samorodnitsky (2004), R. and Samorodnitsky (2008), Owada and Samorodnitsky (2015a), Sarkar and R. (2018), Athreya, Mj and R. (2019)
- Extremal point processes: Resnick and Samorodnitsky (2004), R. (2010), Sarkar and R. (2018)
- Statistical aspects: Bhattacharya and R. (2018)
- Uniform Hölder continuity of paths: Panigrahi, R. and Xiao (2018)

Properties of stationary stable fields via the action

Various probabilistic facets of a stable random field have been connected to the ergodic theoretic properties of the underlying nonsingular action:

- Mixing features: Rosinski and Samorodnitsky (1996), Samorodnitsky (2005), Roy (2007, 2012), Wang, R. and Stoev (2013)
- Large deviations issues: Mikosch and Samorodnitsky (2000), Fasen and R. (2016)
- Growth of maxima: Samorodnitsky (2004), R. and Samorodnitsky (2008), Owada and Samorodnitsky (2015a), Sarkar and R. (2018), Athreya, Mj and R. (2019)
- Extremal point processes: Resnick and Samorodnitsky (2004), R. (2010), Sarkar and R. (2018)
- Statistical aspects: Bhattacharya and R. (2018)
- Uniform Hölder continuity of paths: Panigrahi, R. and Xiao (2018)

The present work carries this link forward to the realm of von Neumann algebras via Murray and von Neumann (1936)'s crossed product construction.

A Crash Course on von Neumann Algebras

Topologies on operators

$\mathcal{B}(\mathcal{H}) :=$ all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

Topologies on operators

$\mathcal{B}(\mathcal{H}) :=$ all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\| \le 1} \|(T_\alpha - T)\xi\| \to 0$.

Topologies on operators

$\mathcal{B}(\mathcal{H}) :=$ all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\| \leq 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]

Topologies on operators

$\mathcal{B}(\mathcal{H}) :=$ all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\| \le 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]
  - Too strong and restrictive.

Topologies on operators

$\mathcal{B}(\mathcal{H}) :=$ all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\|\leq 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]
    - Too strong and restrictive.
    - $\mathcal{B}(\mathcal{H})$ may not be separable.

Topologies on operators

$\mathcal{B}(\mathcal{H}) :=$ all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\| \leq 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]
  - Too strong and restrictive.
  - $\mathcal{B}(\mathcal{H})$ may not be separable.
  - Difficult to carry out sophisticated analysis.

Topologies on operators

$\mathcal{B}(\mathcal{H})$:= all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\| \leq 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]
  - Too strong and restrictive.
  - $\mathcal{B}(\mathcal{H})$ may not be separable.
  - Difficult to carry out sophisticated analysis.

Topologies on operators

$\mathcal{B}(\mathcal{H})$:= all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\|\leq 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]
  - Too strong and restrictive.
  - $\mathcal{B}(\mathcal{H})$ may not be separable.
  - Difficult to carry out sophisticated analysis.

$\vee$

- **Strong operator topology** (not metrizable): $T_\alpha \to T$ in SOT iff $\|(T_\alpha - T)\xi\| \to 0$ for all $\xi \in \mathcal{H}$.

$\mathcal{B}(\mathcal{H})$:= all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\|\leq 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]
  - Too strong and restrictive.
  - $\mathcal{B}(\mathcal{H})$ may not be separable.
  - Difficult to carry out sophisticated analysis.

$\vee$

- **Strong operator topology** (not metrizable): $T_\alpha \to T$ in SOT iff $\|(T_\alpha - T)\xi\| \to 0$ for all $\xi \in \mathcal{H}$. [Topology of pointwise convergence on ($\mathcal{H}$, inner-product topology).]

Topologies on operators

$\mathcal{B}(\mathcal{H})$:= all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\| \le 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]
  - Too strong and restrictive.
  - $\mathcal{B}(\mathcal{H})$ may not be separable.
  - Difficult to carry out sophisticated analysis.

$\vee$

- **Strong operator topology** (not metrizable): $T_\alpha \to T$ in SOT iff $\|(T_\alpha - T)\xi\| \to 0$ for all $\xi \in \mathcal{H}$. [Topology of pointwise convergence on ($\mathcal{H}$, inner-product topology).]

$\vee$

Topologies on operators

$\mathcal{B}(\mathcal{H})$:= all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\| \le 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]
  - Too strong and restrictive.
  - $\mathcal{B}(\mathcal{H})$ may not be separable.
  - Difficult to carry out sophisticated analysis.

$\vee$

- **Strong operator topology** (not metrizable): $T_\alpha \to T$ in SOT iff $\|(T_\alpha - T)\xi\| \to 0$ for all $\xi \in \mathcal{H}$. [Topology of pointwise convergence on ($\mathcal{H}$, inner-product topology).]

$\vee$

- **Weak operator topology** (not metrizable): $T_\alpha \to T$ in SOT iff $\langle (T_\alpha - T)\xi, \eta \rangle \to 0$ for all $\xi, \eta \in \mathcal{H}$.

Topologies on operators

$\mathcal{B}(\mathcal{H}) :=$ all bdd linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

- **Norm topology** (metrizable): $T_\alpha \to T$ in NT iff $\|T_\alpha - T\| := \sup_{\|\xi\|\leq 1} \|(T_\alpha - T)\xi\| \to 0$. [Topology of uniform convergence on bounded subsets of ($\mathcal{H}$, inner-product topology).]
  - Too strong and restrictive.
  - $\mathcal{B}(\mathcal{H})$ may not be separable.
  - Difficult to carry out sophisticated analysis.

$\vee$

- **Strong operator topology** (not metrizable): $T_\alpha \to T$ in SOT iff $\|(T_\alpha - T)\xi\| \to 0$ for all $\xi \in \mathcal{H}$. [Topology of pointwise convergence on ($\mathcal{H}$, inner-product topology).]

$\vee$

- **Weak operator topology** (not metrizable): $T_\alpha \to T$ in SOT iff $\langle (T_\alpha - T)\xi, \eta\rangle \to 0$ for all $\xi, \eta \in \mathcal{H}$. [Topology of pointwise convergence on ($\mathcal{H}$, weak topology).]

Bicommutant theorem of von Neumann

Theorem (von Neumann)

Suppose M is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1, the identity operator. Then the following are equivalent:

1. *M is closed in weak operator topology.*
2. *M is closed in strong operator topology.*
3. $M = (M')' =: M''$.

Here $M' := \{T \in \mathcal{B}(\mathcal{H}) : TA = AT \text{ for all } A \in M\}$ is the commutant of M.

Bicommutant theorem of von Neumann

Theorem (von Neumann)

Suppose M is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1, the identity operator. Then the following are equivalent:

1. *M is closed in weak operator topology.*
2. *M is closed in strong operator topology.*
3. $M = (M')' =: M''$.

Here $M' := \{T \in \mathcal{B}(\mathcal{H}) : TA = AT \text{ for all } A \in M\}$ is the commutant of M.

The first two are analytic properties while the third one is an algebraic one.

Bicommutant theorem of von Neumann

Theorem (von Neumann)

Suppose M is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1, the identity operator. Then the following are equivalent:

1. *M is closed in weak operator topology.*
2. *M is closed in strong operator topology.*
3. $M = (M')' =: M''$.

Here $M' := \{T \in \mathcal{B}(\mathcal{H}) : TA = AT$ for all $A \in M\}$ is the commutant of M.

The first two are analytic properties while the third one is an algebraic one.

Definition (see, e.g, Sunder (1987), Jones (2009), Peterson (2013))

*A unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ satisfying one (and hence all) of the above equivalent conditions is called a **von Neumann algebra** (or a $\mathbf{W}^*$**-algebra**).*

The central decomposition

Note that if M is a von Neumann algebra, then so is M'. We now define a very important class (building blocks) of von Neumann algebras.

Definition

A von Neumann algebra M is called a factor if $Z(M) := M \cap M' := \{T \in M : TA = AT \text{ for all } A \in M\} = \mathbb{C}1$ (i.e., the centre is trivial).

The central decomposition

Note that if M is a von Neumann algebra, then so is M'. We now define a very important class (building blocks) of von Neumann algebras.

Definition

A von Neumann algebra M is called a factor if $Z(M) := M \cap M' := \{T \in M : TA = AT \text{ for all } A \in M\} = \mathbb{C}1$ (i.e., the centre is trivial).

Theorem (von Neumann)

Any von Neumann algebra can be decomposed as a direct sum (or more generally, "direct integral") of factors: there exists a measure space $(Y, \mathcal{Y}, \rho)$ such that

$$M = \int_Y M_y \, \rho(dy) \quad (\textit{direct integral; see Knudby (2011)}),$$

where M_y is a factor for ρ-almost all $y \in Y$.

The central decomposition

Note that if M is a von Neumann algebra, then so is M'. We now define a very important class (building blocks) of von Neumann algebras.

Definition

A von Neumann algebra M is called a factor if $Z(M) := M \cap M' := \{T \in M : TA = AT \text{ for all } A \in M\} = \mathbb{C}1$ (i.e., the centre is trivial).

Theorem (von Neumann)

Any von Neumann algebra can be decomposed as a direct sum (or more generally, "direct integral") of factors: there exists a measure space $(Y, \mathcal{Y}, \rho)$ such that

$$M = \int_Y M_y \, \rho(dy) \quad \text{(direct integral; see Knudby (2011))},$$

where M_y is a factor for ρ-almost all $y \in Y$.

Enough (for a von Neumann algebraist) to study and classify factors.

Type II_1 factors

"Definition"

A factor M is of type II_1 if M is inifinite-dimensional and it admits a normalized trace, i.e., there exists a "continuous" linear functional $tr : M \to \mathbb{C}$ satisfying $tr(1) = 1$, $tr(ab) = tr(ba)$ and $tr(a^*a) \geq 0$ for all $a, b \in M$.

Type II_1 factors

"Definition"

A factor M is of type II_1 if M is inifinite-dimensional and it admits a normalized trace, i.e., there exists a "continuous" linear functional $tr : M \to \mathbb{C}$ satisfying $tr(1) = 1$, $tr(ab) = tr(ba)$ and $tr(a^*a) \geq 0$ for all $a, b \in M$.

Definition (R. (2020+))

A von Neumann algebra M is said to admit no II_1 factor in its central decomposition if M has a central decomposition

$$M = \int_Y M_y \, \rho(dy) \;\; \textit{(direct integral)},$$

such that for ρ-almost all $y \in Y$, M_y is not a factor of type II_1.

Type II_1 factors

"Definition"

A factor M is of type II_1 if M is inifinite-dimensional and it admits a normalized trace, i.e., there exists a "continuous" linear functional $tr : M \to \mathbb{C}$ satisfying $tr(1) = 1$, $tr(ab) = tr(ba)$ and $tr(a^*a) \geq 0$ for all $a, b \in M$.

Definition (R. (2020+))

A von Neumann algebra M is said to admit no II_1 factor in its central decomposition if M has a central decomposition

$$M = \int_Y M_y \, \rho(dy) \quad \textit{(direct integral)},$$

such that for ρ-almost all $y \in Y$, M_y is not a factor of type II_1.

If Y is countable with ρ being the counting measure, then the direct integral becomes a direct sum $(M = \oplus_{y \in Y} M_y)$ of factors.

Type II_1 factors

"Definition"

A factor M is of type II_1 if M is inifinite-dimensional and it admits a normalized trace, i.e., there exists a "continuous" linear functional $tr : M \to \mathbb{C}$ satisfying $tr(1) = 1$, $tr(ab) = tr(ba)$ and $tr(a^*a) \geq 0$ for all $a, b \in M$.

Definition (R. (2020+))

A von Neumann algebra M is said to admit no II_1 factor in its central decomposition if M has a central decomposition

$$M = \int_Y M_y \, \rho(dy) \quad \textit{(direct integral)},$$

such that for ρ-almost all $y \in Y$, M_y is not a factor of type II_1.

If Y is countable with ρ being the counting measure, then the direct integral becomes a direct sum ($M = \oplus_{y \in Y} M_y$) of factors. In this special case, the above definition is equivalent to saying no M_y is a type II_1 factor.

Recall: nonsingular G-action

Let $(G, \cdot)$ be a countable group with identity element e. $\{\phi_t\}_{t \in G}$ is called a nonsingular (also known as quasi-invariant) G-action on a σ-finite standard measure space $(S, \mathcal{S}, \mu)$ if

- $\phi_t : S \to S$ is a measurable map for each $t \in G$,
- $\phi_e(s) = s$ for all $s \in S$,
- $\phi_{t_1 . t_2} = \phi_{t_2} \circ \phi_{t_1}$ for all $t_1, t_2 \in G$,
- $\mu \circ \phi_t \sim \mu$ for all $t \in G$.

"Group measure space construction"

- $(G, \cdot)$ is a countable group with identity element e.
- $(S, \mathcal{S}, \mu)$ is a σ-finite standard measure space
- $\{\phi_t\}_{t \in G}$ is a nonsingular G-action on $(S, \mathcal{S}, \mu)$

“Group measure space construction”

- $(G, \cdot)$ is a countable group with identity element e.
- $(S, \mathcal{S}, \mu)$ is a σ-finite standard measure space
- $\{\phi_t\}_{t \in G}$ is a nonsingular G-action on $(S, \mathcal{S}, \mu)$

“Definition” (see, e.g, Jones (2009), Peterson (2013))

Following the work of Murray and von Neumann (1936) (in the measure-preserving case), one can construct a von Neumann algebra (as a subalgebra of $\mathcal{B}(\ell^2(G) \otimes \mathcal{L}^2(S, \mu))$) that “encodes the ergodic theoretic features” of $\{\phi_t\}_{t \in G}$ by internalizing a crossed product relation that normalizes $\mathcal{L}^\infty(S, \mu)$ inside $\mathcal{B}(\mathcal{L}^2(S, \mu))$ through the Koopman representation. This von Neumann algebra is called group measure space construction.

"Group measure space construction"

- $(G, \cdot)$ is a countable group with identity element e.
- $(S, \mathcal{S}, \mu)$ is a σ-finite standard measure space
- $\{\phi_t\}_{t \in G}$ is a nonsingular G-action on $(S, \mathcal{S}, \mu)$

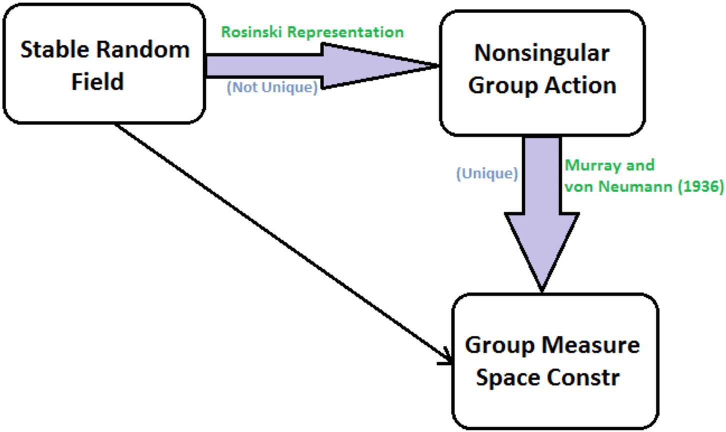
"Definition" (see, e.g, Jones (2009), Peterson (2013))

Following the work of Murray and von Neumann (1936) (in the measure-preserving case), one can construct a von Neumann algebra (as a subalgebra of $\mathcal{B}(\ell^2(G) \otimes \mathcal{L}^2(S, \mu))$) that "encodes the ergodic theoretic features" of $\{\phi_t\}_{t \in G}$ by internalizing a crossed product relation that normalizes $\mathcal{L}^\infty(S, \mu)$ inside $\mathcal{B}(\mathcal{L}^2(S, \mu))$ through the Koopman representation. This von Neumann algebra is called group measure space construction.

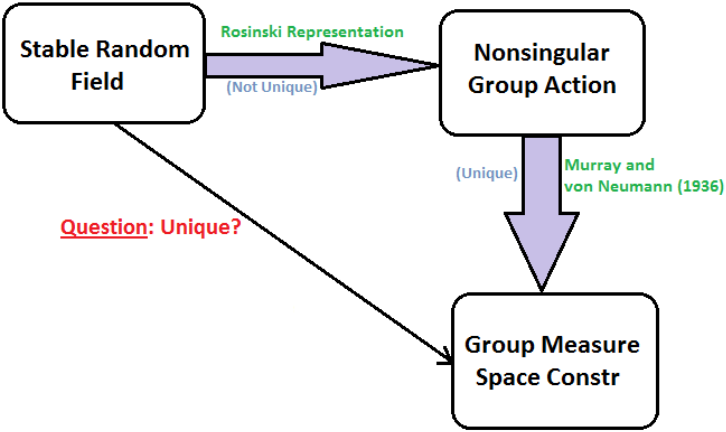
Notation: $\mathcal{L}^\infty(S, \mu) \rtimes_{\{\phi_t\}} G$ or simply $\mathcal{L}^\infty(S, \mu) \rtimes G$.

Linking Stable Random Fields with von Neumann Algebras

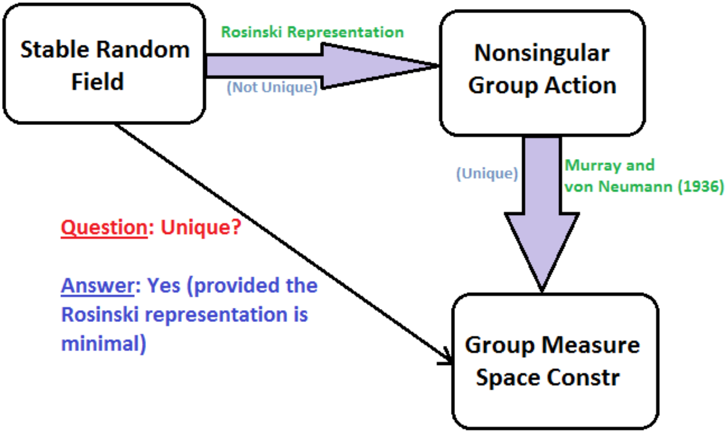
How good is the connection?



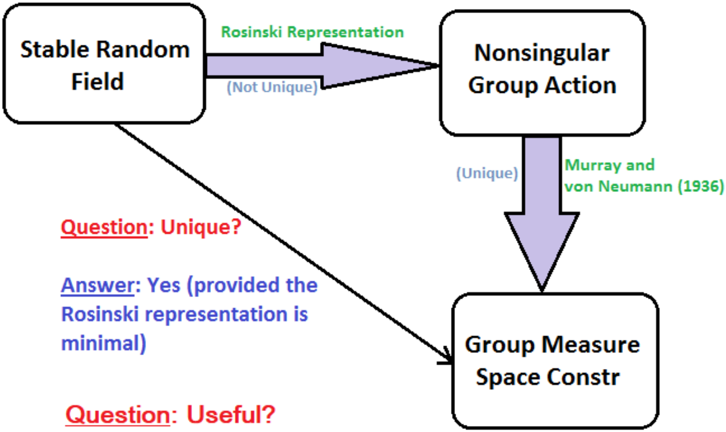
How good is the connection?



How good is the connection?



How good is the connection?



Question: Unique?

Answer: Yes (provided the Rosinski representation is minimal)

Question: Useful?

The minimal group measure space construction

Theorem (R. (2020+))

Suppose $\{X_t\}_{t\in G}$ is a (left) stationary $S\alpha S$ random field indexed by a countable group G. Let $\{\phi_t^{(1)}\}_{t\in G}$ and $\{\phi_t^{(2)}\}_{t\in G}$ be two nonsingular G-actions (on $(S^{(1)},\mu^{(1)})$ and $(S^{(2)},\mu^{(2)})$, respectively) obtained from two minimal (and hence Rosinski) representations. Then

$$\mathcal{L}^\infty(S^{(1)},\mu^{(1)}) \rtimes G \cong \mathcal{L}^\infty(S^{(2)},\mu^{(2)}) \rtimes G$$

as von Neumann algebras. In particular, group measure space construction is an invariant for any minimal representation of a fixed stationary $S\alpha S$ random field.

The minimal group measure space construction

Theorem (R. (2020+))

Suppose $\{X_t\}_{t\in G}$ *is a (left) stationary* $S\alpha S$ *random field indexed by a countable group* G*. Let* $\{\phi_t^{(1)}\}_{t\in G}$ *and* $\{\phi_t^{(2)}\}_{t\in G}$ *be two nonsingular* G*-actions (on* $(S^{(1)},\mu^{(1)})$ *and* $(S^{(2)},\mu^{(2)})$*, respectively) obtained from two minimal (and hence Rosinski) representations. Then*

$$\mathcal{L}^\infty(S^{(1)},\mu^{(1)}) \rtimes G \cong \mathcal{L}^\infty(S^{(2)},\mu^{(2)}) \rtimes G$$

as von Neumann algebras. In particular, group measure space construction is an invariant for any minimal representation of a fixed stationary $S\alpha S$ *random field.*

Sketch of proof.

The minimal group measure space construction

Theorem (R. (2020+))

Suppose $\{X_t\}_{t \in G}$ is a (left) stationary $S\alpha S$ random field indexed by a countable group G. Let $\{\phi_t^{(1)}\}_{t \in G}$ and $\{\phi_t^{(2)}\}_{t \in G}$ be two nonsingular G-actions (on $(S^{(1)}, \mu^{(1)})$ and $(S^{(2)}, \mu^{(2)})$, respectively) obtained from two minimal (and hence Rosinski) representations. Then

$$\mathcal{L}^\infty(S^{(1)}, \mu^{(1)}) \rtimes G \cong \mathcal{L}^\infty(S^{(2)}, \mu^{(2)}) \rtimes G$$

as von Neumann algebras. In particular, group measure space construction is an invariant for any minimal representation of a fixed stationary $S\alpha S$ random field.

Sketch of proof.

$\{\phi_t^{(1)}\} \cong \{\phi_t^{(2)}\}$ as group actions (extension of Theorem 3.6 of Rosinski (1995))

The minimal group measure space construction

Theorem (R. (2020+))

Suppose $\{X_t\}_{t\in G}$ is a (left) stationary $S\alpha S$ random field indexed by a countable group G. Let $\{\phi_t^{(1)}\}_{t\in G}$ and $\{\phi_t^{(2)}\}_{t\in G}$ be two nonsingular G-actions (on $(S^{(1)},\mu^{(1)})$ and $(S^{(2)},\mu^{(2)})$, respectively) obtained from two minimal (and hence Rosinski) representations. Then

$$\mathcal{L}^\infty(S^{(1)},\mu^{(1)}) \rtimes G \cong \mathcal{L}^\infty(S^{(2)},\mu^{(2)}) \rtimes G$$

as von Neumann algebras. In particular, group measure space construction is an invariant for any minimal representation of a fixed stationary $S\alpha S$ random field.

Sketch of proof.

$\{\phi_t^{(1)}\} \cong \{\phi_t^{(2)}\}$ as group actions (extension of Theorem 3.6 of Rosinski (1995))
$\Rightarrow$ they are "orbit equivalent"

The minimal group measure space construction

Theorem (R. (2020+))

Suppose $\{X_t\}_{t\in G}$ *is a (left) stationary* $S\alpha S$ *random field indexed by a countable group* G*. Let* $\{\phi_t^{(1)}\}_{t\in G}$ *and* $\{\phi_t^{(2)}\}_{t\in G}$ *be two nonsingular* G*-actions (on* $(S^{(1)},\mu^{(1)})$ *and* $(S^{(2)},\mu^{(2)})$*, respectively) obtained from two minimal (and hence Rosinski) representations. Then*

$$\mathcal{L}^\infty(S^{(1)},\mu^{(1)}) \rtimes G \cong \mathcal{L}^\infty(S^{(2)},\mu^{(2)}) \rtimes G$$

as von Neumann algebras. In particular, group measure space construction is an invariant for any minimal representation of a fixed stationary $S\alpha S$ *random field.*

Sketch of proof.

$\{\phi_t^{(1)}\} \cong \{\phi_t^{(2)}\}$ as group actions (extension of Theorem 3.6 of Rosinski (1995))
$\Rightarrow$ they are "orbit equivalent" $\Rightarrow$ $\mathcal{L}^\infty(S^{(1)},\mu^{(1)}) \rtimes G \cong \mathcal{L}^\infty(S^{(2)},\mu^{(2)}) \rtimes G$

The minimal group measure space construction

Theorem (R. (2020+))

Suppose $\{X_t\}_{t\in G}$ is a (left) stationary $S\alpha S$ random field indexed by a countable group G. Let $\{\phi_t^{(1)}\}_{t\in G}$ and $\{\phi_t^{(2)}\}_{t\in G}$ be two nonsingular G-actions (on $(S^{(1)},\mu^{(1)})$ and $(S^{(2)},\mu^{(2)})$, respectively) obtained from two minimal (and hence Rosinski) representations. Then

$$\mathcal{L}^\infty(S^{(1)},\mu^{(1)}) \rtimes G \cong \mathcal{L}^\infty(S^{(2)},\mu^{(2)}) \rtimes G$$

as von Neumann algebras. In particular, group measure space construction is an invariant for any minimal representation of a fixed stationary $S\alpha S$ random field.

Sketch of proof.

$\{\phi_t^{(1)}\} \cong \{\phi_t^{(2)}\}$ as group actions (extension of Theorem 3.6 of Rosinski (1995)) $\Rightarrow$ they are "orbit equivalent" $\Rightarrow$ $\mathcal{L}^\infty(S^{(1)},\mu^{(1)}) \rtimes G \cong \mathcal{L}^\infty(S^{(2)},\mu^{(2)}) \rtimes G$ by a seminal result of Singer (1955). $\square$

A W^*-rigidity question

Summary: Minimal group measure space construction is an invariant for any stationary $S\alpha S$ random field.

A W^*-rigidity question

Summary: Minimal group measure space construction is an invariant for any stationary $S\alpha S$ random field.

Same holds for stationary max-stable fields by an extension of Proposition 6.1 in Wang and Stoev (2009).

A W^*-rigidity question

Summary: Minimal group measure space construction is an invariant for any stationary $S\alpha S$ random field.

Same holds for stationary max-stable fields by an extension of Proposition 6.1 in Wang and Stoev (2009).

Question: How much does this invariant "remember" the random field?

A W^*-rigidity question

Summary: Minimal group measure space construction is an invariant for any stationary $S\alpha S$ random field.

Same holds for stationary max-stable fields by an extension of Proposition 6.1 in Wang and Stoev (2009).

Question: How much does this invariant "remember" the random field?

In other words, if two stationary $S\alpha S$ random fields (not necessarily indexed by the same group) have isomorphic minimal group measure space constructions, then do they have similar probabilistic properties?

A W^*-rigidity question

Summary: Minimal group measure space construction is an invariant for any stationary $S\alpha S$ random field.

Same holds for stationary max-stable fields by an extension of Proposition 6.1 in Wang and Stoev (2009).

Question: How much does this invariant "remember" the random field?

In other words, if two stationary $S\alpha S$ random fields (not necessarily indexed by the same group) have isomorphic minimal group measure space constructions, then do they have similar probabilistic properties?

This question parallels the theory of W^*-rigidity (a term coined by Sorin Popa - see the survey of Ioana (2018)) for group actions.

A W^*-rigidity question

Summary: Minimal group measure space construction is an invariant for any stationary $S\alpha S$ random field.

Same holds for stationary max-stable fields by an extension of Proposition 6.1 in Wang and Stoev (2009).

Question: How much does this invariant "remember" the random field?

In other words, if two stationary $S\alpha S$ random fields (not necessarily indexed by the same group) have isomorphic minimal group measure space constructions, then do they have similar probabilistic properties?

This question parallels the theory of W^*-rigidity (a term coined by Sorin Popa - see the survey of Ioana (2018)) for group actions.

See also the ICM 2018 lecture of Adrian Ioana from YouTube.

What about any Rosinski representation?

What about any Rosinski representation?

- **Another W^*-rigidity question:** How much does a Rosinski (not necessarily the minimal) group measure space construction remember the stable random field?

What about any Rosinski representation?

- **Another W^*-rigidity question:** How much does a Rosinski (not necessarily the minimal) group measure space construction remember the stable random field?

- Since any Rosinski representation "can be written in terms of" any minimal representation, we conjecture that many von Neumann algebraic aspects of the corresponding group measure space construction will become invariants and many stochastic properties of the field will be remembered.

What about any Rosinski representation?

- **Another W^*-rigidity question:** How much does a Rosinski (not necessarily the minimal) group measure space construction remember the stable random field?

- Since any Rosinski representation "can be written in terms of" any minimal representation, we conjecture that many von Neumann algebraic aspects of the corresponding group measure space construction will become invariants and many stochastic properties of the field will be remembered.

- We have exhibited two such instances in this work when $G = \mathbb{Z}^d$

What about any Rosinski representation?

- **Another W^*-rigidity question:** How much does a Rosinski (not necessarily the minimal) group measure space construction remember the stable random field?

- Since any Rosinski representation "can be written in terms of" any minimal representation, we conjecture that many von Neumann algebraic aspects of the corresponding group measure space construction will become invariants and many stochastic properties of the field will be remembered.

- We have exhibited two such instances in this work when $G = \mathbb{Z}^d$ - both ergodicity and complete non-ergodicity are W^*-rigid properties for stable random fields.

What about any Rosinski representation?

- **Another W^*-rigidity question:** How much does a Rosinski (not necessarily the minimal) group measure space construction remember the stable random field?

- Since any Rosinski representation "can be written in terms of" any minimal representation, we conjecture that many von Neumann algebraic aspects of the corresponding group measure space construction will become invariants and many stochastic properties of the field will be remembered.

- We have exhibited two such instances in this work when $G = \mathbb{Z}^d$ - both ergodicity and complete non-ergodicity are W^*-rigid properties for stable random fields.

- From now on $G = \mathbb{Z}^d$ (unless mentioned otherwise).

Operator Algebraic Characterization of Ergodicity for Stable Random Fields ($G = \mathbb{Z}^d$)

Ergodicity of $\mathbb{Z}^d$-indexed stable fields

Recall that any stationary SαS random field $\mathbf{X} = \{X_t\}_{t \in \mathbb{Z}^d}$ induces a measure-preserving shift action (of $\mathbb{Z}^d$) on $\left(\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}_{\mathbf{X}}\right)$, where

$$\mathbb{P}_{\mathbf{X}} = \text{ law of } \mathbf{X} := \mathbb{P}\Big(\big\{\omega \in \Omega : \big(X_t(\omega) : t \in \mathbb{Z}^d\big) \in \cdot\big\}\Big).$$

Ergodicity of $\mathbb{Z}^d$-indexed stable fields

Recall that any stationary SαS random field $\mathbf{X} = \{X_t\}_{t \in \mathbb{Z}^d}$ induces a measure-preserving shift action (of $\mathbb{Z}^d$) on $\left(\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}_{\mathbf{X}}\right)$, where

$$\mathbb{P}_{\mathbf{X}} = \text{ law of } \mathbf{X} := \mathbb{P}\Big(\big\{\omega \in \Omega : \big(X_t(\omega) : t \in \mathbb{Z}^d\big) \in \cdot\big\}\Big).$$

Definition

$\{X_t\}_{t \in \mathbb{Z}^d}$ is called ergodic if the above shift action is so, i,e., the shift-invariant subsets of $\mathbb{R}^{\mathbb{Z}^d}$ are $\mathbb{P}_{\mathbf{X}}$-trivial.

Ergodicity of $\mathbb{Z}^d$-indexed stable fields

Recall that any stationary SαS random field $\mathbf{X} = \{X_t\}_{t \in \mathbb{Z}^d}$ induces a measure-preserving shift action (of $\mathbb{Z}^d$) on $\left(\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}_{\mathbf{X}}\right)$, where

$$\mathbb{P}_{\mathbf{X}} = \text{ law of } \mathbf{X} := \mathbb{P}\Big(\big\{\omega \in \Omega : \big(X_t(\omega) : t \in \mathbb{Z}^d\big) \in \cdot\big\}\Big).$$

Definition

$\{X_t\}_{t \in \mathbb{Z}^d}$ is called ergodic if the above shift action is so, i,e., the shift-invariant subsets of $\mathbb{R}^{\mathbb{Z}^d}$ are $\mathbb{P}_{\mathbf{X}}$-trivial.

Question: When is $\{X_t\}_{t \in \mathbb{Z}^d}$ ergodic?

Ergodicity of $\mathbb{Z}^d$-indexed stable fields

Recall that any stationary SαS random field $\mathbf{X} = \{X_t\}_{t \in \mathbb{Z}^d}$ induces a measure-preserving shift action (of $\mathbb{Z}^d$) on $\left(\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}_{\mathbf{X}}\right)$, where

$$\mathbb{P}_{\mathbf{X}} = \text{ law of } \mathbf{X} := \mathbb{P}\Big(\big\{\omega \in \Omega : \big(X_t(\omega) : t \in \mathbb{Z}^d\big) \in \cdot\big\}\Big).$$

Definition

$\{X_t\}_{t \in \mathbb{Z}^d}$ is called ergodic if the above shift action is so, i,e., the shift-invariant subsets of $\mathbb{R}^{\mathbb{Z}^d}$ are $\mathbb{P}_{\mathbf{X}}$-trivial.

Question: When is $\{X_t\}_{t \in \mathbb{Z}^d}$ ergodic? [Helps in proving limit theorems.]

Ergodicity of $\mathbb{Z}^d$-indexed stable fields

Recall that any stationary SαS random field $\mathbf{X} = \{X_t\}_{t \in \mathbb{Z}^d}$ induces a measure-preserving shift action (of $\mathbb{Z}^d$) on $\left(\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}_{\mathbf{X}}\right)$, where

$$\mathbb{P}_{\mathbf{X}} = \text{ law of } \mathbf{X} := \mathbb{P}\Big(\big\{\omega \in \Omega : \big(X_t(\omega) : t \in \mathbb{Z}^d\big) \in \cdot\big\}\Big).$$

Definition

$\{X_t\}_{t \in \mathbb{Z}^d}$ is called ergodic if the above shift action is so, i,e., the shift-invariant subsets of $\mathbb{R}^{\mathbb{Z}^d}$ are $\mathbb{P}_{\mathbf{X}}$-trivial.

Question: When is $\{X_t\}_{t \in \mathbb{Z}^d}$ ergodic? [Helps in proving limit theorems.]

- $d = 1$: Samorodnitsky (2005) gave a criterion based on the ergodic theoretic properties of the underlying action.

Ergodicity of $\mathbb{Z}^d$-indexed stable fields

Recall that any stationary SαS random field $\mathbf{X} = \{X_t\}_{t\in\mathbb{Z}^d}$ induces a measure-preserving shift action (of $\mathbb{Z}^d$) on $\left(\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}_{\mathbf{X}}\right)$, where

$$\mathbb{P}_{\mathbf{X}} = \text{ law of } \mathbf{X} := \mathbb{P}\Big(\big\{\omega \in \Omega : \big(X_t(\omega) : t \in \mathbb{Z}^d\big) \in \cdot\big\}\Big).$$

Definition

$\{X_t\}_{t\in\mathbb{Z}^d}$ is called ergodic if the above shift action is so, i,e., the shift-invariant subsets of $\mathbb{R}^{\mathbb{Z}^d}$ are $\mathbb{P}_{\mathbf{X}}$-trivial.

Question: When is $\{X_t\}_{t\in\mathbb{Z}^d}$ ergodic? [Helps in proving limit theorems.]

- $d = 1$: Samorodnitsky (2005) gave a criterion based on the ergodic theoretic properties of the underlying action.
- $d > 1$: Wang, R. and Stoev (2013) extended the above work.

Ergodicity of $\mathbb{Z}^d$-indexed stable fields

Recall that any stationary SαS random field $\mathbf{X} = \{X_t\}_{t \in \mathbb{Z}^d}$ induces a measure-preserving shift action (of $\mathbb{Z}^d$) on $\left(\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}_{\mathbf{X}}\right)$, where

$$\mathbb{P}_{\mathbf{X}} = \text{ law of } \mathbf{X} := \mathbb{P}\Big(\big\{\omega \in \Omega : \big(X_t(\omega) : t \in \mathbb{Z}^d\big) \in \cdot\big\}\Big).$$

Definition

$\{X_t\}_{t \in \mathbb{Z}^d}$ is called ergodic if the above shift action is so, i,e., the shift-invariant subsets of $\mathbb{R}^{\mathbb{Z}^d}$ are $\mathbb{P}_{\mathbf{X}}$-trivial.

Question: When is $\{X_t\}_{t \in \mathbb{Z}^d}$ ergodic? [Helps in proving limit theorems.]

- $d = 1$: Samorodnitsky (2005) gave a criterion based on the ergodic theoretic properties of the underlying action.
- $d > 1$: Wang, R. and Stoev (2013) extended the above work.
- R. (2020+): New characterization using group measure space construction for $d \geq 1$.

Ergodicty via von Neumann algebas

Theorem (R. (2020+))

Suppose $\{X_t\}_{t \in \mathbb{Z}^d}$ is a stationary $S\alpha S$ random field generated by a free nonsingular action $\{\phi_t\}_{t \in \mathbb{Z}^d}$. Then $\{X_t\}_{t \in \mathbb{Z}^d}$ is ergodic (equiv., weakly mixing) if and only if the corresponding group measure space construction admits no II_1 factor in its central decomposition.

Ergodicty via von Neumann algebas

Theorem (R. (2020+))

Suppose $\{X_t\}_{t\in\mathbb{Z}^d}$ is a stationary $S\alpha S$ random field generated by a free nonsingular action $\{\phi_t\}_{t\in\mathbb{Z}^d}$. Then $\{X_t\}_{t\in\mathbb{Z}^d}$ is ergodic (equiv., weakly mixing) if and only if the corresponding group measure space construction admits no II_1 factor in its central decomposition.

Corollary (R. (2020+))

"Admitting no II_1 factor in the central decomposition" is an invariant for any "free Rosinski group measure space construction" of a fixed stationary stable random field indexed by $\mathbb{Z}^d$.

Ergodicty via von Neumann algebas

Theorem (R. (2020+))

Suppose $\{X_t\}_{t\in\mathbb{Z}^d}$ is a stationary $S\alpha S$ random field generated by a free nonsingular action $\{\phi_t\}_{t\in\mathbb{Z}^d}$. Then $\{X_t\}_{t\in\mathbb{Z}^d}$ is ergodic (equiv., weakly mixing) if and only if the corresponding group measure space construction admits no II_1 factor in its central decomposition.

Corollary (R. (2020+))

"Admitting no II_1 factor in the central decomposition" is an invariant for any "free Rosinski group measure space construction" of a fixed stationary stable random field indexed by $\mathbb{Z}^d$.

In other words, if a free Rosinski group measure space construction of such a random field admits no II_1 factor in its central decomposition, then the same is true about any free Rosinski group measure space construction of that random field.

Connection to orbit equivalence

Corollary (R. (2020+) - Ergodicity is W^*-rigid and hence OE-rigid)

If two stationary $S\alpha S$ random fields indexed by $\mathbb{Z}^d$ (possibly with two different d's) have isomorphic free Rosinski group measure space constructions, then one is ergodic (equiv., weakly mixing) if and only if the other one is so.

Connection to orbit equivalence

Corollary (R. (2020+) - Ergodicity is W^*-rigid and hence OE-rigid)

If two stationary $S\alpha S$ random fields indexed by $\mathbb{Z}^d$ (possibly with two different d's) have isomorphic free Rosinski group measure space constructions, then one is ergodic (equiv., weakly mixing) if and only if the other one is so. In particular, if they are generated by orbit equivalent free actions, then also the same conclusion holds.

Connection to orbit equivalence

Corollary (R. (2020+) - Ergodicity is W^*-rigid and hence OE-rigid)

If two stationary $S\alpha S$ random fields indexed by $\mathbb{Z}^d$ (possibly with two different d's) have isomorphic free Rosinski group measure space constructions, then one is ergodic (equiv., weakly mixing) if and only if the other one is so. In particular, if they are generated by orbit equivalent free actions, then also the same conclusion holds.

The indexing groups having possibly different ranks (as $\mathbb{Z}$-modules) is actually very useful in the context of orbit equivalence.

Connection to orbit equivalence

Corollary (R. (2020+) - Ergodicity is W^*-rigid and hence OE-rigid)

If two stationary $S\alpha S$ random fields indexed by $\mathbb{Z}^d$ (possibly with two different d's) have isomorphic free Rosinski group measure space constructions, then one is ergodic (equiv., weakly mixing) if and only if the other one is so. In particular, if they are generated by orbit equivalent free actions, then also the same conclusion holds.

The indexing groups having possibly different ranks (as $\mathbb{Z}$-modules) is actually very useful in the context of orbit equivalence.

The seminal result of Connes, Feldman and Weiss (1981) states that any nonsingular action of $\mathbb{Z}^d$ (more generally, of any amenable group) is orbit equivalent to a nonsingular $\mathbb{Z}$-action.

Connection to orbit equivalence

Corollary (R. (2020+) - Ergodicity is W^*-rigid and hence OE-rigid)

If two stationary $S\alpha S$ random fields indexed by $\mathbb{Z}^d$ (possibly with two different d's) have isomorphic free Rosinski group measure space constructions, then one is ergodic (equiv., weakly mixing) if and only if the other one is so. In particular, if they are generated by orbit equivalent free actions, then also the same conclusion holds.

The indexing groups having possibly different ranks (as $\mathbb{Z}$-modules) is actually very useful in the context of orbit equivalence.

The seminal result of Connes, Feldman and Weiss (1981) states that any nonsingular action of $\mathbb{Z}^d$ (more generally, of any amenable group) is orbit equivalent to a nonsingular $\mathbb{Z}$-action.

Therefore, it is now possible to associate a stationary $S\alpha S$ process to any stationary $S\alpha S$ random field indexed by $\mathbb{Z}^d$ in an ergodicity-preserving manner.

Connection to orbit equivalence

Corollary (R. (2020+) - Ergodicity is W^*-rigid and hence OE-rigid)

If two stationary $S\alpha S$ random fields indexed by $\mathbb{Z}^d$ (possibly with two different d's) have isomorphic free Rosinski group measure space constructions, then one is ergodic (equiv., weakly mixing) if and only if the other one is so. In particular, if they are generated by orbit equivalent free actions, then also the same conclusion holds.

The indexing groups having possibly different ranks (as $\mathbb{Z}$-modules) is actually very useful in the context of orbit equivalence.

The seminal result of Connes, Feldman and Weiss (1981) states that any nonsingular action of $\mathbb{Z}^d$ (more generally, of any amenable group) is orbit equivalent to a nonsingular $\mathbb{Z}$-action.

Therefore, it is now possible to associate a stationary $S\alpha S$ process to any stationary $S\alpha S$ random field indexed by $\mathbb{Z}^d$ in an ergodicity-preserving manner. This may help in classification of such fields.

Sketch of proof

- Can we prove it when the action $\{\phi_t\}_{t \in \mathbb{Z}^d}$ is also ergodic? **Yes we can.**
  - a fact from von Neumann Algebras: *if $\{\phi_t\}_{t \in G}$ is free and ergodic, then the factor $\mathcal{L}^\infty(S, \mu) \rtimes G$ is of type II_1 if and only if there exists a $\{\phi_t\}$-invariant finite measure $\nu \sim \mu$*, and
  - Theorem 4.1 of Wang, R. and Stoev (2013) (probabilistic input).
- What about the general case? Use
  - ergodic decomposition (Schmidt (1976), Corollary 6.9) for a nonsingular action on a standard measure space, and
  - its *canonical connection to the central decomposition of the corresponding group measure space construction* (another operator algebraic tool).

Sketch of proof

- Can we prove it when the action $\{\phi_t\}_{t \in \mathbb{Z}^d}$ is also ergodic? **Yes we can.**
  - a fact from von Neumann Algebras: *if* $\{\phi_t\}_{t \in G}$ *is free and ergodic, then the factor* $\mathcal{L}^\infty(S, \mu) \rtimes G$ *is of type* II_1 *if and only if there exists a* $\{\phi_t\}$*-invariant finite measure* $\nu \sim \mu$, and
  - Theorem 4.1 of Wang, R. and Stoev (2013) (probabilistic input).
- What about the general case? Use
  - ergodic decomposition (Schmidt (1976), Corollary 6.9) for a nonsingular action on a standard measure space, and
  - its *canonical connection to the central decomposition of the corresponding group measure space construction* (another operator algebraic tool).
- From the proof, it transpires that

Sketch of proof

- Can we prove it when the action $\{\phi_t\}_{t \in \mathbb{Z}^d}$ is also ergodic? **Yes we can.**
  - a fact from von Neumann Algebras: *if $\{\phi_t\}_{t \in G}$ is free and ergodic, then the factor $\mathcal{L}^\infty(S, \mu) \rtimes G$ is of type II_1 if and only if there exists a $\{\phi_t\}$-invariant finite measure $\nu \sim \mu$*, and
  - Theorem 4.1 of Wang, R. and Stoev (2013) (probabilistic input).
- What about the general case? Use
  - ergodic decomposition (Schmidt (1976), Corollary 6.9) for a nonsingular action on a standard measure space, and
  - its *canonical connection to the central decomposition of the corresponding group measure space construction* (another operator algebraic tool).
- From the proof, it transpires that
  - $\{X_t\}_{t \in \mathbb{Z}^d}$ is fully non-ergodic iff (almost) all the factors are of type II_1 and

Sketch of proof

- Can we prove it when the action $\{\phi_t\}_{t \in \mathbb{Z}^d}$ is also ergodic? **Yes we can.**
  - a fact from von Neumann Algebras: *if* $\{\phi_t\}_{t \in G}$ *is free and ergodic, then the factor* $\mathcal{L}^\infty(S, \mu) \rtimes G$ *is of type* II_1 *if and only if there exists a* $\{\phi_t\}$*-invariant finite measure* $\nu \sim \mu$, and
  - Theorem 4.1 of Wang, R. and Stoev (2013) (probabilistic input).
- What about the general case? Use
  - ergodic decomposition (Schmidt (1976), Corollary 6.9) for a nonsingular action on a standard measure space, and
  - its *canonical connection to the central decomposition of the corresponding group measure space construction* (another operator algebraic tool).
- From the proof, it transpires that
  - $\{X_t\}_{t \in \mathbb{Z}^d}$ is fully non-ergodic iff (almost) all the factors are of type II_1 and
  - same characterization of ergodicity holds for max-stable fields.

Future directions

- View more known examples through our lens - Cohen and Samorodnitsky (2006) (known to be ergodic), Dombry and Guillotin-Plantard (2009), Owada and Samorodnitsky (2015a), etc.

Future directions

- View more known examples through our lens - Cohen and Samorodnitsky (2006) (known to be ergodic), Dombry and Guillotin-Plantard (2009), Owada and Samorodnitsky (2015a), etc.

- When will a stationary $S\alpha S$ random field be mixing? Connection to Dombry and Kabluchko (2017) (for max-stable fields).

Future directions

- View more known examples through our lens - Cohen and Samorodnitsky (2006) (known to be ergodic), Dombry and Guillotin-Plantard (2009), Owada and Samorodnitsky (2015a), etc.

- When will a stationary $S\alpha S$ random field be mixing? Connection to Dombry and Kabluchko (2017) (for max-stable fields). Will mixing be a W^*-rigid property (like ergodicity) for $G = \mathbb{Z}^d$?

Future directions

- View more known examples through our lens - Cohen and Samorodnitsky (2006) (known to be ergodic), Dombry and Guillotin-Plantard (2009), Owada and Samorodnitsky (2015a), etc.
- When will a stationary $S\alpha S$ random field be mixing? Connection to Dombry and Kabluchko (2017) (for max-stable fields). Will mixing be a W^*-rigid property (like ergodicity) for $G = \mathbb{Z}^d$?
- What is the role of type III factors in various probabilistic properties of stationary $S\alpha S$ random fields? How about the stable random field of Sarkar and R. (2018) generated by the action of $\mathbb{F}_d$ on its Furstenberg-Poisson boundary?

Future directions

- View more known examples through our lens - Cohen and Samorodnitsky (2006) (known to be ergodic), Dombry and Guillotin-Plantard (2009), Owada and Samorodnitsky (2015a), etc.

- When will a stationary $S\alpha S$ random field be mixing? Connection to Dombry and Kabluchko (2017) (for max-stable fields). Will mixing be a W^*-rigid property (like ergodicity) for $G = \mathbb{Z}^d$?

- What is the role of type III factors in various probabilistic properties of stationary $S\alpha S$ random fields? How about the stable random field of Sarkar and R. (2018) generated by the action of $\mathbb{F}_d$ on its Furstenberg-Poisson boundary?

- Ergodicity for stationary $S\alpha S$ random fields indexed by $G \neq \mathbb{Z}^d$?

Future directions

- View more known examples through our lens - Cohen and Samorodnitsky (2006) (known to be ergodic), Dombry and Guillotin-Plantard (2009), Owada and Samorodnitsky (2015a), etc.
- When will a stationary $S\alpha S$ random field be mixing? Connection to Dombry and Kabluchko (2017) (for max-stable fields). Will mixing be a W^*-rigid property (like ergodicity) for $G = \mathbb{Z}^d$?
- What is the role of type III factors in various probabilistic properties of stationary $S\alpha S$ random fields? How about the stable random field of Sarkar and R. (2018) generated by the action of $\mathbb{F}_d$ on its Furstenberg-Poisson boundary?
- Ergodicity for stationary $S\alpha S$ random fields indexed by $G \neq \mathbb{Z}^d$? The hindrance is not operator algebraic but ergodic theoretic

Future directions

- View more known examples through our lens - Cohen and Samorodnitsky (2006) (known to be ergodic), Dombry and Guillotin-Plantard (2009), Owada and Samorodnitsky (2015a), etc.
- When will a stationary $S\alpha S$ random field be mixing? Connection to Dombry and Kabluchko (2017) (for max-stable fields). Will mixing be a W^*-rigid property (like ergodicity) for $G = \mathbb{Z}^d$?
- What is the role of type III factors in various probabilistic properties of stationary $S\alpha S$ random fields? How about the stable random field of Sarkar and R. (2018) generated by the action of $\mathbb{F}_d$ on its Furstenberg-Poisson boundary?
- Ergodicity for stationary $S\alpha S$ random fields indexed by $G \neq \mathbb{Z}^d$? The hindrance is not operator algebraic but ergodic theoretic - unavailability of ergodic theorem for nonsingular actions of groups

Future directions

- View more known examples through our lens - Cohen and Samorodnitsky (2006) (known to be ergodic), Dombry and Guillotin-Plantard (2009), Owada and Samorodnitsky (2015a), etc.

- When will a stationary $S\alpha S$ random field be mixing? Connection to Dombry and Kabluchko (2017) (for max-stable fields). Will mixing be a W^*-rigid property (like ergodicity) for $G = \mathbb{Z}^d$?

- What is the role of type III factors in various probabilistic properties of stationary $S\alpha S$ random fields? How about the stable random field of Sarkar and R. (2018) generated by the action of $\mathbb{F}_d$ on its Furstenberg-Poisson boundary?

- Ergodicity for stationary $S\alpha S$ random fields indexed by $G \neq \mathbb{Z}^d$? The hindrance is not operator algebraic but ergodic theoretic - unavailability of ergodic theorem for nonsingular actions of groups - extension of Lindenstrauss (2001)?

- View more known examples through our lens - Cohen and Samorodnitsky (2006) (known to be ergodic), Dombry and Guillotin-Plantard (2009), Owada and Samorodnitsky (2015a), etc.
- When will a stationary $S\alpha S$ random field be mixing? Connection to Dombry and Kabluchko (2017) (for max-stable fields). Will mixing be a W^*-rigid property (like ergodicity) for $G = \mathbb{Z}^d$?
- What is the role of type III factors in various probabilistic properties of stationary $S\alpha S$ random fields? How about the stable random field of Sarkar and R. (2018) generated by the action of $\mathbb{F}_d$ on its Furstenberg-Poisson boundary?
- Ergodicity for stationary $S\alpha S$ random fields indexed by $G \neq \mathbb{Z}^d$? The hindrance is not operator algebraic but ergodic theoretic - unavailability of ergodic theorem for nonsingular actions of groups - extension of Lindenstrauss (2001)? For discrete Heisenberg groups, use Jarrett (2019)?

Thank You Very Much for Your Patience.

arXiv:2007.14821

Supplementaries: Technicalities + References

Koopman representation

G-action $\{\phi_t\}$ lifts to the space of all real-valued measurable functions on S by

$$\sigma_t g = g \circ \phi_t, \ \ t \in G.$$

This lifted action preserves the $\mathcal{L}^\infty$-norm but not other $\mathcal{L}^p$-norms.

However, for each $t \in G$, $\pi_t : \mathcal{L}^2(S, \mu) \to \mathcal{L}^2(S, \mu)$ given by

$$(\pi_t g)(s) = g \circ \phi_t(s) \left(\frac{d\mu \circ \phi_t}{d\mu}(s) \right)^{1/2}, \ s \in S$$

defines an isometry. The unitary representation $\{\pi_t\}_{t \in G}$ of G inside $\mathcal{L}^2(S, \mu)$ is called the Koopman representation.

The crossed product relation

Using the cocycle relationship

$$\frac{d\mu \circ \phi_{uv}}{d\mu} = \frac{d\mu \circ \phi_u}{d\mu}\, \sigma_u \left(\frac{d\mu \circ \phi_v}{d\mu} \right), \quad u, v \in G,$$

one gets that for all $a \in \mathcal{L}^\infty(S, \mu)$ (thought of as acting on $\mathcal{L}^2(S, \mu)$ by multiplication), for all $t \in G$ and for all $g \in \mathcal{L}^2(S, \mu)$,

$$(\pi_t\, a\, \pi_{t^{-1}} g)(s) = ((\sigma_t a) g)(s), \quad s \in S. \tag{2}$$

In other words, the Koopman representation "normalizes" $\mathcal{L}^\infty(S, \mu)$ inside $\mathcal{B}(\mathcal{L}^2(S, \mu))$. The group measure space construction is a space, where the crossed product relation (2) is internalized.

Group measure space construction

Consider the von Neumann algebra

$$\mathcal{B}(l^2(G) \otimes \mathcal{L}^2(S,\mu)) = \overline{\mathcal{B}(l^2(G)) \otimes \mathcal{B}(\mathcal{L}^2(S,\mu))}$$

(with the closure being taken with respect to the weak/strong operator topology). Define a representation of G by $t \mapsto u_t := \lambda_t \otimes \pi_t$, where $\{\lambda_t\}$ is the left regular representation and $\{\pi_t\}$ is the Koopman representation. We also represent $\mathcal{L}^\infty(S,\mu)$ by $a \mapsto 1 \otimes \mathcal{M}_a$, where $\mathcal{M}_a$ is the multiplication (by a) operator on $\mathcal{L}^2(S,\mu)$. It can be checked that the following "internal" crossed product relation holds:

$$u_t(1 \otimes \mathcal{M}_a)u_{t^{-1}} = 1 \otimes \mathcal{M}_{\sigma_t a}\,.$$

Define the *group measure space construction* (also known as *crossed product construction*) as

$$\mathcal{L}^\infty(S,\mu) \rtimes G := \{u_t, 1 \otimes \mathcal{M}_a : t \in G,\ a \in \mathcal{L}^\infty(S,\mu)\}''.$$

Connections to ergodic theory

It can be shown that the internal crossed product relation implies that any $x \in \mathcal{L}^\infty(S,\mu) \rtimes G$ can be uniquely written as $x = \sum_{t \in G} a_t u_t$ with $\{a_t : t \in G\} \subseteq \mathcal{L}^\infty(S,\mu)$. Thus, we can view x as a $|G| \times |G|$ matrix with entries coming from $\mathcal{L}^\infty(S,\mu)$ that are the same along each left group-diagonal; see, e.g, Jones (2009).

Theorem (see, e.g, Peterson (2013))

The following results hold for a nonsingular G-action $\{\phi_t\}$ and the corresponding group measure space construction defined above.

1. *If the action $\{\phi_t\}_{t \in G}$ is free and ergodic, then $\mathcal{L}^\infty(S,\mu) \rtimes G$ is a factor.*
2. *If $\mathcal{L}^\infty(S,\mu) \rtimes G$ is a factor, then $\{\phi_t\}_{t \in G}$ is ergodic.*
3. *If $\{\phi_t\}_{t \in G}$ is free and ergodic, then the factor $\mathcal{L}^\infty(S,\mu) \rtimes G$ is of type II_1 if and only if $\{\phi_t\}_{t \in G}$ is a positive action.*

Furthermore, if the two nonsingular actions (not necessarily of the same group) are orbit-equivalent, then the corresponding group measure space constructions are isomorphic as von Neumann algebras

References

J. Aaronson (1997): *An Introduction to Infinite Ergodic Theory*. American Mathematical Society, Providence.

J. Athreya, M. Mj and P. Roy (2019): Stable Random Fields, Patterson-Sullivan measures and Extremal Cocycle Growth. *arXiv:1809.08295*.

C. Béguin, A. Valette and A. Zuk (1997): On the spectrum of a random walk on the discrete Heisenberg group and the norm of Harper's operator. *J. Geom. Phys.* 21:337–356.

A. Bhattacharya and P. Roy (2018): A large sample test for the length of memory of stationary symmetric stable random fields via nonsingular $\mathbb{Z}^d$-actions. *J. Appl. Probab.* 55:179–195.

O. Bratteli and D. W. Robinson (1987): *Operator algebras and quantum statistical mechanics. 1*. Texts and Monographs in Physics. Springer-Verlag, New York, 2nd edition. C^*- and W^*-algebras, symmetry groups, decomposition of states.

S. Cohen and G. Samorodnitsky (2006): Random rewards, fractional Brownian local times and stable self-similar processes. *Ann. Appl. Probab.* 16:1432–1461.

A. Connes (1976): Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. *Ann. of Math. (2)* 104:73–115.

A. Connes, J. Feldman and B. Weiss (1981): An amenable equivalence relation is generated by a single transformation. *Ergodic Theory Dynam. Systems* 1:431–450 (1982).

C. Dombry and N. Guillotin-Plantard (2009): Discrete approximation of a stable self-similar stationary increments process. *Bernoulli* 15:195–222.

C. Dombry and Z. Kabluchko (2017): Ergodic decompositions of stationary max-stable processes in terms of their spectral functions. *Stochastic Process. Appl.* 127:1763–1784.

V. Fasen and P. Roy (2016): Stable random fields, point processes and large deviations. *Stochastic Processes and their Applications* 126:832 – 856.

D. Gretete (2011): Random walk on a discrete Heisenberg group. *Rend. Circ. Mat. Palermo (2)* 60:329–335.

C. Hardin Jr. (1981): Isometries on subspaces of L^p. *Indiana Univ. Math. J.* 30:449–465.

C. Hardin Jr. (1982): On the spectral representation of symmetric stable processes. *Journal of Multivariate Analysis* 12:385–401.

M. Hochman (2010): A ratio ergodic theorem for multiparameter non-singular actions. *J. Eur. Math. Soc. (JEMS)* 12:365–383.

A. Ioana (2011): W^*-superrigidity for Bernoulli actions of property (T) groups. *J. Amer. Math. Soc.* 24:1175–1226.

A. IOANA (2018): Rigidity for von Neumann algebras. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*. World Sci. Publ., Hackensack, NJ, pp. 1639–1672.

K. JARRETT (2019): An ergodic theorem for nonsingular actions of the Heisenberg groups. *Trans. Amer. Math. Soc.* 372:5507–5529.

V. F. R. JONES (2009): *Von Neumann Algebras*. Lectures Notes, University of California, Berkeley.
https://math.berkeley.edu/ vfr/MATH20909/VonNeumann2009.pdf.

P. JUNG, T. OWADA and G. SAMORODNITSKY (2017): Functional central limit theorem for a class of negatively dependent heavy-tailed stationary infinitely divisible processes generated by conservative flows. *Ann. Probab.* 45:2087–2130.

S. KNUDBY (2011): *Disintegration theory for von Neumann algebras*. Graduate Project, University of Copenhagen.

U. KRENGEL (1985): *Ergodic Theorems*. De Gruyter, Berlin, New York.

E. LINDENSTRAUSS (2001): Pointwise theorems for amenable groups. *Invent. Math.* 146:259–295.

T. MIKOSCH and G. SAMORODNITSKY (2000): Ruin probability with claims modeled by a stationary ergodic stable process. *Ann. Probab.* 28:1814–1851.

F. J. MURRAY and J. VON NEUMANN (1936): On rings of operators. *Ann. of Math. (2)* 37:116–229.

T. Owada and G. Samorodnitsky (2015a): Functional central limit theorem for heavy tailed stationary infinitely divisible processes generated by conservative flows. *Ann. Probab.* 43:240–285.

T. Owada and G. Samorodnitsky (2015b): Maxima of long memory stationary symmetric α-stable processes, and self-similar processes with stationary max-increments. *Bernoulli* 21:1575–1599.

S. Panigrahi, P. Roy and Y. Xiao (2018): Maximal moments and uniform modulus of continuity for stable random fields. *arXiv:1709.07135*.

J. Peterson (2010): Examples of group actions which are virtually W^*-superrigid. *arXiv:1002.1745*.

J. Peterson (2013): *Notes on von Neumann algebras*. Lectures Notes, Vanderbilt University. https://math.vanderbilt.edu/peters10/teaching/spring2013/vonNeumannAlgeb

S. Popa (2006): Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups. II. *Invent. Math.* 165:409–451.

S. Popa and S. Vaes (2010): Group measure space decomposition of II_1 factors and W^*-superrigidity. *Invent. Math.* 182:371–417.

S. Popa and S. Vaes (2014): Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups. *Acta Math.* 212:141–198.

S. Resnick (1992): *Adventures in Stochastic Processes*. Birkhäuser, Boston.

S. Resnick and G. Samorodnitsky (2004): Point processes associated with stationary stable processes. *Stochastic Process. Appl.* 114:191–209.

J. Rosiński (1994): On uniqueness of the spectral representation of stable processes. *J. Theoret. Probab.* 7:615–634.

J. Rosiński (1995): On the structure of stationary stable processes. *Ann. Probab.* 23:1163–1187.

J. Rosiński (2000): Decomposition of stationary α-stable random fields. *Ann. Probab.* 28:1797–1813.

J. Rosiński and G. Samorodnitsky (1996): Classes of mixing stable processes. *Bernoulli* 2:3655–378.

E. Roy (2007): Ergodic properties of Poissonian ID processes. *Ann. Probab.* 35:551–576.

E. Roy (2012): Maharam extension and stationary stable processes. *Ann. Probab.* 40:1357–1374.

P. Roy (2010): Ergodic theory, abelian groups and point processes induced by stable random fields. *Ann. Probab.* 38:770–793.

P. Roy (2017): Maxima of stable random fields, nonsingular actions and finitely generated abelian groups: a survey. *Indian J. Pure Appl. Math.* 48:513–540.

P. Roy and G. Samorodnitsky (2008): Stationary symmetric α-stable discrete parameter random fields. *J. Theoret. Probab.* 21:212–233.

G. Samorodnitsky (2004): Extreme value theory, ergodic theory, and the boundary between short memory and long memory for stationary stable processes. *Ann. Probab.* 32:1438–1468.

G. Samorodnitsky (2005): Null flows, positive flows and the structure of stationary symmetric stable processes. *Ann. Probab.* 33:1782–1803.

G. Samorodnitsky and M. S. Taqqu (1994): *Stable non-Gaussian random processes*. Stochastic Modeling. Chapman & Hall, New York. Stochastic models with infinite variance.

S. Sarkar and P. Roy (2018): Stable random fields indexed by finitely generated free groups. *Ann. Probab.* 46:2680 – 2714.

K. Schmidt (1977): *Cocycles on ergodic transformation groups*. Macmillan Company of India, Ltd., Delhi. Macmillan Lectures in Mathematics, Vol. 1.

I. M. Singer (1955): Automorphisms of finite factors. *Amer. J. Math.* 77:117–133.

S. Stoev and M. S. Taqqu (2005): Extremal stochastic integrals: a parallel between max–stable processes and $\alpha-$stable processes. *Extremes* 8:237–266.

V. S. Sunder (1987): *An invitation to von Neumann algebras*. Universitext. Springer-Verlag, New York.

D. Surgailis, J. Rosiński, V. Mandrekar and S. Cambanis (1993): Stable mixed moving averages. *Probab. Theory Related Fields* 97:543–558.

A. A. Tempel'man (1972): Ergodic theorems for general dynamical systems. *Trudy Moskov. Mat. Obšč.* 26:95–132.

S. Vaes (2014): Normalizers inside amalgamated free product von Neumann algebras. *Publ. Res. Inst. Math. Sci.* 50:695–721.

V. Varadarajan (1970): *Geometry of Quantum Theory*, volume 2. Van Nostrand Reinhold, New York.

Y. Wang, P. Roy and S. A. Stoev (2013): Ergodic properties of sum- and max-stable stationary random fields via null and positive group actions. *Ann. Probab.* 41:206–228.

Y. Wang and S. A. Stoev (2009): On the structure and representations of max-stable processes. Technical Report 487, Department of Statistics, University of Michigan. *arXiv:0903.3594*.

Y. Wang and S. A. Stoev (2010a): On the association of sum-and max-stable processes. *Statistics & Probability Letters* 80:480–488.

Y. Wang and S. A. Stoev (2010b): On the structure and representations of max-stable processes. *Advances in Applied Probability* 42:855–877.

R. Zimmer (1984): *Ergodic Theory and Semisimple Groups*. Birkhäuser, Boston.