Interacting Urns on Finite Graphs

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Joint work with Gursharn Kaur (University of Virginia).

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Urn models have been used for modeling clinical trials. More recently, they have been applied to studying opinion dynamics, reinforcement learning, evolutionary models, ant walks etc.

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- The model consisted of an urn which initially contains a finite number of balls of different colours.
- At every discrete time-step t, a ball is drawn from the urn uniformly at random and it is replaced in the urn along with $a > 0$ balls of the same colour.

Classical Urn Models

Theorem (*Eggenberger and Pólya, 1923*)

Let $Z(t)$ be the fraction of white balls in the urn after time t (or after t draws). Then, as $t \to \infty$,

$$Z(t) \xrightarrow{a.s.} Z$$

such that $Z \sim \beta\left(Z(0), 1 - Z(0)\right)$, where $\beta(\cdot,\cdot)$ denotes Beta distribution.

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Bagchi-Pal Urn: If the drawn ball is of white colour, it is replaced in the urn with a balls of the same colour and b balls of black colour and if it is of black colour, it is replaced in the urn with c balls of the same colour and d balls of white colour. Assuming the urn is balanced, that is $a + b = c + d$ and $a - c \leq (a + b)/2$, $Z(t) \to \frac{c}{b+c}$ as $t \to \infty$.

Further Extensions

- Multicolour Urns (Amites Dasgupta, Krishanu Maulik, Arup Bose, Irene Crimaldi).
- Infinite Colour Urns (Antar Bandyopadhyay, Debleena Thacker, Svante Janson, Cécile Mailler).
- Random Replacement (Svante Janson, Rafik Aguech, Irene Crimaldi).

Interacting Urns

A general two-colour interacting urn model

N urns such that the reinforcement in each urn depends on all the urns or on a non-trivial subset of the given set of N urns.

We restrict our discussion to two-colour balanced urn schemes.

More precisely, suppose there are N urns with configurations $(W_i(t), B_i(t))$, where $W_i(t)$ and $B_i(t)$ denote the number of white balls and black balls respectively, at time $t \geq 0$ in the i^{th} urn, for every $i \in [N]$.

More precisely, suppose there are N urns with configurations $(W_i(t), B_i(t))$, where $W_i(t)$ and $B_i(t)$ denote the number of white balls and black balls respectively, at time $t \geq 0$ in the i^{th} urn, for every $i \in [N]$.

The reinforcement in the i^{th} urn at time t is given by $(I_i^W(t), I_i^B(t)) = (W_i(t), B_i(t)) - (W_i(t-1), B_i(t-1))$. We consider non-negative and finite reinforcement.

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If the evolution of the i^{th} urn depends on urns at $\{i_1, \ldots, i_{k_i}\} \subseteq [N]$, then the conditional distribution of $(I_i^W(t), I_i^B(t))$ is determined by $W_{i_1}(t-1), \ldots, W_{i_{k_i}}(t-1)$.

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We call the set $\{i_1, \ldots, i_{k_i}\}$ the dependency set of the i^{th} urn. In particular, in a graph based general two-colour interacting urn model, a natural choice for the dependency set of an urn is the collection of urns in its neighbourhood.

Interacting Pólya or Friedman Urns

P. Dai Pra, I. G. Minelli, P. Y. Louis and I. Crimaldi studied the following Pólya interacting urn model (2014 & 2016).

They consider N two-colours urns in which the reinforcement of each urn depends on all the other urns.

- At time 0, each urn contains a white and b black balls with $a \geq 1, b \geq 1$.
- At each time $t+1$, given the fraction of balls of white colour in each urn at time t, independently of what happens in all the other urns, a new white ball is replaced in urn i with conditional probability $\alpha Z(t) + (1-\alpha) Z_i(t)$, where $Z(t) = \frac{1}{N} \sum_{i=1}^{N} Z_i(t)$ and $\alpha \in [0,1]$.

Main Results

- $\lim_{n\to\infty} Z_i(t) = \lim_{n\to\infty} Z(t) =: Z$ almost surely.
- $\mathbb{E}[(Z_i(t) - Z(t))^2] = \begin{cases} t^{-2\alpha} & \text{for } 0 < \alpha < 1/2 \\ t^{-1}\log t & \text{for } \alpha = 1/2 \\ t^{-1} & \text{for } 1/2 < \alpha \leq 1. \end{cases}$.
- $\sqrt{t}(Z(t) - Z) \to \mathcal{N}\left(0, \frac{1}{N}(Z(1-Z)\right)$.
- For $\alpha > 1/2$,
  $\sqrt{t}(Z_j(t) - Z) \to \mathcal{N}\left(0, \left(\frac{1}{N} + \frac{1-1/N}{2\alpha-1}\right)(Z(1-Z)\right)$.
- For $\alpha = 1/2$, $\sqrt{\frac{t}{\log t}}(Z_j(t) - Z) \to \mathcal{N}\left(0, \left(1 - \frac{1}{N}\right)(Z(1-Z)\right)$.
- For $0 < \alpha < 1/2$, $t^{\alpha}(Z_j(t) - Z)$ converges almost surely (and in $\mathcal{L}^1$) to an almost surely non-zero random variable.

Interacting Urns on a Finite Directed Graph

(Joint work with Gursharn Kaur)

Graph Structure

Consider a directed network $\mathcal{G} = (V, \mathcal{E})$, where $V = [N]$ is the set of nodes and $\mathcal{E}$ is the set of directed edges.

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The vertex set of the graph is divided into two disjoint sets, the set of stubborn vertices (zero in-degree) and the set of flexible vertices (non-zero in-degree). These are denoted by S and F respectively.

- No isolated vertices in the graph.
- No vertices with only a single self loop.

So, $V = S \cup F$ with $S = \{i \in V : d_i^{\mathrm{in}} = 0\}$ and $F = \{i \in V : d_i^{\mathrm{in}} > 0\}$.

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Without loss of generality, we assume that the vertices labelled $1, \ldots, |F|$ are flexible, while the rest are stubborn.

With this labelling, the adjacency matrix A (with $A_{i,j} = \mathbb{I}_{\{i \to j\}}$) is of the form.

$$\begin{pmatrix} A_F & \mathbf{0} \\ A_{SF} & \mathbf{0} \end{pmatrix}.$$

Reinforcement dynamics

Suppose each vertex $i \in V$ has an urn that contains balls of two colours, white and black. Let $(W_i(t), B_i(t))$ be the configuration of the urn at vertex i at time $t \geq 0$.

Reinforcement dynamics

Suppose each vertex $i \in V$ has an urn that contains balls of two colours, white and black. Let $(W_i(t), B_i(t))$ be the configuration of the urn at vertex i at time $t \geq 0$.

A ball is selected uniformly at random from all the urns simultaneously and independently of every other urn. The colours of these balls are noted and they are replaced into their respective urns. For every $i \in V$, if the colour of the ball selected from the i^{th} urn is white, α_i white and $m_i - \alpha_i$ black balls are added to each urn j, such that $i \to j$; and if the colour of the ball selected from the i^{th} urn is black then $m_i - \beta_i$ white balls and β_i black balls are added to each urn j, such that $i \to j$.

In other words, the i^{th} urn reinforces its out-neighbours according to the following reinforcement matrix

$$R_i = \begin{pmatrix} \alpha_i & m_i - \alpha_i \\ m_i - \beta_i & \beta_i \end{pmatrix}.$$

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2. Non-Pólya type: if $0 < \alpha_i + \beta_i < 2m_i$.

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What can we say about the liming fraction of balls of colour white in each urn?

Notation

- $\mathbf{Z}(t) = (Z_1(t), \ldots, Z_N(t))$.
- $a_i = \alpha_i / m_i$ and $b_i = \beta_i / m_i$. Let $\mathrm{a} = (a_1, \ldots, a_N), \mathrm{b} = (b_1, \ldots, b_N)$ and $\hat{m}_i = \sum_{j \in N(i)} m_j$.
- Diagonal matrices:
  $B = \mathrm{Diag}(a_1 + b_1 - 1, \ldots, a_N + b_N - 1)$,
  $\mathbf{T}(t) = \mathrm{Diag}\left(T_1(t), \ldots, T_N(t)\right)$,
  $M = \mathrm{Diag}(m_1, \ldots, m_N)$ and
  $\hat{M} = \mathrm{Diag}(\hat{m}_1, \ldots, \hat{m}_{|F|}, 0, \ldots, 0)$.
- $W = BMA\hat{M}^{-1}$ where $\hat{M}^{-1} = \begin{bmatrix} \hat{M}_F^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Convergence of $\mathbf{Z}(n)$

Theorem (G. Kaur and N. S.)

Suppose for every $f \in F$ there exists a node v such that $v \rightsquigarrow f$ and either $v \in S$ or $R_v \neq m_v I$. Then as $t \to \infty$, $I - W_F$ is invertible and

$$\mathbf{Z}_F(t) \xrightarrow{a.s.} z^\star := \left[\mathbf{1} - (bMA\hat{M}^{-1})_F + \mathbf{Z}_S(0)W_{SF}\right](I - W_F)^{-1}.$$

Synchronization

Corollary

Suppose conditions for convergence as in the above theorem hold. Then, under the synchronization conditions **SC1** *and* **SC2** *given below, as $t \to \infty$, for every $i \in F$*

$$Z_i(t) \xrightarrow{a.s.} \frac{m^F - \beta^F + m^S - m^{Z_S(0)} + \alpha^{Z_S(0)} + \beta^{Z_S(0)}}{2m^F + m^S - \alpha^F - \beta^F}.$$

In particular,

1. *If $Z_j(0) = w$ for every $j \in S$ and synchronization conditions* **SC1** *and* **SC2** *hold, then $Z_i(t) \xrightarrow{a.s.} \frac{m^F + m^S - \beta^F + w(\alpha^S + \beta^S - m^S)}{2m^F + m^S - \alpha^F - \beta^F}$, as $t \to \infty$ for every $i \in F$.*
2. *If $S = \emptyset$ and synchronization condition* **SC1** *holds, then, $Z_i(t) \xrightarrow{a.s.} \frac{m^F - \beta^F}{2m^F - \alpha^F - \beta^F}$, as $t \to \infty$ for every $i \in V$.*

Conditions for Synchronization

(SC1) There exist $\alpha^F, \beta^F, m^F, m^S \in \mathbb{R}$ with $\alpha^F + \beta^F < 2m^F + m^S$, such that for every $i \in F$,

$$\sum_{j \in N_i \cap F} R_j = \begin{pmatrix} \alpha^F & m^F - \alpha^F \\ m^F - \beta^F & \beta^F \end{pmatrix}.$$

(SC2) If $S \neq \emptyset$, there exist $\alpha^{Z_S(0)}, \beta^{Z_S(0)}, m^{Z_S(0)} \in \mathbb{R}$ such that for every $i \in F$, $\sum_{j \in N_i \cap S} m_i = m^S$ and

$$\sum_{j \in N_i \cap S} Z_j(0) R_j = \begin{pmatrix} \alpha^{Z_S(0)} & m^{Z_S(0)} - \alpha^{Z_S(0)} \\ m^{Z_S(0)} - \beta^{Z_S(0)} & \beta^{Z_S(0)} \end{pmatrix}$$

Uniform Reinforcement

When $a_i = a, b_i = b, m_i = m$ for all $i \in V$, $W = BMA\hat{M}^{-1}$ reduces to $(a+b-1)\tilde{A}$ where $\tilde{A} = \begin{bmatrix} A_F & \mathbf{0} \\ A_{SF} & \mathbf{0} \end{bmatrix} \begin{bmatrix} D_F^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Synchronization conditions reduce to $Z_S(0)\tilde{A}_{SF} = c_1\mathbf{1}$ and $\mathbf{1}\tilde{A}_F = c_2\mathbf{1}$. In that case, the limiting fraction is given by

$$\frac{(1-b) + (a+b-1)c_1}{1-(a+b-1)c_2}\mathbf{1}$$

and it equals

$$\frac{1-b}{2-a-b}\mathbf{1}$$

for $S = \emptyset$.

Idea of the Proof: Stochastic Approximation Theory

A stochastic approximation scheme in $\mathbb{R}^d$ is given by:

$$x(t+1) = x(t) + a(t+1)[h(x(t)) + M(t+1)], t \geq 0,$$

where $sup_t \|x(t)\| < \infty$ almost surely, and:

- The map $h : R^d \to R^d$ is Lipschitz.
- $\sum_t a(t) = \infty$ and $\sum_t a(t)^2 < \infty$.
- $\{M(t)\}_{t \geq 0}$ is a Martingale difference sequence with respect to the increasing family of σ-fields given by $\sigma(x(m), M(m))_{m \leq t}$. $\{M(t)\}$ are square-integrable with

$$E[\|M(t+1)\|^2 | \mathcal{F}_t] \leq K(1 + \|x(t)\|^2)$$

for some constant $K > 0$.

The main result of the stochastic approximation theory says that the iterates of the recursion for $x_t \in \mathbb{R}^d$ satisfying

$$x(t+1) = x(t) + a(t+1)[h(x(t)) + M(t+1)], t \geq 0,$$

along with the given conditions, converge almost surely to the stable limit points of the solutions of the ordinary differential equation given by $\dot{x}(t) = h(x(t))$.

Proof of Convergence

Let Let $Y_i(t)$ denote the indicator of the event that a white ball is drawn from the i^{th} urn at time $t \geq 1$.

$$\begin{aligned}
Z_i(t+1) &= \frac{1}{T_i(t+1)} W_i(t+1) \\
&= \frac{T_i(t)}{T_i(t+1)} Z_i(t) \\
&\quad + \frac{1}{T_i(t+1)} \sum_{j \in N_i} \left[\alpha_j Y_j(t+1) + (m_j - \beta_j)(1 - Y_j(t+1))\right] \\
&= Z_i(t) - \frac{\hat{m}_i}{T_i(t+1)} Z_i(t) \\
&\quad + \frac{1}{T_i(t+1)} \sum_{j \in N_i} m_j \left[(a_j + b_j - 1) Z_j(t) + 1 - b_j\right] \\
&\quad + \frac{1}{T_i(t+1)} \sum_{j \in N_i} m_j (a_j + b_j - 1) \Delta Y_j(t+1)
\end{aligned}$$

$\Delta Y_j(t+1) = Y_j(t+1) - \mathbb{E}[Y_j(t+1)|\mathcal{F}_t]$ is a Martingale difference sequence. We can write the recursion for $\mathbf{Z}_F(t)$.

$$
\begin{aligned}
\mathbf{Z}_F(t+1) = \mathbf{Z}_F(t) + (\Delta Y(t+1)\ W\hat{M})_F \mathbf{T}_F(n+1)^{-1} \\
+ \left[-\mathbf{Z}(t) + \mathbf{Z}(t)W + \ (\mathbf{1} - \mathrm{b})MA\hat{M}^{-1}\right]_F \hat{M}_F \mathbf{T}_F(t+1)^{-1}.
\end{aligned}
$$

where $W = BMA\hat{M}^{-1}$ is of the form

$$
\begin{aligned}
W &= \begin{bmatrix} (BM)_F & \mathbf{0} \\ \mathbf{0} & (BM)_S \end{bmatrix} \begin{bmatrix} A_F & \mathbf{0} \\ A_{SF} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{M}_F^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} (BM)_F\ A_F\ \hat{M}_F^{-1} & \mathbf{0} \\ (BM)_S\ A_{SF}\ \hat{M}_F^{-1} & \mathbf{0} \end{bmatrix}.
\end{aligned}
$$

Since $\mathbf{T}(t) = \mathbf{T}(0) + t\hat{M}$, therefore $\hat{M}_F \mathbf{T}_F^{-1}(t) = \mathcal{O}(1/t)$ and

$$h(z) = -z + zW_F + \mathbf{Z}_S(0)W_{SF} + \mathbf{1} - (\mathrm{b}MA\hat{M}^{-1})_F.$$

Hence, the unique equilibrium point is given by

$$z^\star = \left[\mathbf{1} - (\mathrm{b}MA\hat{M}^{-1})_F + \mathbf{Z}_S(0)W_{SF}\right](I - W_F)^{-1},$$

whenever $I - W_F$ is invertible.

The flexible set F can be partitioned into strongly connected components $F_1, \ldots, F_k$ and W_F can be written as an upper block triangular matrix as follows

$$I - W_F = \begin{pmatrix} I_1 - W_{F_1} & -W_{F_1,F_2} & \ldots & -W_{F_1,F_k} \\ 0 & I_2 - W_{F_2} & \ldots & -W_{F_2,F_k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_k - W_{F_k} \end{pmatrix},$$

where $W_{F_i,F_j} = (BM)_{F_i} A_{F_i,F_j} (\hat{M}^{-1})_{F_j}$ is a $|F_i| \times |F_j|$ matrix and $W_{F_i} = W_{F_i,F_i}$. It is enough to show that each block on diagonal is invertible.

Note that $(i,j)^{\text{th}}$ element of W_F is

$$[W_F]_{i,j} = \frac{(\alpha_i + \beta_i - m_i)\mathbb{I}_{i \to j}}{\hat{m}_j} = \frac{(\alpha_i + \beta_i - m_i)\mathbb{I}_{i \to j}}{\sum_{i \in N(j)} m_i}$$

Therefore the j-th column sum of $W_{F'}$ is given by

$$\sum_{i \in F'} [W_{F'}]_{i,j} = \frac{\sum_{i \in N(j) \cap F'} (\alpha_i + \beta_i - m_i)}{\sum_{i \in N(j)} m_i} \leq \frac{\sum_{i \in N(j) \cap F'} m_i}{\sum_{i \in N(j)} m_i},$$

where the last inequality holds since $\alpha_i + \beta_i \leq 2m_i$ for every $i \in V$. Note that the j-th column sum is strictly less than 1 under the conditions of the theorem.

Fluctuation Theorems

A similar argument gives that the real parts of all the eigenvalues of $I - W_{F'}$ are all positive. Thus, $z^\star$ is a stable equilibrium.

What can we say about $Z(t) - Z^\star \mathbf{1}$?

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(A1) F is strongly connected.

(A2) $W = BMA\hat{M}^{-1}$ is diagonalisable.

Fluctuation Theorems: Notation

That is, there exists an invertible matrix U such that with $V = U^{-1}$, $W = U \Lambda V = U \, \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) V$, where $\lambda_1, \ldots, \lambda_N$ are the N eigenvalues of W such that $\Re(\lambda_1) \geq \Re(\lambda_2) \geq \cdots \geq \Re(\lambda_N)$. Let $u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ be the right and left eigenvectors of the eigenvalues $\lambda_1, \ldots, \lambda_N$ respectively.

Define $H \coloneqq I - W_F$ and $\rho \coloneqq \lambda_{\min}(H)$, where I is a $|F| \times |F|$ identity matrix. Define Θ is the $N \times N$ diagonal matrix such that

$$\Theta_{i,i} = \begin{cases} z_i^\star (1 - z_i^\star) & i \in F, \\ Z_i(0)(1 - Z_i(0)) & i \in S. \end{cases}$$

Fluctuation Theorems

Theorem (G. Kaur and N. S.)

Suppose $\mathbf{Z}_F(n) \longrightarrow z^\star$ almost surely as $t \to \infty$. Then,

1. *for $\rho > 1/2$, as $t \to \infty$*

$$\sqrt{t}\left(\mathbf{Z}_F(t) - z^\star\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \Sigma\right)$$

   with $\Sigma_{ij} = \sum_{k \in F} \sum_{\ell \in F} \frac{\lambda_k \lambda_\ell}{1-\lambda_k-\lambda_\ell}(u_k^\top \Theta u_\ell) v_{ki} v_{lj}, \forall i,j \in F$.

2. *for $\rho = 1/2$, as $t \to \infty$*

$$\sqrt{\frac{t}{\log t}}\left(\mathbf{Z}_F(t) - z^\star\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \Sigma\right),$$

   with $\Sigma_{ij} = \frac{1}{4}(u_1^\top \Theta u_1) v_{1i} v_{1j}, \forall i,j \in F$.

Special Cases

Suppose $\mathbf{Z}_F(t) \xrightarrow{a.s.} z^\star \mathbf{1}$.

1. Suppose $W = W^\top$. Then,
   - For $\rho > 1/2$, $\Sigma = z^\star(1 - z^\star) W^2 (I - 2W)^{-1}$.
   - For $\rho = 1/2$, $\Sigma = z^\star(1 - z^\star) W^2 U^\top \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} U$.
2. Under the synchronization condition **SC1**,
   - For $\rho > 1/2$, $\Sigma = \dfrac{(m^F - \beta^F)(m^F - \alpha^F)}{(2m^F - \beta^F - \alpha^F)^2} W^2 (I - 2W)^{-1}$.
   - For $\rho = 1/2$, $\Sigma = \frac{(m^F - \beta^F)(m^F - \alpha^F)}{N(m^F)^2} J$.

Special Cases: Uniform Reinforcement

$\tilde{A} = \tilde{A}^\top$. Define $C(a,b) = \frac{(a+b-1)^2(1-a)(1-b)}{2-a-b)^2}$.

- For $\rho > 1/2$, $C(a,b)\tilde{A}_F^2(I - 2(a+b-1)\tilde{A}_F)^{-1}$.
- For $\rho = 1/2$, $\frac{C(a,b)}{N}J$.

Friedman type: When $a = b$, $\mathbf{Z}_F(n) \xrightarrow{a.s.} \frac{1}{2}\mathbf{1}$, and

$$C(a,b) = \left(a - \frac{1}{2}\right)^2.$$

Special Cases: Uniform Reinforcement

$H = I - (a+b-1)\tilde{A}_F$ which implies

$$\rho = \begin{cases} 1-(a+b-1)\lambda_{max}(\tilde{A}_F) & \text{when } a+b-1>0 \\ 1-(a+b-1)\lambda_{min}(\tilde{A}_F) & \text{when } a+b-1<0. \end{cases}$$

Pólya type Reinforcement

What happens when $S = \emptyset$ and there exists v such that for all $w \rightsquigarrow v$, w is of Pólya type?

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BC1: There exists is constant $r > 0$ such that $\mathbf{1}MA = r\mathbf{1}$ and $MA\mathbf{1}^\top = r\mathbf{1}^\top$.

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What happens when $S = \emptyset$ and there exists v such that for all $w \rightsquigarrow v$, w is of Pólya type?

Suppose the graph is strongly connected and the following balance conditions hold.

BC1: There exists is constant $r > 0$ such that $\mathbf{1}MA = r\mathbf{1}$ and $MA\mathbf{1}^\top = r\mathbf{1}^\top$.

BC2: The initial total number of balls in each urn is the same.

Pólya type Reinforcement

Since there are no stubborn vertices we denote the vector $(Z_1(t), \ldots, Z_N(t))$ by $\mathbf{Z}_t$.

Theorem (G. Kaur and N. S.)

Assume that the graph is strongly connected and **BC1**, **BC2** *hold. Then, there exists a finite random variable* Z^∞ *such that* $\mathbf{Z}_t \xrightarrow{a.s.} Z^\infty \mathbf{1}$.

CLT for Pólya type Reinforcement

Theorem (G. Kaur and N. S.)

Suppose $\mathbf{Z}_t \to Z^\infty \mathbf{1}$ *almost surely. The following hold.*

1. $\Re(\lambda_2) < 1/2$,
   $\sqrt{t}(\mathbf{Z}_t - Z^\infty \mathbf{1}) \xrightarrow{d} \mathcal{N}\left(0, Z^\infty(1 - Z^\infty)\left(\frac{1}{N}J + \Sigma\right)\right)$, *where* $\Sigma = USU^\top$, $S_{h,j} = \frac{1}{1-\lambda_h-\lambda_j} v_h^\top v_j$, *for* $1 \le h, j \le N$.
2. $\Re(\lambda_2) = 1/2$, $\sqrt{\frac{t}{\log(t)}}(\mathbf{Z}_t - Z^\infty \mathbf{1}) \xrightarrow{d} \mathcal{N}(0, Z^\infty(1 - Z^\infty)\Sigma)$, *where* $\Sigma = USU$ *with*

$$S_{h,j} = \begin{cases} v_h^\top v_j & \lambda_h + \lambda_j = 1 \\ 0 & \lambda_h + \lambda_j \neq 1 \end{cases}, \qquad \text{for } 1 \le h, j \le N.$$

Recursion for $\mathbf{Z}_n$

For Pólya type reinforcement we have $\mathrm{a} = \mathrm{b} = \mathbf{1}, B = I, \hat{M} = rI$, and thus $W = BMA\hat{M}^{-1} = \frac{1}{r}MA$. We have $(Y_1(t), \dots, Y_N(t))$ by Y_t. Under conditions **BC1** and **BC2** we have $T_i(t) = T_j(t) =: T_t$ for all $i, j \in V$ and the recursion becomes

$$\mathbf{Z}_{t+1} = \left(1 - \frac{r}{T_{t+1}}\right)\mathbf{Z}_t + \frac{1}{T_{t+1}} Y_{t+1} MA.$$

Let $\Upsilon_t := I - \frac{r}{T_t}\left(I - \frac{1}{r}MA\right)$, then we can write

$$\begin{aligned}\mathbf{Z}_{t+1} &= \mathbf{Z}_t \Upsilon_{t+1} + \frac{1}{T_{t+1}} \Delta Y_{t+1} \\ &= \mathbf{Z}_0 \prod_{k=1}^{t+1} \Upsilon_k + \sum_{j=1}^{t+1} \frac{1}{T_j} \Delta Y_j \prod_{k=j+1}^{t+1} \Upsilon_k\end{aligned}$$

Sketch of the Proof for Convergence

Proposition

Under the balance conditions **BC1** *and* **BC2**, $\bar{Z}_t \coloneqq \frac{1}{N} Z_t \mathbf{1}^\top$ *is a Martingale and* $\bar{Z}_t \to Z^\infty$ *for some finite random variable* Z^∞.

Proof.

Using the balance condition **BC1**

$$\mathbb{E}[\bar{Z}_{t+1} | \mathcal{F}_t] = \frac{1}{N} \mathbb{E}[Z_{t+1} \mathbf{1}^\top | \mathcal{F}_t] = \frac{1}{N} Z_t \Upsilon_{t+1} \mathbf{1}^\top = \frac{1}{N} Z_t \mathbf{1}^\top = \bar{Z}_t$$

Thus $\bar{Z}_t$ is a bounded martingale. Thus, there exists a random variable Z^∞ taking values in $[0, 1]$ such that $\bar{Z}_t \to Z^\infty$ almost surely. $\square$

Let $D_t = Z_t - \bar{Z}_t \mathbf{1} = Z_t \left(I - \frac{1}{N} J \right)$. Next, we show that under the two balance conditions **BC1** and **BC2**, $D_t \to 0$ in $\mathcal{L}^2$ and almost surely. We do this by proving that $\lim_{t \to \infty} \mathbb{E}[\|D_t\|^2] = 0$ and that $\|D_t\|^2$ admits an almost sure limit as $n \to \infty$.

Sketch of the Proof: Notation

Let $\Lambda_t := I - \frac{r}{T_t}(I - \Lambda)$ and $K := I - \frac{1}{N}J$, then

$$\Upsilon_t = P\left(I - \frac{r}{T_t}(I - \Lambda)\right)P^{-1} = P\Lambda_t P^{-1} \quad \text{and} \quad K = P\Psi P^{-1},$$

where

$$\Psi = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and,

$$\Lambda_t = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 - \frac{r}{T_t}(1 - \lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - \frac{r}{T_t}(1 - \lambda_t) \end{pmatrix}.$$

Sketch of the Proof

Note that

$$\begin{aligned} D_t &= \sum_{j=0}^{t} \frac{1}{T_j} U_j \left(\prod_{k=j+1}^{t} \Upsilon_k \right) K \\ &= \sum_{j=0}^{t} \frac{1}{T_j} U_j P \left[\left(\prod_{k=j+1}^{t} \Lambda_k \right) \Psi \right] P^{-1}. \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[\|D_t\|^2\right] &\leq \mathbb{E}\left[\sum_{j=0}^{t} \frac{1}{T_j^2} \left\| U_j P \left(\prod_{k=j+1}^{t} \Lambda_k \right) \Psi P^{-1} \right\|^2 \right] \\
&\leq \sum_{j=0}^{t} \mathbb{E}\left[\|U_j\|^2\right] \frac{1}{T_j^2} \left\| P \left(\prod_{k=j+1}^{t} \Lambda_k \right) \Psi P^{-1} \right\|^2 \\
&\leq \sum_{j=0}^{t} \frac{\tilde{C}}{T_j^2} \left\| \left(\prod_{k=j+1}^{t} \Lambda_k \right) \Psi \right\|^2 \\
&\leq \sum_{j=0}^{t} \frac{\tilde{C}}{T_j^2} \prod_{k=j+1}^{t} \left(\max_i \left| 1 - \frac{r}{T_k} (1 - \lambda_i) \right| \right)^2 \\
&\approx \sum_{j=0}^{t} \frac{1}{j^2} \left(\frac{t}{j} \right)^{-2(1-\lambda_2)} \\
&= t^{-2(1-\lambda_2)} \sum_{j=0}^{t} j^{-2\lambda_2}.
\end{aligned}
$$

This gives:

$$\mathbb{E}\left[\|D_t\|^2\right] = \begin{cases} \mathcal{O}(1/t) & \text{for } -1 \leq \lambda_2 < 1/2 \\ \mathcal{O}\left(\log(t)/t\right) & \text{for } \lambda_2 = 1/2 \\ \mathcal{O}(t^{-2(1-\lambda_2)}) & \text{for } 1/2 < \lambda_2 < 1 \end{cases}$$

Note that $\mathbb{E}[D_{t+1}|\mathcal{F}_n] = \mathbb{E}[Z_{t+1}|\mathcal{F}_t]K = Z_t\Upsilon_t K = D_t\Upsilon_t$, since K and Υ_t commute with each other under the balance condition **BC1**.

$$
\begin{aligned}
\mathbb{E}[\|D_{t+1}\|^2|\mathcal{F}_t] &= \mathbb{E}[D_{t+1}(D_{t+1})^\top|\mathcal{F}_t] \\
&= \mathbb{E}[D_{t+1}|\mathcal{F}_t]\mathbb{E}[D_{t+1}|\mathcal{F}_t]^\top + \mathbb{E}[\|\Delta D_{t+1}\|^2|\mathcal{F}_t] \\
&= D_t\Upsilon_t(D_t\Upsilon_t^\top) + \mathbb{E}[\|\Delta D_{t+1}\|^2|\mathcal{F}_t] \\
&= \|D_t\|^2 - D_t(I - \Upsilon_t\Upsilon_t^\top)D_t^\top + \mathbb{E}[\|\Delta D_{t+1}\|^2|\mathcal{F}_t] \\
&\leq \|D_t\|^2 + \mathbb{E}[\|\Delta D_{t+1}\|^2|\mathcal{F}_t] \\
&\leq \|D_t\|^2 + \frac{C}{(t+1)^2}
\end{aligned}
$$

A Theorem of Robbins and Siegmund

Theorem (Almost Super-Martingales)

[1] *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_t\}$ be a filtration of sub σ-fields of $\mathcal{F}$. Let U_t, β_t, γ_t be non-negative measurable random sequences such that for all $t \geq 1$,*

$$\mathbb{E}[U_{t+1}|\mathcal{F}_t] \leq (1 + \beta_t)U_t + \nu_t - \gamma_t$$

Then on the set $\{\sum_t \beta_t < \infty, \sum_t \nu_t < \infty\}$, U_t converges almost surely to a random variable and $\sum_t \gamma_t < \infty$ almost surely.

---

[1] An application of a theorem of Robbins and Siegmund, D. Anbar, The Annals of Statistics (1976) and A convergence theorem for nonnegative almost supermartingales and some applications, H. Robbins and D. Siegmund, Optimizing Methods in Statistics (1971).

CLT for Pólya type

- Show that $\bar{Z}_t - Z^{\infty}$, scaled appropriately, converges to a Gaussian.
- Show that $\mathbf{Z}_t - \bar{Z}_t\mathbf{1}$, , scaled appropriately, converges to a Gaussian.

Thank you!