Some non-existence results for the stochastic wave equation

The effect of noise on blow-up of deterministic systems

Mohammud Foondun

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University of Strathclyde

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Some References

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The heat equation problem

An ODE

- Consider the following ODE

$$\frac{dy}{dt} = b(y), \quad y(0) = y.$$

- The solution blows up in finite time if and only if

$$T = \int_y^\infty \frac{1}{b(s)} ds$$

is finite.

The SDE

- We now consider the following SDE:

$$X_t = X_0 + \int_0^t b(X_s)ds + B_t.$$

- The solution blows up almost surely if and only if

$$\int_{\cdot}^{\infty} \frac{1}{b(s)} ds < \infty.$$

- This is a special case of Feller's test for explosion.
- The intuition is that the Brownian motion pushes the solution up so that the deterministic part makes the solution blows up.

The deterministic heat equation

- We now consider the heat equation with Dirichlet boundary condition.
$$\frac{\partial u}{\partial t} = \Delta u + b(u) \quad \text{on} \quad [0,1]$$
- The solution will blow-up in finite time if
$$\int_{\cdot}^{\infty} \frac{1}{b(s)} ds < \infty$$
and the initial condition u_0 is large enough.
- The point is that the presence of the Dirichlet Laplacian makes it that the above integral condition is not enough for blow-up.

The stochastic heat equation

- We now look at the following stochastic heat equation

$$\frac{\partial u}{\partial t} = \Delta u + b(u) + \dot{W} \quad \text{on} \quad [0,1]$$

- It turns out since the noise terms pushes the solution up, the following condition guarantees blow-up no matter what the initial condition is

$$\int_{\cdot}^{\infty} \frac{1}{b(s)} ds < \infty.$$

- Question: What kind of condition do be have for the stochastic wave equation?

Proof ideas

- Look at $y(t) := \int_0^1 u(t,x)\phi(x)dx$
- Use comparison arguments.
- For the deterministic case, compare with a non-linear ODE which can be analysed
- For the stochastic case, compare with an SDE which blows up according to Feller's test.
- If $y(t)$ blows up, then $\sup_{x\in[0,1]} |u(t,x)|$ blows up as well.

The wave equation

An ODE

- Consider the second order ODE:

$$\frac{d^2y}{dt^2} = b(y) \quad y(0) = \alpha, \quad y'(0) = \beta.$$

- This is equivalent to

$$y(t) = \alpha + \beta t + \int_0^t (t-s)b(y(s))\,ds$$

- The solution blows up if and only if

$$\int_\alpha^\infty \frac{1}{[\beta^2 + 2\int_\alpha^s b(r)dr]^{1/2}} ds < \infty$$

- What is the role of this condition on blow-up properties of stochastic equations?

The proof

- $$y''(t)y'(t) = b(y(t))y'(t) \quad t \geq 0$$
- This is equivalent to
$$y'(t)^2 - y'(0)^2 = 2\int_0^t b(y(s))\,dy(s) = 2\int_\alpha^{y(t)} b(r)\,dr.$$
- This is of the form $y'(t) = F(y(t))$. So that the integral condition for the first order ODE applies and we obtain the required condition.

- Consider the following wave equation with Dirichlet boundary condition.

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + b(u) \quad \text{on} \quad [0,1]$$

- Set

$$\alpha = \int_0^1 \phi(x) u(x,0) dx \quad \beta = \int_0^1 \phi(x) u_t(x,0) dx$$

- The solution blows up in finite time provided that

$$T = \int_\alpha^\infty \left[\mu\alpha^2 + \beta^2 - \mu s^2 + 2\int_\alpha^s b(r) dr \right]^{-1/2} ds < \infty$$

The Stochastic wave equation

- We look at

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + b(u) + \dot{W} \quad \text{on} \quad [0,1]$$

with Dirichlet boundary conditions.

- Suppose that for $\alpha, \beta > 0$, we have

$$T(\alpha, \beta) := \int_\alpha^\infty \frac{1}{[\beta^2 + 2\int_\alpha^s b(r)dr]^{1/2}} ds < \infty$$

- Then the solution blows up with a positive probability.

The idea behind the proof

- We look at the integral formulation of the solution.

$$u(t,x) = \int_0^1 G(t,x,y) v_0(y)\, dy + \frac{\partial}{\partial t}\left(\int_0^1 G(t,x,y) u_0(y)\, dy\right)$$
$$+ \int_0^t \int_0^1 G(t-s,x,y)\, W(ds\, dy)$$
$$+ \int_0^t \int_0^1 G(t-s,x,y) b(u(s,y))\, ds\, dy \quad \text{a.s.}$$

- A major difficulty is that we do not have a comparison principle for the wave equation.
- But an important observation is that for small times we can still use some kind of comparison argument together with the fact that the stochastic part gets large with a positive probability.

- The kernel $G(t,x,y)$ is given by the following

$$G(t,x,y) := \sum_{n=1}^{\infty} \frac{\sin(n\pi t)}{n\pi} \varphi_n(x)\varphi_n(y),$$

  where $\varphi_n(x) = \sqrt{2}\sin(n\pi x)$, $n \geq 1$.
- We look at $y(t) = \int_0^1 u(t,x)\phi_1(x)dx$.
- The idea is to compare $y(t)$ with an integral equation for a certain time interval only.
- The integral condition will then follow.

- The stochastic part is given by

$$\int_0^t \int_0^1 \int_0^1 G(t-s,x,y)\varphi_1(x)W(ds\,dy)\,dx,$$

- This can be rewritten as

$$\begin{aligned} M(t) &= \int_0^t \int_0^1 \sin(\pi(t-s))\varphi_1(y)\,W(dy\,ds) \\ &= C\int_0^t \sin(\pi(t-s))\,dB_s \\ &= C\pi\int_0^t \cos(\pi(t-s))B_s\,ds. \end{aligned}$$

- Using this and the support theorem for Brownian motion, we can show that in a certain time interval the above quantity can get large with a positive probability.

- The function $y(t)$ can then be compared to an integral equation of the form

$$y(t) = A + B(t - t_0) + 2\int_{t_0}^{t} (t-s) b(y(s))\, ds + L \quad t \in [t_0, T].$$

- For blow-up we need

$$\int_{A+L}^{\infty} \frac{1}{[B^2 + 2\int_{A+L}^{s} b(r)\, dr]^{1/2}}\, ds < \infty,$$

- Consider

$$y(t) = A + Bt + \int_0^t (t-s)b(y(s))\,ds + G(t) \quad t \in [0,T].$$

- $G(t)$ grows to infinity as $t \to \infty$
- For blow-up we need

$$\int_\alpha^\infty \frac{1}{[\beta^2 + 2\int_\alpha^s b(r)\,dr]^{1/2}}\,ds < \infty,$$

The wave equation on the whole line

- We look at
$$\frac{\partial^2 u}{\partial t^2} = \Delta u + b(u) + \dot{W}$$
on the whole line
- Suppose that for $\alpha, \beta > 0$, we have
$$T(\alpha,\beta) := \int_\alpha^\infty \frac{1}{[\beta^2 + 2\int_\alpha^s b(r)dr]^{1/2}} ds < \infty$$
- Then the solution blows up almost surely.

The idea behind the proof

- We look at the integral formulation of the solution.

$$
\begin{aligned}
u(t,x) = &\int_{-\infty}^{\infty} G(t,x,y) v_0(y)\, dy + \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} G(t,x,y) u_0(y)\, dy\right) \\
&+ \int_{-\infty}^{\infty}\int_0^1 G(t-s,x,y)\, W(ds\, dy) \\
&+ \int_{-\infty}^{\infty}\int_0^t G(t-s,x,y) b(u(s,y))\, ds\, dy \quad \text{a.s.}
\end{aligned}
$$

- The following is a Gaussian process

$$
g(t,\, x) := \int_0^t \int_{-\infty}^{\infty} G(t-s,\, x-y) W(dy\, ds).
$$

- For fixed $x \in \mathbf{R}$, almost surely,

$$\limsup_{t\to\infty} \frac{g(t,x)}{t\sqrt{\log\log t}} > 1.$$

- This requires a bit of Gaussian theory to prove.
- We can then compare the mild solution to an integral equation which blows up.