Moderate deviation estimates in stationary last passage percolation

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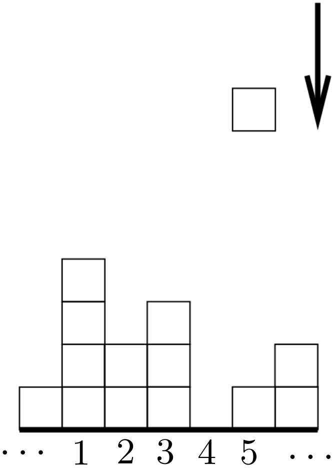
Falling blocks

- Blocks falling on $\mathbb{Z}$ with space-time i.i.d. interarrival times.
- At time t, interface height $\sim t$ and fluctuations $\sim \sqrt{t}$.
- Model yielding a smoother surface?

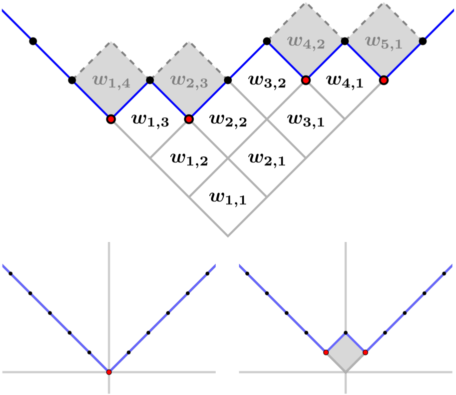


Corner growth model

- $w_{i,j}$ are i.i.d. .
- Change a valley to a trough after the corresponding time $w_{i,j}$.
- $T_{i,j}$: the time of absorbing (i, j) satisfies the recursion

$$T_{i,j} = \max\{T_{i-1,j}, T_{i,j-1}\} + w_{i,j}.$$

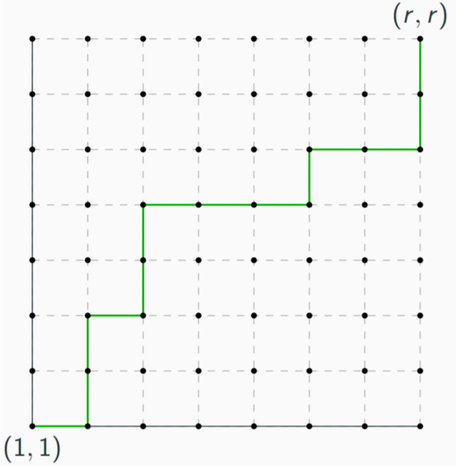
- Expect linear growth with $t^{1/3}$ fluctuations in the surface height.



Last Passage Percolation on $\mathbb{Z}^2$

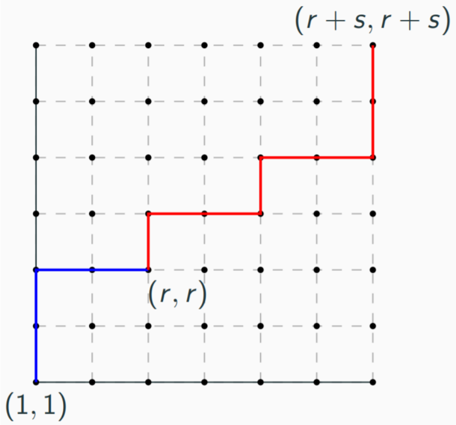
- Have i.i.d. random variables $w_{i,j}$ on the vertices. The weight of a path is the sum of the values of the traversed vertices in $\mathbb{Z}^2$.
- $G(u, v)$ is the maximum weight of up-right paths going from u to v. The almost surely unique path attaining $G(u, v)$ is called the geodesic.
- For convenience, $G(n) = G((1, 1), (n, n))$.
- Satisfies the same recursion

$$G(u, v) = \max\{G(u, v - e_1), G(u, v - e_2)\} + w_v.$$



LPP on $\mathbb{Z}^2$: first order behaviour

- Linear growth in all directions: $\frac{G(\mathbf{0},\alpha(m,n))}{\alpha} \to c(m,n) \in (0,\infty)$ as $\alpha \to \infty$ almost surely.
- A consequence of superadditivity: $G(r+s) \geq G(r) + G(s)'$ and Fekete's lemma/Kingman's theorem.



Exponential LPP

- Each vertex in $\mathbb{Z}^2$ carries an i.i.d. $\mathrm{Exp}(1)$ variable.

Theorem (Johansson'99)

For exponential LPP and $m \geq n$,

$$\mathbb{P}(G(m,n) \leq t) = \frac{1}{Z_{m,n}} \int_{[0,t]^n} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{j=1}^{n} x_j^{m-n} e^{-x_j} d^n x.$$

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- weight of geodesic $\leftrightarrow$ length of longest increasing subsequence in a random generalized permutation $\leftrightarrow$ length of top row in a pair of random Young Tableaux.
- Theorem implies that $G(m,n)$ has same distribution as the largest eigenvalue of X^*X where X is an $m \times n$ matrix of i.i.d. standard complex Gaussian random variables.

Exponential LPP: Properties

- Limit shape: $\frac{\mathbb{E}G(\mathbf{0},\alpha(m,n))}{\alpha} \to (\sqrt{m}+\sqrt{n})^2$ as $\alpha \to \infty$.
- $\frac{G(n)-4n}{n^{1/3}}$ converges in distribution to a multiple of the GUE Tracy-Widom distribution.
- [Ledoux, Rider '10]: For all $y < \delta n^{2/3}$ and for all large n,

$$\mathbb{P}(G(n)-4n > yn^{1/3}) \leq C_1 e^{-c_1 y^{3/2}},$$

$$\mathbb{P}(G(n)-4n > yn^{1/3}) \geq C_3 e^{-c_3 y^{3/2}},$$

$$\mathbb{P}(G(n)-4n < -yn^{1/3}) \leq C_2 e^{-c_2 y^{3}}.$$

- [Basu, Ganguly, Hegde, Krishnapur '19] Lower tail lower bound

$$\mathbb{P}(G(n)-4n < -yn^{1/3}) \geq C_4 e^{-c_4 y^{3}}.$$

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- Works for all directions away from the boundaries.
- Optimal in the exponent.

Exponential LPP: Transversal Fluctuations

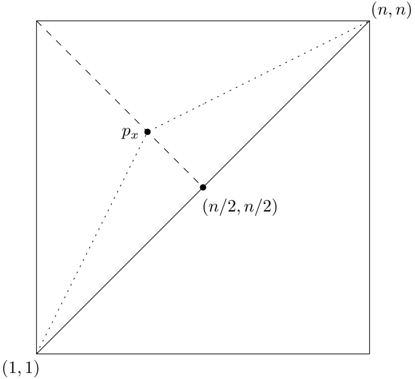
- Let A_α be the event that the geodesic for $G(n)$ stays in a strip of width α about the line $\{x = y\}$.
- [Johansson '99] $\mathbb{P}(A_{n^{2/3+\epsilon}}) \to 1$ and $\mathbb{P}(A_{n^{2/3-\epsilon}}) \to 0$ as $n \to \infty$.
- [Basu, Sidoravicius, Sly '16] $\mathbb{P}((A_{rn^{2/3}})^c) \leq C_1 e^{-c_1 r^3}$.
- [Hammond, Sarkar '18] $\mathbb{P}((A_{rn^{2/3}})^c) \geq C_2 e^{-c_2 r^3}$.

Exponential LPP: Transversal Fluctuations

- How far does the geodesic for $G(n)$ venture from the line $\{x = y\}$?.
- By the limit shape result, for $p_x = (n/2 - x, n/2 + x)$,

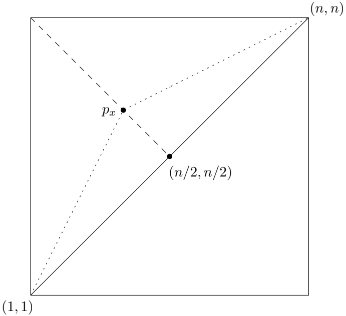
$$\mathbb{E}G\left((1,1), p_x\right) + \mathbb{E}G\left(p_x, (n,n)\right) \sim 4n - C\frac{x^2}{n}.$$

- For typical transversal fluctutations, heuristically $\frac{x^2}{n} \sim n^{1/3}$ and thus $x \sim n^{2/3}$.



Exponential LPP: Transversal Fluctuations

- Notation $\widetilde{G}(n) = G(n) - \mathbb{E}G(n)$.
- An upper bound on the probability of B_x: the event that $p_x = (n/2 - x, n/2 + x)$ lies on the geodesic.
- An application of the point-to-point moderate deviation estimates. Here $x = rn^{2/3}$.



$$\mathbb{P}(B_x) = \mathbb{P}(G((1,1),p_x)) + G(p_x,(n,n)) = G(n))$$

$$\leq \mathbb{P}\left(\widetilde{G}((1,1),p_x)) \geq \frac{C}{4}r^2 n^{1/3}\right) + \mathbb{P}\left(\widetilde{G}(p_x,(n,n)) \geq \frac{C}{4}r^2 n^{1/3}\right)$$

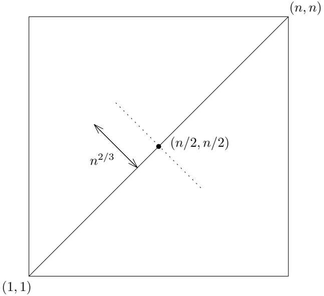
$$+ \mathbb{P}\left(\widetilde{G}(n) \leq -\frac{C}{2}r^2 n^{1/3}\right)$$

$$\leq e^{-c_1 r^3} + e^{-c_2 r^3} + e^{-c_3 r^6} \leq e^{-cr^3}.$$

Exponential LPP: Transversal fluctuations

- By technical results, can obtain

$$\mathbb{P}\left(\max_{x\in(-n^{2/3},n^{2/3})}\{\widetilde{G}(p_x)\}\geq yn^{1/3}\right)\leq e^{-cy^{3/2}}.$$



- Summing up over r would give an e^{-cr^3} bound for the geodesic going a distance more than $rn^{2/3}$ transversally at the midpoint.

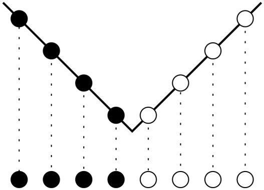
The TASEP

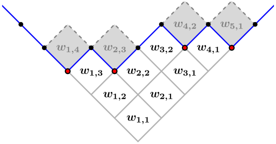
- Totally Asymmetric Exclusion Process.
- Start with a configuration of particles and holes on $\mathbb{Z} + \frac{1}{2}$.
- Vertices have i.i.d. $\mathrm{Exp}(1)$ clocks which signal the respective particle to attempt a jump to its right.
- A jump is successful if there is a hole to the right of a particle.

Connection between TASEP and exponential LPP

- Start the TASEP with the step initial condition: all particles to the left of 0 and all holes to the right of 0.
- If a particle moves from $i + \frac{1}{2}$ to $(i+1) + \frac{1}{2}$, then flip the wedge on the line $\{x = i + 1\}$.
- The time taken for the particle at $-m + \frac{1}{2}$ to jump n steps to the right has the same distribution as $G(m, n)$.
- Both satisfy the same recusion:

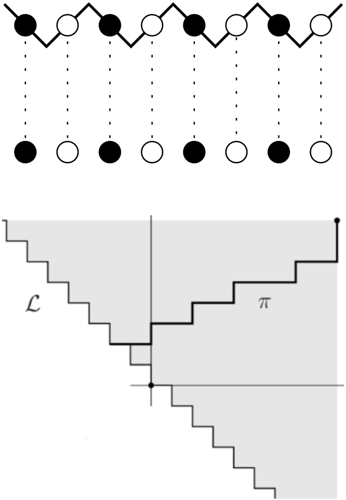
$$G((m, n)) = \max\{G((m-1, n)), G((m, n-1))\} + w_{(m,n)}.$$





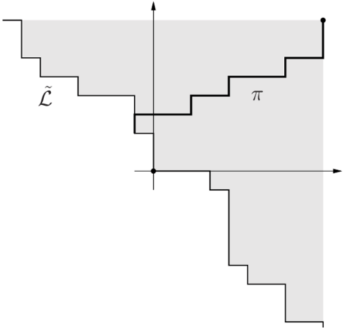
Other initial conditions for the TASEP: Flat

- The flat initial condition: particles at $(i + \frac{1}{2})$ for all odd i.
- Time taken to add (m, n) is distributed as the point-to-line passage time $G_\ell(m, n)$ from $\{x + y = 0\}$ to (m, n) in exponential LPP.
- [Baik, Rains] Same as the distribution of $\frac{1}{2}\lambda_{2n+1}$ where λ_{2n+1} is the largest eigenvalue of $X^T X$, and X is a $(2n + 2) \times (2n + 1)$ matrix of i.i.d. standard normal variables.



Other initial conditions for the TASEP: Stationary

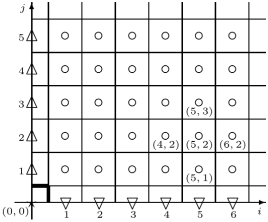
- [Liggett '76] Product $\mathrm{Ber}(\varrho)$ measures are the extremal stationary measures for the TASEP.
- Corresponds to the point-to-line passage time from a random line fluctuating about the deterministic line $\{y = -\frac{\varrho}{1-\varrho}x\}$.



Stationary LPP

- Start the TASEP with a particle at $0 + \frac{1}{2}$, a hole at $-1 + \frac{1}{2}$ and a product $\mathrm{Ber}(\varrho)$ measure elsewhere.
- Burke Property: Interpreting particles as servers and holes as customers, the particle at $0 + \frac{1}{2}$ moves according to a Poisson process of jump rate $(1 - \varrho)$.

$$
\begin{aligned}
&\omega_{00} = 0, && \text{where the } \star \text{ is,}\\
&\omega_{i0} \sim \mathrm{Exp}(1-\varrho),\ i \geq 1, && \text{where the } \nabla\text{'s are,}\\
&\omega_{0j} \sim \mathrm{Exp}(\varrho),\ j \geq 1, && \text{where the } \triangle\text{'s are,}\\
&\omega_{ij} \sim \mathrm{Exp}(1),\ i, j \geq 1, && \text{where the } \circ\text{'s are.}
\end{aligned}
$$



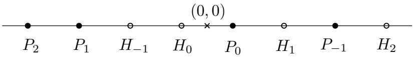


Figure from [Bálazs, Cator, Seppäläinen '06].

Stationary LPP

- [Bálazs, Cator, Seppäläinen '06] Time for the n^{th} particle to the left of 0 crossing the m^{th} hole to the right of 0 is distributed according to the passage time $G_{\text{stat}}(m, n)$.
- Time has same distribution on the boundaries. For the interior, the recursion is the same.

$$
\begin{aligned}
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&\omega_{i0} \sim \mathrm{Exp}(1-\varrho),\ i \geq 1, && \text{where the } \nabla\text{'s are,}\\
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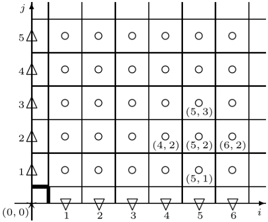


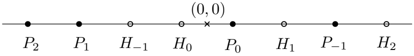
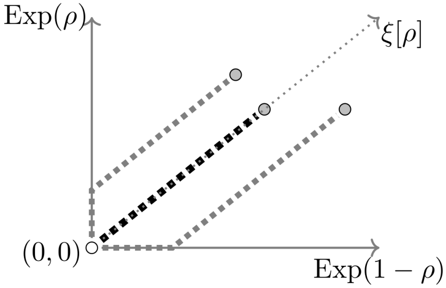


Figure from [Bálazs, Cator, Seppäläinen '06].

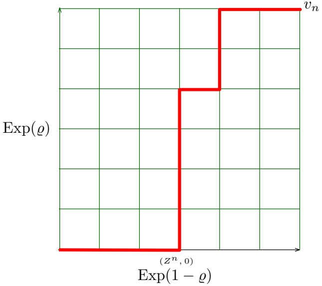
The characteristic direction

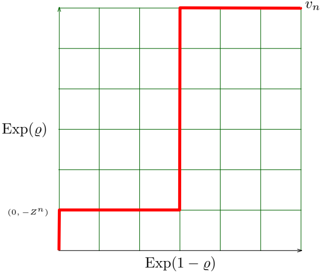
- A first order calculation indicates that the geodesic in the direction $((1-\varrho)^2, \varrho^2)$ spends a negligible amount of time on the boundary.
- Finding the vector (θ_1, θ_2) such that $\frac{x}{1-\varrho} + (\sqrt{\theta_1 - x} + \sqrt{\theta_2})^2$ and $\frac{x}{\varrho} + (\sqrt{\theta_1} + \sqrt{\theta_2 - x})^2$ are maximized at $x = 0$.



Exit time

- A general point in the characteristic direction $v_n = ((1-\varrho)^2 n, \varrho^2 n)$.
- Exit time: Z^n the point at which the geodesic for $G_{\text{stat}}(v_n)$ exits the coordinate axes.

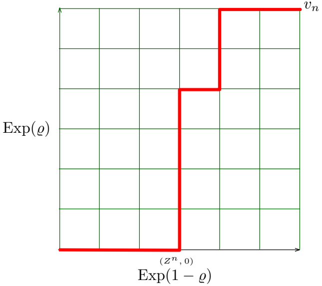


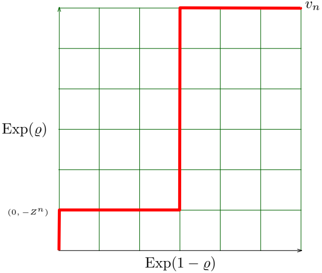


Upcoming

- Z^n: the exit time in the characteristic direction fluctuates at the scale $n^{2/3}$.
- Deviation estimates for the exit time at the correct scale.

Exit time





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Expected weight of geodesics with exit time x

- $v_n = ((1-\varrho)^2 n, \varrho^2 n)$, a generic point in the characteristic direction.
- Recall that $\frac{x}{1-\varrho} + (\sqrt{\theta_1 - x} + \sqrt{\theta_2})^2$ and $\frac{x}{\varrho} + (\sqrt{\theta_1} + \sqrt{\theta_2 - x})^2$ are maximised at $x = 0$ for the characteristic direction $(\theta_1, \theta_2) = ((1-\varrho)^2, \varrho^2)$.
- Taylor expansion:

$$\frac{x}{1-\varrho} + (\sqrt{\theta_1 n - x} + \sqrt{\theta_2 n})^2 \sim n - \frac{4\varrho}{(1-\varrho)^3}\frac{x^2}{n}$$

- For $\varrho = 1/2$, the above is $n - \frac{x^2}{n}$.
- For typical exit times, heuristically $\frac{x^2}{n} \sim n^{1/3}$ and thus $x \sim n^{2/3}$.

Exit time tail estimates

- Upper tail estimates?
- [Bálazs, Cator, Seppäläinen '06] Power law tail: $\mathbb{P}(|Z^n| \geq rn^{2/3}) \leq \frac{C}{r^3}$.
- [Ferrari, Occelli '18] $\mathbb{P}(|Z^n| \geq rn^{2/3}) \leq Ce^{-cr^2}$.
- [Seppäläinen, Shen '19] $\mathbb{P}(|Z^n| \geq rn^{2/3}) \geq Ce^{-cr^3}$.

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- Lower tail: [Seppäläinen, Shen '19] $\mathbb{P}(|Z^n| \leq \delta n^{2/3}) \leq C\delta|\log \delta|^{2/3}$.
- [???] $\mathbb{P}(|Z^n| \leq \delta n^{2/3}) \geq C\delta$.

Theorem ([B. '20],[Emrah, Janjigian, Seppäläinen '20])

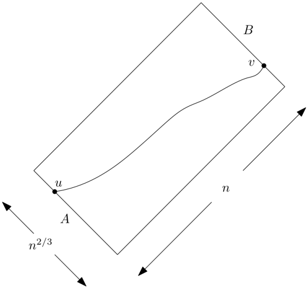
$$\mathbb{P}(|Z^n| \geq rn^{2/3}) \leq Ce^{-cr^3}$$

Technical estimates

- Recall that for all $x < \delta n^{2/3}$,
$$\mathbb{P}(G(n) - 4n > xn^{2/3}) \leq C_1 e^{-c_1 x^{3/2}}.$$

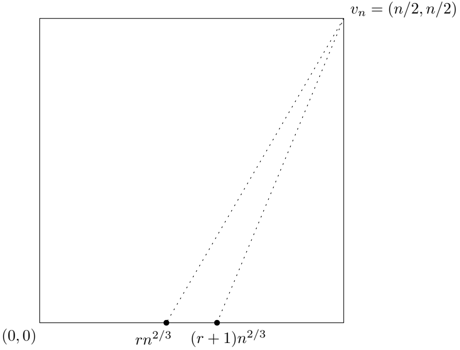
- [Basu, Sidoravicius, Sly '16]
$$\mathbb{P}\left(\max_{u \in A, v \in B} G(u,v) \geq 4n + xn^{1/3}\right) \leq C_2 e^{-c_2 x^{3/2}}.$$

- Works with uniform constants as long as the long sides are bounded away from being horizontal/vertical.



Upper tail for the exit time

- Take $\varrho = 1/2$.
- Will show $\mathbb{P}(Z^n \in (rn^{2/3}, (r+1)n^{2/3}) \leq Ce^{-cr^3}$ with uniform constants as long as $0 < r < (1-\epsilon)\frac{n^{1/3}}{2}$.
- The case $Z^n \geq (1-\epsilon)\frac{n^{1/3}}{2}$ can be handled separately. We skip it.
- Then $\mathbb{P}(Z^n \geq rn^{2/3}) \leq \sum_r^\infty e^{-cx^3} \sim e^{-cr^3}$.



Bounding $\mathbb{P}(Z^n \in (rn^{2/3}, (r+1)n^{2/3})$

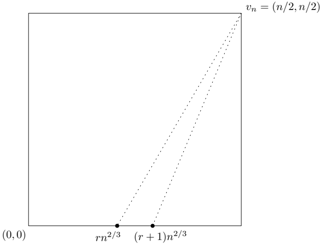
- Let w_r denote $(rn^{2/3}, 0)$. For $\mathbf{x}$ in (w_r, w_{r+1}), we have $\mathbb{E}\left(G_{\mathrm{stat}}(\mathbf{x}) + G(\mathbf{x}, v_n)\right) \leq n - r^2 n^{1/3}$.

$$\mathbb{P}(Z^n \in (w_r, w_{r+1})) = \mathbb{P}\left(\max_{\mathbf{x}\in(w_r,w_{r+1})}\{G_{\mathrm{stat}}(\mathbf{x}) + G(\mathbf{x}, v_n)\} = G_{\mathrm{stat}}(v_n)\right)$$

$$\leq \mathbb{P}\left(G_{\mathrm{stat}}(v_n) < n - \frac{r^2 n^{1/3}}{2}\right) + \mathbb{P}\left(\max\{G_{\mathrm{stat}}(\mathbf{x}) + G(\mathbf{x}, v_n)\} > n - \frac{r^2 n^{1/3}}{2}\right)$$

$$\leq \mathbb{P}\left(\max\left\{\widetilde{G}(\mathbf{x}, v_n)\right\} \geq \frac{r^2 n^{1/3}}{4}\right) + \mathbb{P}\left(\max\left\{\widetilde{G}_{\mathrm{stat}}(\mathbf{x})\right\} \geq \frac{r^2 n^{1/3}}{4}\right)$$

$$\leq e^{-cr^3} + ().$$



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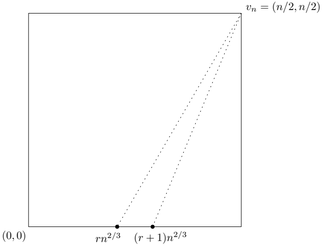
$$\mathbb{P}(Z^n \in (w_r, w_{r+1})) = \mathbb{P}\Big(\max_{\mathbf{x}\in(w_r,w_{r+1})} \{G_{\mathrm{stat}}(\mathbf{x}) + G(\mathbf{x}, v_n)\} = G_{\mathrm{stat}}(v_n)\Big)$$

$$\leq \mathbb{P}\Big(G_{\mathrm{stat}}(v_n) < n - \frac{r^2 n^{1/3}}{2}\Big) + \mathbb{P}\Big(\max\{G_{\mathrm{stat}}(\mathbf{x}) + G(\mathbf{x}, v_n)\} > n - \frac{r^2 n^{1/3}}{2}\Big)$$

$$\leq \mathbb{P}\Big(\max\left\{\widetilde{G}(\mathbf{x}, v_n)\right\} \geq \frac{r^2 n^{1/3}}{4}\Big) + \mathbb{P}\Big(\max\left\{\widetilde{G}_{\mathrm{stat}}(\mathbf{x})\right\} \geq \frac{r^2 n^{1/3}}{4}\Big)$$

$$\leq e^{-cr^3} + ().$$

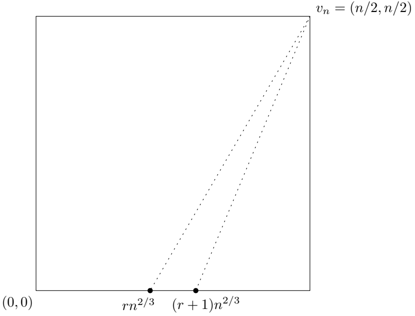
- Notice that $G_{\mathrm{stat}}(v_n) \geq G((1,1), v_n)$.
- Moderate deviations for exponential LPP gives e^{-cr^6} upper bound for the red term.



The term coming from the boundary weights

- A random walk estimate.

$$\mathbb{P}\left(\frac{G_{\mathrm{stat}}(w_r) - 2w_r}{\sqrt{r}n^{1/3}} \geq \frac{r^{3/2}}{4}\right) \leq e^{-cr^3}.$$



Moderate deviations for the stationary passage time

- Bounding the tails

$$\mathbb{P}(G_{\mathrm{stat}}(v_n) - n \geq yn^{1/3}),$$
$$\mathbb{P}(G_{\mathrm{stat}}(v_n) - n \leq -yn^{1/3}).$$

- First, compare to tails of exponential LPP.

Tails for exponential LPP

- [Ledoux, Rider '10]: For all $y < \delta n^{2/3}$ and for all large n,

$$\mathbb{P}(G(n) - 4n > yn^{1/3}) \leq C_1 e^{-c_1 y^{3/2}},$$

$$\mathbb{P}(G(n) - 4n > yn^{1/3}) \geq C_3 e^{-c_3 y^{3/2}},$$

$$\mathbb{P}(G(n) - 4n < -yn^{1/3}) \leq C_2 e^{-c_2 y^{3}}.$$

- [Basu, Ganguly, Hegde, Krishnapur '19] Lower tail lower bound

$$\mathbb{P}(G(n) - 4n < -yn^{1/3}) \geq C_4 e^{-c_4 y^{3}}.$$

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$$\mathbb{P}(G(n) - 4n < -yn^{1/3}) \geq C_4 e^{-c_4 y^{3}}.$$

- Works for Laguerre ensembles generally:

$$\mathbb{P}(G_\ell(n) - 4n < -yn^{1/3}) \geq C_5 e^{-c_5 y^{3}}.$$

- Recall that $G_\ell(m, n)$ is the line-to-point passage time from $\{x + y = 0\}$ to (m, n) and $G_\ell(n) = G_\ell(n, n)$.

Upper tail in stationary LPP

Theorem ([B. '20],[Emrah, Janjigian, Seppäläinen '20])

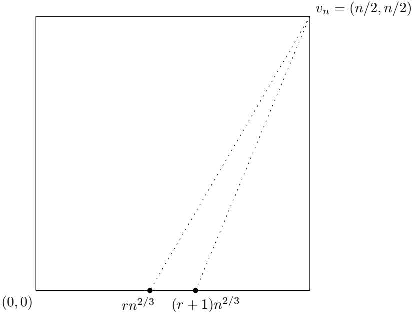
For all $y < \delta n^{2/3}$ and n large enough,

$$C_1 e^{-c_1 y^{3/2}} \le \mathbb{P}(G_{\mathrm{stat}}(v_n) - n \ge y n^{1/3}) \le C_2 e^{-c_2 y^{3/2}}.$$

- $G_{\mathrm{stat}}(v_n) \ge G((1,1), v_n)$ gives lower bound.
- Do $\varrho = 1/2$ for convenience. Recall $v_n = (n/2, n/2)$.
- Need to bound $\mathbb{P}(\max_{\mathbf{x} \in (-n/2, n/2)} \{G_{\mathrm{stat}}(\mathbf{x}) + G(\mathbf{x}, v_n)\} \ge n + y n^{1/3})$.
- Paths with large exit times can be handled. Reduces to bounding $\mathbb{P}(\max_{\mathbf{x} \in (-n/4, n/4)} \{G_{\mathrm{stat}}(\mathbf{x}) + G(\mathbf{x}, v_n)\} \ge n + y n^{1/3})$.

Upper tail in stationary LPP

- Recall $w_r = (rN^{2/3}, 0)$ and that for $\mathbf{x}$ in (w_r, w_{r+1}), we have $\mathbb{E}\left(G_{\mathrm{stat}}(\mathbf{x}) + G(\mathbf{x}, v_n)\right) \leq n - r^2 n^{1/3}$.
  $\mathbb{P}\big(\max_{\mathbf{x}\in(w_r, w_{r+1})} \{G_{\mathrm{stat}}(\mathbf{x}) + G(\mathbf{x}, v_n)\} \geq n + yn^{1/3}\big) \leq Ce^{-c(y+r^2)^{3/2}}$.
- $\sum_{r=-\infty}^{\infty} e^{-c(y+r^2)^{3/2}} \sim e^{-cy^{3/2}}$.



Lower tail in stationary LPP

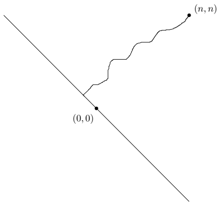
Theorem ([B. '20])

For $\varrho = 1/2$ and all $y < \delta n^{2/3}$ and n large enough,

$$C_1 e^{-c_1 y^3} \leq \mathbb{P}(G_{\text{stat}}(v_n) - n \leq -yn^{1/3}) \leq C_2 e^{-c_2 y^3}.$$

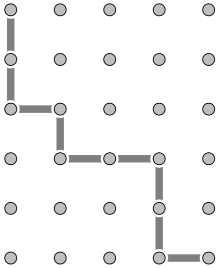
- $G_{\text{stat}}(v_n) \geq G((1,1), v_n)$ gives the upper bound for all ϱ.
- For the lower bound, we will use

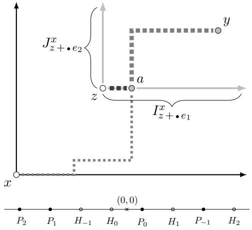
$$\mathbb{P}(G_\ell(n) - 4n < -yn^{1/3}) \geq Ce^{-cy^3}.$$



Strong Burke property of stationary LPP

- The increments along any down-right path are independent. The vertical and horizontal increments are distributed as $\mathrm{Exp}(\varrho)$ and $\mathrm{Exp}(1-\varrho)$ respectively.
- Increment stationarity of the model: $G^x_{\mathrm{stat}}(\cdot) - G^x_{\mathrm{stat}}(z)$ is distributed as $G^z_{\mathrm{stat}}(\cdot)$.
- Gives a coupling where $G^z_{\mathrm{stat}}(\cdot)$ is computed by using boundary weights given by $I^x_{z+\cdot e_1}$ and $J^x_{z+\cdot e_2}$.





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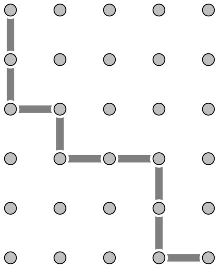
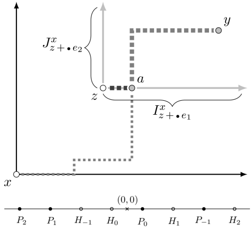
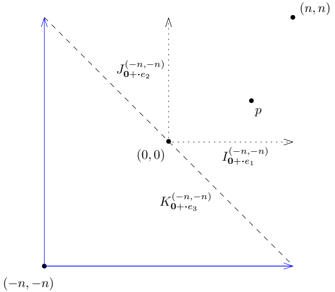
- Gives a coupling where $G^z_{\mathrm{stat}}(\cdot)$ is computed by using boundary weights given by $I^x_{z+\bullet e_1}$ and $J^x_{z+\bullet e_2}$.
- $I^x_{z+ie_1} = G^x_{\mathrm{stat}}(ie_1) - G^x_{\mathrm{stat}}((i-1)e_1)$: horizontal increments of $G^x_{\mathrm{stat}}(\cdot)$ on the line $z + \mathbb{Z}e_1$.
- $J^x_{z+ie_2} = G^x_{\mathrm{stat}}(ie_2) - G^x_{\mathrm{stat}}((i-1)e_2)$: vertical increments of $G^x_{\mathrm{stat}}(\cdot)$ on the line $z + \mathbb{Z}e_2$.
- a is the unique point maximising $G^x_{\mathrm{stat}}(a) + G(a,y)$ or equivalently $G^x_{\mathrm{stat}}(a) - G^x_{\mathrm{stat}}(z) + G(a,y)$.

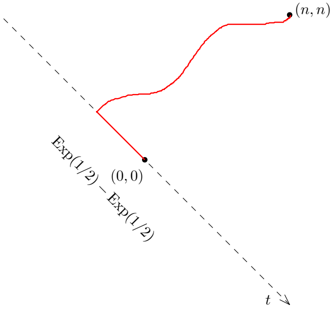




An equivalent stationary LPP model

- $G^{(-n,-n)}_{\text{stat}}(p) - G^{(-n,-n)}_{\text{stat}}(\mathbf{0}) = G^{\mathbf{0}}_{\text{stat}}(p)$ using I,J as boundary weights.
- Also, $G^{(-n,-n)}_{\text{stat}}(p) - G^{(-n,-n)}_{\text{stat}}(\mathbf{0}) = \underline{G}^{\mathbf{0}}_{\text{stat}}(p)$. Hence $\underline{G}^{\mathbf{0}}_{\text{stat}}(p) = G^{\mathbf{0}}_{\text{stat}}(p)$.
- $e_3 = e_1 - e_2$.
- Define $T(t) = \sum_{j=0}^{t} K^{(-n,-n)}_{\mathbf{0}+ie_3}$.
- $\underline{G}^{\mathbf{0}}_{\text{stat}}(p) = \max_t \{T(t) + G(te_3, p)\}$.
- Each $K^{(-n,-n)}_{\mathbf{0}+te_3} = X_i - Y_i$ where X_i and Y_i are all mutually independent with marginals $\mathrm{Exp}(1/2)$. Hence each $K^{(-n,-n)}_{\mathbf{0}+te_3}$ has mean zero.





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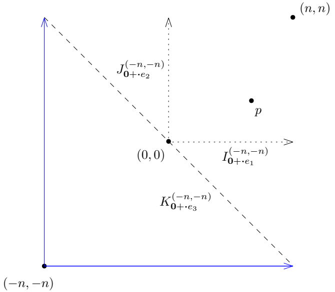
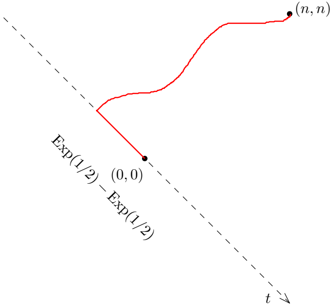
- Each $K^{(-n,-n)}_{\mathbf{0}+te_3} = X_i - Y_i$ where X_i and Y_i are all mutually independent with marginals $\mathrm{Exp}(1/2)$. Hence each $K^{(-n,-n)}_{\mathbf{0}+te_3}$ has mean zero.
- Similar exit-time result: Geodesic leaves line outside $(-xn^{2/3}, xn^{2/3})$ with probability at most e^{-cx^3}.





Lower tail lower bound

- Only works for $\varrho = 1/2$. Recall $v_n = (n/2, n/2)$.

$$\mathbb{P}(\underline{G}^{\mathbf{0}}_{\mathrm{stat}}(v_n) \leq n - yn^{1/3}) = \mathbb{P}\left(\max_t \{ T(t) + G(te_3, v_n)\} \leq n - yn^{1/3}\right)$$

- By exit-time result, the stationary geodesic exits the line $\{x + y = 0\}$ in $t \notin [-y^2 n^{2/3}, y^2 n^{2/3}]$ with probability at most e^{-cy^6}.
- Need to lower bound $\mathbb{P}\left(\max_{t \in (-y^2 n^{2/3}, y^2 n^{2/3})} \{ T(t) + G(te_3, v_n)\} \leq n - yn^{1/3}\right)$.

Lower tail lower bound

$$\mathbb{P}\left(\max_{t\in(-y^2n^{2/3},y^2n^{2/3})}\{T(t)+G(te_3,v_n)\}\leq n-yn^{1/3}\right)$$
$$\geq \mathbb{P}\left(\max_{t\in(-y^2n^{2/3},y^2n^{2/3})}T(t)\leq -\frac{yn^{1/3}}{2}\right)\mathbb{P}\left(\max\{G(te_3,v_n)\}\leq n-\frac{yn^{1/3}}{2}\right)$$
$$\geq C_1\mathbb{P}\left(G_\ell(v_n)\leq n-\frac{yn^{1/3}}{2}\right)\geq Ce^{-cy^3}.$$

- First term: a Brownian motion estimate. Second term: point-to-line LPP lower tail lower bound.

Some remarks

- We use the random matrix estimates for exponential LPP.
- Since we only used the point-to-point moderate deviations, the same method should give bounds for other models like stationary versions of geometric and Poissonian LPP.
- The work by [Emrah, Janjigian, Seppäläinen '20] uses the Burke property of stationary LPP to derive an exact formula for the l.m.g.f. of $G_{\mathrm{stat}}(m, n)$ which is used to give the exit-time bounds.

Questions?