Gaussian complex zeros:

Conditional distribution on rare events

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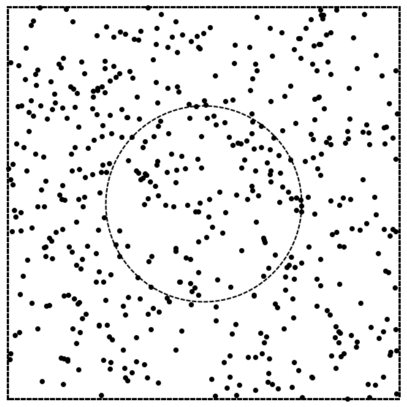
Joint works with Subhroshekhar Ghosh (NUS)
and with Aron Wennman (TAU)

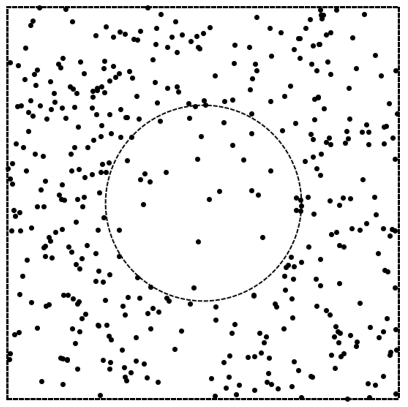
arXiv:1609.00084, 2009.08774

The setting

Homogeneous Poisson point process in the plane.



In 'equilibrium'



After thinning

Plan

1. Invariant point processes in $\mathbb{C}$
2. Rare events
3. Conditional limiting distribution and Potential theory - constrained minimizers

Invariant point processes in $\mathbb{C}$

Invariant\Stationary point processes in $\mathbb{C}$

- Point process in $\mathbb{C}$: $\mathfrak{X} = \{z_j\}_{j \in I}$
- Number of points in a set $\mathcal{G} \subset \mathbb{C}$: $n(\mathcal{G}) = n_{\mathfrak{X}}(\mathcal{G})$.
- Assume the distribution of $\mathfrak{X}$ is invariant with respect to isometries of $\mathbb{C}$ (rotations, translations, reflections).
- Examples:
    - Homogeneous Poisson Point Process

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- Examples:
    - Homogeneous Poisson Point Process
    - Infinite Ginibre ensemble ('eigenvalues')
    - Zeros of the Gaussian Entire Function

Ginibre ensemble (random eigenvalues)

Finite ensemble

- Complex eigenvalues of *non-Hermitian* $N \times N$ matrix
- Entries are *independent* standard *complex* Gaussian
- Determinantal point process

Infinite ensemble - limit of finite Ginibre as $N \to \infty$

- Also a determinantal point process
- 'Gas' with particle-particle interactions (repulsion) embedded in uniform background.
- Compare with Poisson p.p. which is a gas with no interactions between the particles.

Zeros of the Gaussian Entire Function (GEF)

- $\{\xi_n\}_{n=0}^{\infty}$ - independent standard complex Gaussians.
- GEF is given by the Gaussian Taylor series:

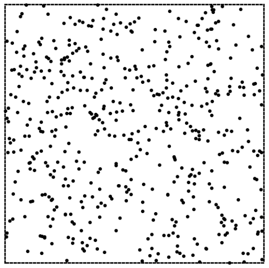
$$F(z) = \sum_{n=0}^{\infty} \xi_n \frac{z^n}{\sqrt{n!}}, \quad z \in \mathbb{C}.$$

  - Infinite radius of convergence (almost surely).
- Zero set: $\mathcal{Z}(F) = F^{-1}(0)$ is a *discrete* set in $\mathbb{C}$.
- Forms a point process which is *not* determinantal. More complicated interactions between the 'particles'.
- On short scales similar to Ginibre (repulsion).

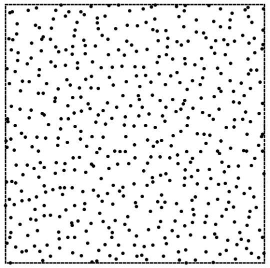
Some pictures...

All processes are normalized to have the same intensity.

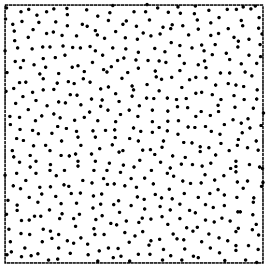
This is how they look like in 'equilibrium':



Poisson Point Process



Ginibre ensemble



Gaussian zeros

Expected number of points in $\mathcal{G}$ is $\frac{1}{\pi}\,\mathrm{Area}(\mathcal{G})$.

Rare events

Point processes - rare events

- $\mathfrak{X}_r$ rescaled process so that

$$\mathbb{E}\left[n_{\mathfrak{X}_r}(\mathcal{G})\right] = \frac{\text{Area}(\mathcal{G})}{\pi} r^2 \qquad (\text{interested in } r \to \infty)$$

- Statistics of GEF zeros are reasonably well-understood.
  - Variance: Forrester-Honner, Sodin-Tsirelson, Shiffman-Zelditch
  - CLT: Sodin-Tsirelson, Nazarov-Sodin
  - Large/Moderate deviations (disk): Krishnapur, Sodin-Tsirelson, Nazarov-Sodin-Volberg
- Consider rare events. Typical examples are:
  - Hole event: $\{n_{\mathfrak{X}_r}(\mathcal{G}) = 0\}$

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    - Deficiency: $\{n_{\mathfrak{X}_r}(\mathcal{G}) < \frac{1}{2}\mathbb{E}\left[n_{\mathfrak{X}_r}(\mathcal{G})\right]\}$

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    - Deficiency: $\{n_{\mathfrak{X}_r}(\mathcal{G}) < \frac{1}{2}\mathbb{E}\left[n_{\mathfrak{X}_r}(\mathcal{G})\right]\}$
    - Overcrowding: $\{n_{\mathfrak{X}_r}(\mathcal{G}) > 2\mathbb{E}\left[n_{\mathfrak{X}_r}(\mathcal{G})\right]\}$

Rare events and conditional distribution

E.g. for the hole event. As $r \to \infty$, what is the asymptotic rate of decay of

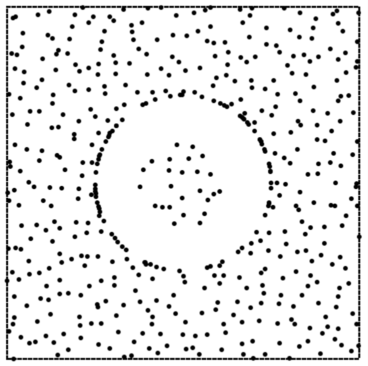
$$\mathbb{P}\left(n_{\mathfrak{X}_r}(\mathcal{G}) = 0\right)?$$

and what is the *liming spatial distribution* of the points *conditioned* on these rare events?

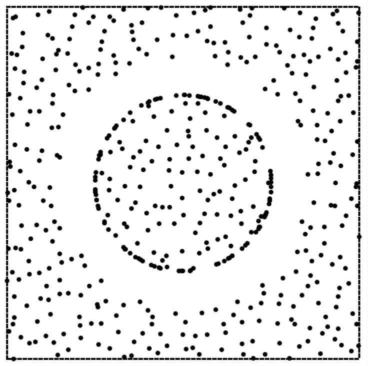
- Poisson process: follows immediately from the definition. Ginibre: determinantal structure, potential theory helps.
- Need to approximate with finite ensembles with N points, where N is a function of r.

Limiting conditional distribution - examples

Ginibre ensemble. $\mathcal{G}$ is the unit disk.



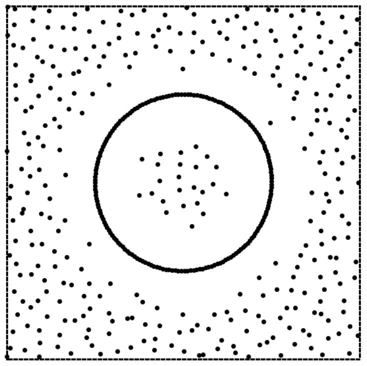
Deficiency



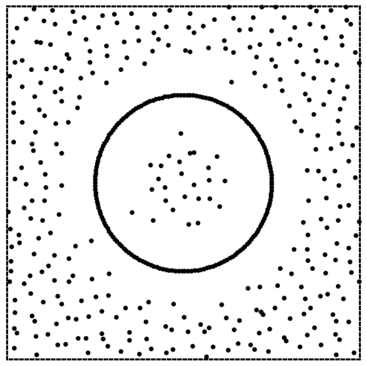
Overcrowding

Limiting conditional distribution - more examples

Zeros of the GEF. $\mathcal{G}$ is the unit disk.



Deficiency



Overcrowding

There is a partial duality (Ghosh-N.).

Large deviations for empirical distribution

Suppose we can approximate in some sense the process $\mathfrak{X}_r$ by a process $\mathfrak{X}^{(N)}$ with N points. We consider the empirical measure of the points of the latter process:

$$\mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{w_j}.$$

A large deviation principle (LDP) for the sequence of empirical measures μ_N roughly means that for nice subsets $\mathcal{C}$ of $M_1(\mathbb{C})$ we have

$$\log \mathbb{P}_{\mu_N \sim \mathfrak{X}^{(N)}} (\mu_N \in \mathcal{C}) \approx -a_N \inf_{\mu \in \mathcal{C}} I_{\mathfrak{X}}(\mu),$$

where $a_N \to \infty$ and $I_{\mathfrak{X}} : M_1(\mathbb{C}) \to \mathbb{R}^+$ is the *rate function*.

LDP - Ginibre ensemble

- Hiai-Petz, Ben Arous-Zeitouni
- Finite Ginibre $\mathfrak{G}^{(N)}$ – eigenvalues of $N \times N$ matrix with i.i.d. complex Gaussian entries (in random uniform order).
- Joint density of (complex) eigenvalues $\{w_1, \ldots, w_N\}$ w.r.t. Lebesgue measure on $\mathbb{C}^N$

$$\propto \prod_{j \neq k} |w_j - w_k| \cdot \exp\left(-N \sum_{j=1}^{N} |w_j|^2\right)$$

- Roughly speaking the probability of a rare event is determined by the maximum of the joint density over all "admissible configurations" of the points $\{w_1, \ldots, w_N\}$.

LDP - Ginibre ensemble - cont.

Write joint density in logarithmic scale:

$$\propto \exp\left(-N^2\left[\frac{1}{N^2}\sum_{j\neq k}\log\frac{1}{|w_j-w_k|}+\frac{1}{N}\sum_{j=1}^{N}|w_j|^2\right]\right)$$

We rewrite

$$\frac{1}{N}\sum_{j=1}^{N}|w_j|^2=\int|w|^2\,\mathrm{d}\mu_N(w)$$

and (disregarding the singularity on the diagonal)

$$\frac{1}{N^2}\sum_{j\neq k}\log\frac{1}{|w_j-w_k|}\asymp\iint\log\frac{1}{|z-w|}\,\mathrm{d}\mu_N(z)\,\mathrm{d}\mu_N(w)=:\mathcal{E}(\mu_N)$$

leading to the rate function:

$$I_{\mathfrak{G}}(\mu)=\int|w|^2\,\mathrm{d}\mu+\mathcal{E}(\mu)+C_{\mathfrak{G}}.$$

LDP - Ginibre ensemble - cont.

For example the hole event $\{n_{\mathfrak{G}_r}(\mathcal{G}) = 0\}$ corresponds to the set of measures

$$\mathcal{M}_{\mathcal{G}} \coloneqq \{\mu \in M_1(\mathbb{C}) : \ \mu(\mathcal{G}) = 0\}$$

(Remark: $\mathcal{M}_{\mathcal{G}}$ is actually *not* a good set of measures in the sense of large deviations theory).

Asymptotically, with $N \propto r^2$

$$\log \mathbb{P}\left(n_{\mathfrak{G}_r}(\mathcal{G}) = 0\right) \asymp \log \mathbb{P}_{\mu_N \sim \mathfrak{G}(N)}\left(\mu_N \in \mathcal{M}_{\mathcal{G}}\right) \asymp -N^2 \inf_{\mu \in \mathcal{M}_{\mathcal{G}}} I_{\mathfrak{G}}(\mu).$$

Rate function $I_{\mathfrak{G}}(\mu) = \int |w|^2 \,\mathrm{d}\mu + \mathcal{E}(\mu) + C_{\mathfrak{G}}$ is sometimes called in potential theory the *weighted logarithmic energy* of μ.

LDP - Gaussian complex zeros

The rescaled GEF is given by the Gaussian Taylor series:

$$F_r(z) = \sum_{n=0}^{\infty} \xi_n \frac{(rz)^n}{\sqrt{n!}}, \quad z \in \mathbb{C},$$

where ξ_n are independent standard complex Gaussians.

Approximate the zeros of F_r by zeros of the polynomials

$$P_N(z) = \sum_{n=0}^{N} \xi_n \frac{(rz)^n}{\sqrt{n!}} = \frac{r^N \xi_N}{\sqrt{N!}} \underbrace{\prod_{j=1}^{N} (z - w_j)}_{=: Q_N(z)}$$

Joint density of the zeros $\{w_1, \ldots, w_N\}$ is more complicated:

$$\propto \prod_{j \neq k} |w_j - w_k| \left(\int_{\mathbb{C}} |Q_N(z)|^2 \left[\frac{1}{\pi} e^{-|z|^2} \right] \mathrm{d}m(z) \right)^{-(N+1)}$$

LDP - Gaussian complex zeros - cont.

Zeitouni and Zelditch proved a large deviations result for the empirical measure of the zeros (in a more general setting). The rate function is given by

$$I_3(\mu) = 2 \sup_{z \in \mathbb{C}} \left\{ U^{\mu}(z) - \frac{|z|^2}{2} \right\} + \mathcal{E}(\mu) + C_3$$

using logarithmic potential and energy

$$U^{\mu}(z) = \int_{\mathbb{C}} \log |z - w| \, \mathrm{d}\mu(w)$$

$$\mathcal{E}(\mu) = \iint_{\mathbb{C}\times\mathbb{C}} \log \frac{1}{|z-w|} \mathrm{d}\mu(z) \, \mathrm{d}\mu(w) = -\int_{\mathbb{C}} U^{\mu}(z) \, \mathrm{d}\mu(z) .$$

Conditional limiting distribution and Potential theory - constrained minimizers

Conditional limiting distribution

The following heuristics holds (known as the "Gibbs Conditioning Principle" in large deviations theory):

E.g. in the case of the hole event in the set $\mathcal{G}$, the measure $\mu_{\mathcal{G}}$ in the set

$$\mathcal{M}_{\mathcal{G}} := \{\mu \in M_1(\mathbb{C}) : \ \mu(\mathcal{G}) = 0\},$$

that minimizes the value of the functional $I_{\mathfrak{X}}$ corresponds to the limiting distribution of the processes $\mathfrak{X}^{(N)}$ on the hole event.

Q: How to solve a constrained minimization problem for (convex) functionals on probability measures?

Results for the Ginibre ensemble

Round hole: determinantal structure allows direct computation. $n_{\mathfrak{G}_r}(\mathbb{D})-$ sum of indep. random variables.

- Jacovici-Lebowitz-Manificat - Prediction for finite β ensembles in two and three dimensions.
  - Including the Ginibre ensemble (determinantal).
  - Description of the limiting measure for round hole.
  - Predictions for moderate and large fluctuations (JLM).
- Shirai - infinite Ginibre ensemble.

There are also results for more general cases.

- Adhikari-Reddy - decay rate of the hole probability for general domains.
- Anderson-Serfaty-Zeitouni - description of the limiting measure for deficiency/overcrowding events.

Interlude - some results in one dimension

- Ben Arous-Guionnet - first empirical LDP for GUE.
- Majumdar-Nadal-Scardicchio-Vivo - description of limiting distribution conditioned on a 'gap'.
- Valkó-Virág, Holcomb-Valkó - decay rate of gap probability for Sine_β process.
    - Limiting conditional distribution?
- Basu-Dembo-Feldheim-Zeitouni - exponential concentration around the mean for zeros of stationary Gaussian processes.
- Many other results.

Complex Gaussian Zeros - circular case

There is no determinantal structure! The radial symmetry of Taylor series is helpful for circular domains.

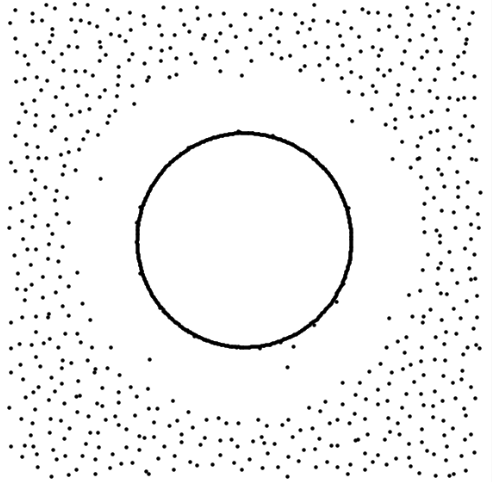
Theorem (Ghosh-N.)
The empirical distribution of the Gaussian zero process, conditioned on having no zeros in a disk of radius r, converges in distribution, as $r \to \infty$, to the Radon measure:

$$\mathrm{d}\mu_H(w) = e \cdot \delta_{\{|w|=1\}} + \mathbf{1}_{\{|w| \geq \sqrt{e}\}} \frac{\mathrm{d}m(w)}{\pi}$$

This affirms a prediction of Nazarov and Sodin – a "forbidden region" appears outside the hole, where the asymptotic density of the zeros vanishes.

Complex Gaussian Zeros - “forbidden region”

In order to have no zeros in a large disk, we have to balance by moving outer zeros to the boundary of the disk.



Complex Gaussian Zeros - circular case - cont.

There is no determinantal structure. The radial symmetry of Taylor series is helpful for circular domains.

- with S. Ghosh we identified the precise logarithmic decay rate of the probability of deficiency and overcrowding for the GEF, and the limiting distributions.
- Before, Sodin and Tsirelson found the correct rate of decay for these events (matching those of Ginibre).
- In addition, Nazarov, Sodin, and Volberg proved that the JLM prediction for large charge fluctuation of Coulomb systems also applies to the zeros.

Complex Gaussian Zeros - non circular case

Now there is no determinantal structure, no radial symmetry, and no connection to familiar objects from potential theory!

- with A. Wennman we found which shapes of the hole lead to a round forbidden region.
- can describe the class of shapes for forbidden regions corresponding to holes with smooth boundary (and more general cases).
- we had to develop some new methods related to free boundary$\backslash$obstacle problems, to describe properties of the constrained minimizing measures.

Complex Gaussian Zeros - disk-like domains

Definition
A simply-connected domain $\mathcal{G} \subset \mathbb{C}$ is *disk-like* with center 0 and radius 1, if the Riemann map $\varphi : \mathcal{G} \to \mathbb{D}$, which maps 0 to 0, with $\varphi'(0) = 1$, satisfies

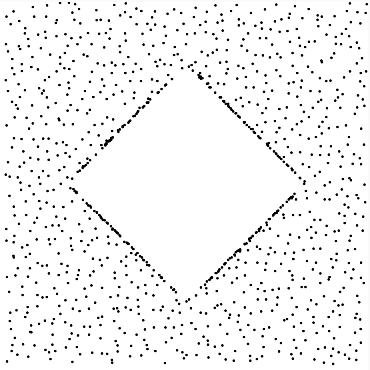
$$|\varphi(z)| \geq |z| \exp\left(-\frac{1}{2e}|z|^2\right), \qquad z \in \mathcal{G}.$$

Theorem (N.-Wennman)
The $\mathcal{G}$ be a sufficiently nice simply-connected domain. The forbidden region is the disk $\mathbb{D}(0, \sqrt{e})$ if and only if $\mathcal{G}$ is disk-like.

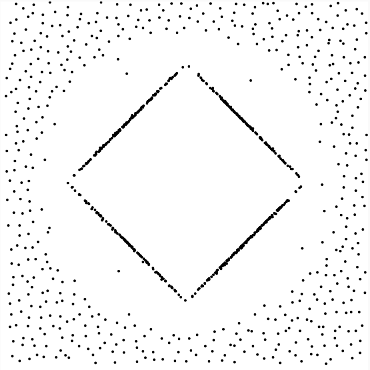
Remark: In this case the measure of the singular component is proportional to the *harmonic measure* of $\mathcal{G}$ from the point 0.

Disk-like domains - examples

One can check that equilateral triangles are *not* disk-like, while squares are. Limiting measures on a square shaped hole:



Ginibre



Zeros of GEF

Some open problems

What happens when $\mathcal{G}$ is not simply-connected?

- Some very partial results (annulus)

How does the singular component on the boundary of the hole looks like when we 'zoom in'?

- Shirai - found profile of the singular component for Ginibre ensemble in the circular case.
- Ginibre ensemble in the non-circular case ??
- Zeros of the GEF ??

Replace Gaussian coefficients with Gaussian processes.

The end

Thank you

for listening