# Extreme value theory for stable random fields indexed by finitely generated free groups

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Joint work with Sourav Sarkar: arXiv:1608.03887
(based on his master's dissertation)

$(G, \cdot) =$ ctble group with identity element $e$
(possibly noncommutative and in most cases, finitely generated)

$(S, \mathcal{S}, \mu) =$ a $\sigma$-finite measure space

**Defn:** $\{\phi_t\}_{t \in G}$ is called a nonsingular / quasi-invariant $G$-action on $(S, \mathcal{S}, \mu)$ if

① each $\phi_t : (S, \mathcal{S}) \longrightarrow (S, \mathcal{S})$ is mble

② $\phi_e = id_S$

③ $\phi_{t_1 \cdot t_2} = \phi_{t_2} \circ \phi_{t_1} \quad \forall\, t_1, t_2 \in G$

These statements can also hold modulo $\mu$-null sets.

④ $\underbrace{\mu \circ \phi_t}$ $\sim \mu \quad \forall\, t \in G$

This is a measure since ② + ③ $\Rightarrow \phi_t = (\phi_{t^{-1}})^{-1}$.

Here "$\sim$" denotes equivalence of measures (i.e., having same null sets).

Clearly, $\boxed{\text{measure-preserving } G\text{-action}} \;\overset{\Longrightarrow}{\not\Longleftarrow}\; \boxed{\text{nonsingular } G\text{-action}}$

$\Updownarrow$

$\mu \circ \phi_t = \mu \quad \forall\, t \in G$

We shall see two examples later (one is measure-preserving and the other is nonsingular but not measure-preserving).

Such group actions arise naturally in many contexts and we shall focus on their relation to "left stationary symmetric $\alpha$-stable ($S\alpha S$) random fields indexed by $G$.

$\boxed{\text{Ergodic theoretic properties of nonsingular } G\text{-actions}} \longleftrightarrow \boxed{\text{Probabilistic properties of stationary } S\alpha S \text{ random fields indexed by } G}$

Qn) <u>What is a random field?</u>

Ans) A random field is a collection of <u>random variables</u> (i.e., real-valued measurable functions) defined on a common probability space and indexed by a $\underbrace{\text{"multi-dimensional object"}}$.

($\mathbb{Z}^d$ or $\mathbb{R}^d$ or a group or a manifold, etc. In our case the indexing set will be $G$.)

$(\Omega, \mathcal{F}, P)$ = a probability space on which all the random variables (in this talk) are defined

$(X_t)_{t\in G}$ = a random field indexed by $G$, i.e., [redacted] for each $t \in G$, $X_t : (\Omega, \mathcal{F}) \xrightarrow{} (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is a mble map.

<u>Defn</u>: $(X_t)$ is called (left) stationary if

$$(X_{s.t})_{t\in G} \stackrel{d}{=} (X_t)_{t\in G},$$

[ "$\stackrel{d}{=}$" denotes equality of finite-dimensional distributions ]

i.e., $\forall\ k \geq 1$, $\forall\ t_1, t_2, \ldots, t_k \in G$, $\forall\ s \in G$ and $\forall\ B \in \mathcal{B}_{\mathbb{R}^k}$,

$$P\Big(\underbrace{\{\omega\in\Omega : (X_{s.t_1}(\omega), \ldots, X_{s.t_k}(\omega)) \in B\}}_{\in \mathcal{F}}\Big) = P\Big(\underbrace{\{\omega\in\Omega : (X_{t_1}(\omega), \ldots, X_{t_k}(\omega)) \in B\}}_{\in \mathcal{F}}\Big)$$

We fix $\alpha \in (0,2)$ for the entire talk.

<u>Defn</u>: A random variable $X : (\Omega, \mathcal{F}) \xrightarrow{\text{mble}} (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is said to follow a <u>symmetric $\alpha$-stable</u> ($S\alpha S$) distribution if $\exists\ \sigma \in (0,\infty)$ such that

$$\underbrace{\int_\Omega \exp\{i\theta X(\omega)\}\,dP(\omega)}_{\text{Fourier transform / characteristic function of } X} = \exp\{-\sigma^\alpha |\theta|^\alpha\}$$

for all $\theta \in \mathbb{R}$.

<u>Defn</u>: $(X_t)_{t\in G}$ is called an $S\alpha S$ random field if each finite linear combination of $X_t$'s follow an $S\alpha S$ distribution.

<u>Remark</u>: $\alpha = 2$ corresponds to the Gaussian case – we shall only be concerned about the non-Gaussian situation.

<u>Thm</u> (Rosinski (1995, 2000)): A random field $(X_t)_{t\in G}$ is a (left) stationary $S\alpha S$ random field if and only if there exist a $\sigma$-finite measure space $(S, \mathcal{S}, \mu)$, a function $f \in L^\alpha(S,\mu) := \{f : (S,\mathcal{S}) \xrightarrow{\text{mble}} (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) : \|f\|_\alpha := \left(\int_S |f(s)|^\alpha \mu(ds)\right)^{1/\alpha} < \infty\}$, ($\|\cdot\|_\alpha$ is a norm when $\alpha \in [1,2)$ but not when $\alpha \in (0,1)$) a nonsingular/quasi-invariant $G$-action $\{\phi_t\}_{t\in G}$ on $(S,\mathcal{S},\mu)$ such that $\forall\, k \geq 1$ and $\forall\, t_1, t_2, \dots, t_k \in G$,

$$(*) \dots \int_\Omega \exp\left\{i \sum_{j=1}^k \theta_j X_{t_j}(\omega)\right\} dP(\omega) = \exp\left\{-\left\|\sum_{j=1}^k \theta_j f_{t_j}\right\|_\alpha^\alpha\right\}, \quad (\theta_1,\dots,\theta_k) \in \mathbb{R}^k,$$

where $\forall\, t \in G$,

$$f_t(s) = \pm\, f \circ \phi_t(s) \left(\frac{d\mu\circ\phi_t}{d\mu}(s)\right)^{1/\alpha}, \quad s \in S.$$

"cocycle" wrt $\{\phi_t\}_{t\in G}$
(just assume it to be $\equiv +1$ for simplicity)

<u>Remarks</u>: ① The above theorem was actually proved by Rosinski (1995) for $G = \mathbb{Z}$ and Rosinski (2000) for $G = \mathbb{Z}^d$. Same proof, with a bit of care about the side of multiplication, applies to any ctble group $G$.

② <u>$(*) + (**) \Rightarrow$ left stationarity of $(X_t)_{t\in G}$</u>

<u>Sketch of proof</u>: Fix $k \geq 1$, $t_1, t_2, \dots, t_k, s \in G$, $\theta_1, \theta_2, \dots, \theta_k \in \mathbb{R}$. Then $(*) + (**) \Rightarrow \int_\Omega \exp\left\{ i \sum_{j=1}^k \theta_j X_{s.t_j}(\omega)\right\} dP(\omega) = \int_\Omega \exp\left\{ i \sum_{j=1}^k \theta_j X_{t_j}(\omega)\right\} dP(\omega)$

$$\xrightarrow[\text{inversion}]{\text{Fourier}} (X_{s.t})_{t\in G} \stackrel{d}{=} (X_t)_{t\in G}.$$

③ Left stationarity of $(X_t)_{t\in G}$ induces a natural action of $G$ on $\mathbb{R}^G$ by "left-translation", which preserves the "law of $(X_t)_{t\in G}$", i.e., the probability measure on $\mathbb{R}^G$ defined by

$$P_X(\cdot) := P(\{\omega \in \Omega : (X_t(\omega) : t \in G) \in \cdot\}).$$

When seen at the Fourier transform level, this action becomes nonsingular.

④ $(**)$ is not unique but there is a rigidity result (of $L^\alpha$-spaces due to Hardin (1981, 1982)) that makes it "more of less unique ergodic theoretically." Therefore, we shall say $(X_t)_{t\in G}$ is generated/induced by $\{\phi_t\}_{t\in G}$ (and $f$).

**Def^n:** Suppose $\{\phi_t\}_{t\in G}$ is a nonsingular $G$-action on $(S,\mathcal{S},\mu)$. $W\in\mathcal{S}$ is called a wandering set for $\{\phi_t\}_{t\in G}$ if $\forall\, t,t'\in G$ with $t\neq t'$, $\phi_t(W)\cap\phi_{t'}(W)=\emptyset$.

**Example:** $\{\phi_t\}_{t\in\mathbb{Z}}$ is translation on $(\mathbb{R},\mathcal{B}_{\mathbb{R}},\mathrm{Leb})$: $\forall\, t\in\mathbb{Z}$,

$$\phi_t(s)=s+t\,,\qquad s\in\mathbb{R}.$$

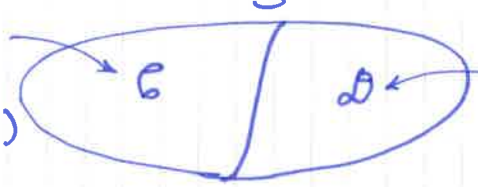
This is a measure-preserving (and hence nonsingular) $\mathbb{Z}$-action. Clearly $W^*=(0,1]$ is a wandering set.

**Hopf decomposition:** (unique modulo $\mu$)

$$\left.\begin{aligned}\phi_t(\mathcal{C})&=\mathcal{C}\\ \phi_t(\mathcal{D})&=\mathcal{D}\end{aligned}\right\}\ \forall\, t\in G$$

Has no wandering subset of +ve $\mu$-measure (conservative part)

$S$



$\mathcal{C}$ $\quad$ $\mathcal{D}$

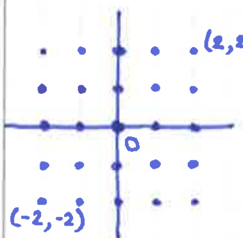
$=\bigcup_{t\in G}\phi_t(W^*)$ for some wandering set $W^*$ (dissipative part)

In the previous example, $\mathcal{C}=\emptyset$ and $\mathcal{D}=\mathbb{R}$ (in fact $W^*=(0,1]$ proves that $\mathcal{D}=\mathbb{R}$).

**Remark:** Roughly speaking, the points in the conservative part tend to come back while the points in the dissipative part tend to move away by the group action.

We first consider $G=\mathbb{Z}^d$ and a stationary $S\alpha S$ random field $(X_t)_{t\in\mathbb{Z}^d}$.

$E_n:=\{t=(t^{(1)},t^{(2)},\dots,t^{(d)})\in\mathbb{Z}^d:\ \text{each } t^{(i)}\in[-n,n]\}$, $n\geq 1$.



$(2,2)$

$0$

$(-2,-2)$

$E_2$ for $d=2$

For $a_n>0$, $b_n>0$ $\forall\, n\geq 1$, we write $a_n=\Theta(b_n)$ to mean $\frac{a_n}{b_n}\to c>0$.

In this notation, $|E_n|=(2n+1)^d=\Theta(n^d)$.

$\forall\, n\geq 1$, define $\underset{\text{mble}}{M_n(\omega)}=\max_{t\in E_n}|X_t(\omega)|$, $\omega\in\Omega$.

**Easy to check:** $M_n(\omega)\nearrow\infty$ (as $n\to\infty$) for $P$-almost all $\omega\in\Omega$.

$(Q_n^{\mathbb{Z}^d})$ How fast does $M_n$ grow as $n\to\infty$?

As we shall see, the answer is connected to the Hopf decomposition of the underlying nonsingular $\mathbb{Z}^d$-action.

Before writing down the anwer to this question, we shall introduce two important notations.

**Def$^n$:** Take $0 < c_n \nearrow \infty$. (i) We write $M_n = \Theta_p(c_n)$ (and say that $M_n$ grows like $c_n$ in probability) if as $n \to \infty$,

$$\frac{M_n}{c_n} \longrightarrow Z_\alpha \text{ (a strictly +ve random variable) in distribution,}$$

i.e., $\forall\ 0 < a < b < \infty$,

$$P\left(\left\{\omega \in \Omega : a < \frac{M_n(\omega)}{c_n} < b\right\}\right) \longrightarrow P(\{\omega \in \Omega : a < Z_\alpha < b\}).$$

In this talk, $Z_\alpha$ will always be (a positive multiple of) an $\alpha$-Frechet random variable. In each situation, the +ve constant has been evaluated although it differs from one result to another. Ignoring this constant, $Z_\alpha$ satisfies

$$P(\{\omega \in \Omega : Z_\alpha(\omega) \le x\}) = \exp\{-x^{-\alpha}\}, \forall\ x > 0.$$

(ii) We write $M_n = o_p(c_n)$ (and say that $M_n$ grows strictly slower than $c_n$ in probability) if $M_n/c_n \longrightarrow 0$ in distribution, i.e., $\forall\ \varepsilon > 0$, $P\left(\left\{\omega \in \Omega : \frac{M_n(\omega)}{c_n} < \varepsilon\right\}\right) \longrightarrow 1$ as $n \to \infty$.

**Remark:** Clearly, (i) and (ii) cannot occur together. In fact, if (i) happens along a subsequence, then also (ii) cannot happen. Conversely, occurence of (ii) prevents (i) from happening even along a subsequence.

**Answer to $(Q_n^{\mathbb{Z}^d})$:**

**Thm:** ($d=1$: Samorodnitsky (2004); $d>1$: R. and Samorodnitsky (2008))

$\{\phi_t\}_{t\in\mathbb{Z}^d}$ has $\mu$-null dissipative part $\Rightarrow$ $M_n = o_p(n^{d/\alpha})$.

$\{\phi_t\}_{t\in\mathbb{Z}^d}$ has nontrivial dissipative part $\Rightarrow$ $M_n = \Theta_p(n^{d/\alpha})$.

**Intuition:** No dissipative part $\Rightarrow$ Stronger dependence among $X_t$'s
$\Rightarrow$ Slower growth of $M_n$.

**Finitely generated free group:** $F_2$ (without loss of generality ☺)

$D = \{a, b, a^{-1}, b^{-1}\}$. (No relations except $a.a^{-1} = a^{-1}.a = b.b^{-1} = b^{-1}.b = e$)

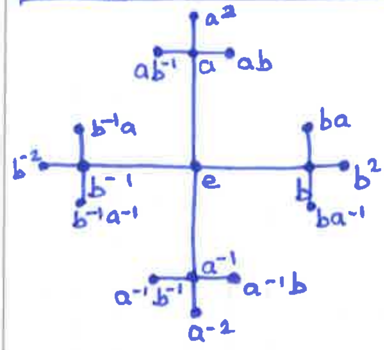
$F_2$ = all possible reduced words that can be formed using $D$

$\cdot$ = "concatenation followed by reduction"

(e.g., $a^2b^3a^{-1}b^2 . b^{-2}a^4b^{-1}a^2 = a^2b^3a^3b^{-1}a^2$

**Easy to check:** $(F_2, \cdot)$ is a noncommutative group.
$e$ = empty word.

## Cayley graph of $F_2$ with generating set $D=\{a,b,a^{-1},b^{-1}\}$



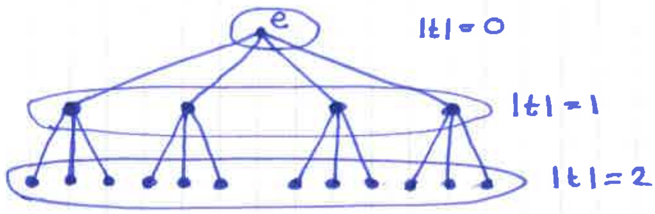

$$V=F_2$$

$$E=\{(g_1,g_2):\ g_1^{-1}g_2\in D\}$$

$D=D^{-1}\ \Rightarrow$ undirected graph

$e\notin D\ \Rightarrow$ no self-loop

Clearly, the Cayley graph is a homogeneous/regular tree of degree 4.




$\forall\, t\in F_2$, $|t|$ = Cayley graph distance between $e$ and $t$,

$E_n \doteq \{t\in F_2 : |t|\le n\}$, $n\ge 1$ $\Rightarrow$ $|E_n| = \Theta(3^n)$.

In general, for any finitely generated ctble group $(G,\cdot)$, with identity element $e$ and a symmetric generating set $D(=D^{-1})\not\supseteq\{e\}$, we can define the Cayley graph similarly.

Let $d(u,v)=$ ~~minimu~~ minimum number of edges required in the Cayley graph to travel from $u$ to $v$, $u,v\in G$.

$=$ Cayley graph distance between $u$ and $v$.

$E_n \doteq \{t\in G : |t| := d(t,e)\le n\}$, $n\ge 1$.

For a stationary S$\alpha$S random field $(X_t)_{t\in G}$, define $\forall\, n\ge 1$,

$$M_n(\omega) = \max_{t\in E_n} |X_t(\omega)|, \quad \omega\in\Omega.$$

Again, $M_n(\omega)\nearrow\infty$ for $P$-almost all $\omega\in\Omega$.

$(Q_n^G)$ How fast does $M_n$ grow as $n\to\infty$?

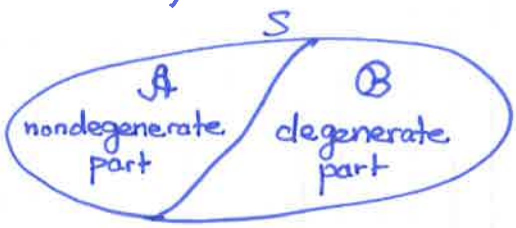
This question is too general to give a specific answer to. We shall state that a general decomposition exists that is connected to this question and establish that for $G=F_2$, this will not coincide with the Hopf decomposition.

Answer to $(Q_n^G)$:

Thm: (Sarkar and R. (2016+)) $\exists$ unique decomp of modulo $\mu$

$\mathcal{D} \subseteq \mathcal{A}$

$\Rightarrow \mathcal{C} \supseteq \mathcal{B}$




$\left.\begin{array}{l}\phi_t(\mathcal{A}) = \mathcal{A} \\ \phi_t(\mathcal{B}) = \mathcal{B}\end{array}\right\} \forall t \in G$

$\{\phi_t\}_{t\in G}$ has $\mu$-null nondegenerate part $\Rightarrow$ $M_n = o_p(|E_n|^{1/\alpha})$.

$\{\phi_t\}_{t\in G}$ has nontrivial nondegenerate part $\Rightarrow$ $M_{n_k} = \Theta_p(|E_{n_k}|^{1/\alpha})$ for some subseq $(n_k)$ of $(n)$.

Quick observation: $G = \mathbb{Z}^d \Rightarrow \mathcal{A} = \mathcal{D}$ and $\mathcal{B} = \mathcal{C}$.

$(Q_n^{F_2})$ What happens (to the decomposition $S = \mathcal{A} \cup \mathcal{B}$) when $G = F_2$?

Why $F_2$ (or $F_d$, $d \geq 2$)?

Twofold motivation:

(A) Ergodic theory becomes more nontrivial as we pass from amenable to nonamenable groups (and $F_d$ is an important test-case): see, for example, Tao (2015) on failure of pointwise and maximal ergodic theorems (in $L^1$) for a measure-preserving action of $F_2$.

(B) By passing to the Cayley graph, we shall obtain a stationary S$\alpha$S random field indexed by a homogeneous tree of even (in this case, four) degree. Such fields are important; see, e.g., Pemantle (1995) for importance of tree indexed processes in probability theory, statistical physics, fractal geometry, branching models, etc. However, no stationary S$\alpha$S random field has been constructed on a tree so far.

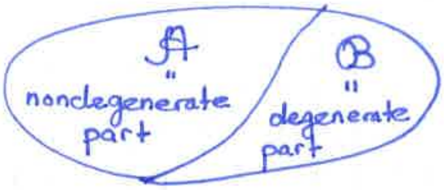
Remarks: (a) We shall see that a different ergodic theoretic phenomenon is obtained in case of $F_2$ due to its nonamenability. ~~This~~ More precisely, the boundary between degenerate and nondegenerate parts will different that the Hopf boundary.

(b) The only known example (to us) of a stable (more generally, regularly varying) fields indexed by a tree is branching random walk (introduced by Durrett (1979, 1983)) with regularly varying steps. But this class is, by construction, nonstationary.

<u>Answer to</u> $(Q_n^{F_2})$: Recall $G = F_2 \Rightarrow |E_n| = \Theta(3^n)$.

<u>Thm</u> (Sarkar and R. (2016+)) If $G = F_2$, then $\exists$ unique (modulo $\mu$) decomposition

$$\mathcal{D} \subsetneqq \mathcal{A} \Downarrow \mathcal{B} \subsetneqq \mathcal{C} \Downarrow \mathcal{C} \cap \mathcal{A} \neq \emptyset$$




$$\left.\begin{array}{l} \phi_t(\mathcal{A}) = \mathcal{A} \\ \phi_t(\mathcal{B}) = \mathcal{B} \end{array}\right\} \forall t \in F_2.$$

$\{\phi_t\}_{t \in F_2}$ has $\mu$-null nondegenerate part $\Rightarrow M_n = o_p(3^{n/\alpha})$

$\{\phi_t\}_{t \in F_2}$ has nontrivial nondegenerate part $\Rightarrow \forall$ subseq $(n_k)$ of $(n)$, $\exists$ a further subseq $(n_{k_l})$ such that $M_{n_{k_l}} = \Theta_p(3^{n_{k_l}/\alpha})$.

We have an example that shows $\mathcal{C} \cap \mathcal{A} \neq \emptyset$.

<u>Example</u>: (The boundary action of $F_2$ and an induced stationary S$\alpha$S random field. This action was pointed out to us by Kingshook Biswas and Mahan Mj.)

Take $S = \partial F_2$ = all infinite length reduced words that can be formed using $D = \{a, b, a^{-1}, b^{-1}\}$.

For all $g \in F_2 \setminus \{e\}$, $C_g \, (\subseteq \partial F_2)$ denotes the set of all infinite length reduced words whose "initial segment" is $g$ (e.g., $C_a$ = the set of all words starting with $a$, etc.)

$$\mathcal{S} = \sigma\Big(\bigcup_{g \in F_2 \setminus \{e\}} C_g\Big).$$

<u>Easy to check (using Kolmogorov Consistency Thm)</u>: $\exists$ unique prob measure $\mu$ on $(S = \partial F_2, \mathcal{S})$ such that

$$\mu(C_g) = \frac{1}{4 \times 3^{|g|-1}} \quad \forall g \in F_2 \setminus \{e\}.$$

$\mu$ is called the uniform measure on $S = \partial F_2$.

<u>Remark</u>: $(S = \partial F_2, \mu)$ is a Furstenberg-Poisson boundary of $F_2$ and $\mu$ is the Patterson-Sullivan measure on the ~~boun~~ boundary $\partial F_2$.

$G = F_2$ acts canonically on $(S = \partial F_2, \mathcal{S}, \mu)$ by "left-concatenation by the inverse followed by reduction": e.g., $\phi_{ba^{-1}}(baba\ldots) = a^2 baba\ldots$.

Remark: Usually one takes "left-concatenation followed by reduction". Here we left-concatenate by the inverse so as to match of def$^n$ of group action: $\phi_{t_1 \cdot t_2} = \phi_{t_2} \circ \phi_{t_1}$. This defn ensures left-stationarity of the induced random field.

Grigorchuk, Kaimanovich and Nagnibeda (2012): $\{\phi_t\}_{t \in F_2}$ defined above is a nonsingular action whose dissipative part is $\mu$-null, i.e., it is a conservative $F_2$-action.

Sarkar and R. (2016+): (a) The degenerate part $\mathcal{B}$ of the above action is $\mu$-null. In other words, the boundary action is nondegenerate.
(b) If $(X_t)_{t \in F_2}$ is a left stationary $S\alpha S$ random field generated by $\{\phi_t\}_{t \in F_2}$ defined above and $f \equiv 1$ (with trivial cocycle), then

$$M_n = \Theta_p(3^{n/\alpha}) \quad \text{even though the action is conservative.}$$

In particular, the degenerate-nondegenerate decomposition is strictly different from the Hopf decomposition for $G = F_2$. In other words, a new transition boundary is obtained here.

Now let us go back to a general stationary $S\alpha S$ random field $(X_t)_{t \in F_2}$ generated by any nonsingular action $\{\phi_t\}_{t \in F_2}$.

Thm: (Sarkar and R. (2016+)) If $\{\phi_t\}_{t \in F_2}$ is dissipative, i.e., its conservative part is $\mu$-null, then $M_n = \Theta_p(3^{n/\alpha})$.

In fact, we have a point process convergence result in this case (from which $M_n = \Theta_p(3^{n/\alpha})$ follows by a continuous mapping argument) and the limiting point process is novel, which we have termed as a "randomly thinned cluster Poisson process".

Remarks: (1) For $G = \mathbb{Z}^d$, a similar point process convergence (see Resnick and Samorodnitsky (2004), R. (2010)) holds but the limit is a vanilla cluster Poisson process with no thinning. This shows that thinning is perhaps a "non-amenable phenomenon".

(2) In our case, the (positive) constant obtained in the limit of $3^{-n/\alpha} M_n$ (in the distributional sense) is also much more ~~dela~~ delicate than the one obtained in the lattice case in Samorodnitsky (2004), R. and Samorodnitsky (2008)

## A "thousand word ~~sy~~ summary" of our results when $G = F_d$, $d \geq 2$.

**Note:** Hopf decomposition and our degenerate-nondegenerate decomposition induce the following paritition: $S = \mathcal{D} \cup (\mathcal{C} \cap \mathcal{A}) \cup \mathcal{B}$.

| $\mathcal{D}$ | $\mathcal{C} \cap \mathcal{A}$ | $\mathcal{B}$ |
|---|---|---|
| Weak conv of point ~~proces~~ process $\Downarrow$ $M_n = \Theta_p(3^{n/\alpha})$ | Tightness of $M_n / 3^{\alpha}$: in fact, given any subseq $(n_k)$ of $(n)$, $\exists$ a furthe subseq $(n_{k_\nu})$ such that $M_{n_{k_\nu}} = \Theta_p(3^{n_{k_\nu}/\alpha})$ (The boundary action examples $\Rightarrow$ this part is nontrivial) | $M_n = o_p(3^{n/\alpha})$ **Open Question:** Exact rate in this case? (See (I) below) |

## Open problems:

(I) In case of $G = \mathbb{Z}^d$, using structure theorem for finitely generated abelian groups, a better rate was obtained in some cases; see R. and Samorodnitsky (2008). Since there is no structure theorem for finitely generated nonabelian groups, this question becomes very challenging.

(II) Samorodnitsky (in a personal communication): Is it possible to characterize all finitely generated c+ble group $G$ for which a new transition boundary will be observed?

(III) Ergodicity, large deviations, etc. in the sum- and ~~max-stable~~ max-stable random fields indexed by $F_d$, $d \geq 2$.

(IV) Stationary S$\alpha$S random fields indexed by more complicated groups, not necessarily c+ble (say, Lie groups)?

(V) What if the underlying tree becomes random (say Galton-Watson)?