A phase transition in the zero count of stationary Gaussian processes

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The setting

SGPs and their zero sets

A **centered SGP** F on $\mathbb{R}$ is a centered Gaussian process whose distribution is translation invariant, i.e., $\forall t_j, s \in \mathbb{R}$

$$(F_{t_1+s}, \ldots, F_{t_n+s}) \overset{d}{=} (F_{t_1}, \ldots, F_{t_n}) \sim \mathcal{N}(0, \Sigma_{t_1,\ldots,t_n}).$$

Spectral measure μ of F (an a.s. continuous SGP) is the unique finite (for us $\mu(\mathbb{R}) = 1$) positive symmetric Borel measure on $\mathbb{R}$ s.t.

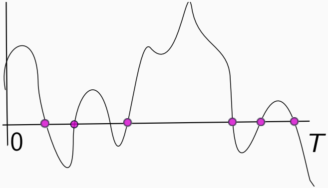
$$\mathbb{E}[F_s F_t] =: k(s-t) = \widehat{\mu}(s-t) = \int_{\mathbb{R}} e^{-i\langle s-t, w\rangle} d\mu(w).$$

Nodal/Zero set of F, $\mathcal{Z}(F) := F^{-1}\{0\}$.

We will always consider F which is C^∞ ($\Leftrightarrow$ all moments of μ are finite).

Bulinskaya's lemma $\Rightarrow$ a.s. F has **no singular zeros**. Hence, every zero has multiplicity 1.

The zero count $\mathcal{N}_T$



$$\mathcal{N}_T := \#\{[0, T] \cap \mathcal{Z}(F)\}$$

Stationarity $\Rightarrow \mathbb{E}[\mathcal{N}_T] = \frac{\alpha}{\pi} T$, where $\alpha = \sqrt{\int_{\mathbb{R}} x^2 d\mu(x)}$.

Heuristic: $\sin(\alpha x)$ has $\frac{\alpha}{\pi} T$ many zeros in $[0, T]$. The zero set of $F \leftrightarrow$ zero set of $\sin(\alpha x)$.

Grand plan: To understand $\mathcal{N}_T$ (i.e., its expectaion, variance, higher moments, concentration properties) in terms of the spectral measure μ.

Our interest: **Overcrowding event** $:= \{\mathcal{N}_T \gg \mathbb{E}[\mathcal{N}_T]\}$.

SGP $\overset{?}{\longleftrightarrow}$ Spectral measure

Ex: Let $\mu = \frac{a}{2}(\delta_{-2} + \delta_2) + \frac{b}{2}(\delta_{-7} + \delta_7) + \frac{c}{2}(\delta_{-30} + \delta_{30})$, w/ $a + b + c = 1$.

$F(t) = (\xi_a \cos 2t + \eta_a \sin 2t) + (\xi_b \cos 7t + \eta_b \sin 7t) + (\xi_c \cos 30t + \eta_c \sin 30t)$,

$$\{\xi_s, \eta_s\} \text{ independent and } \xi_s, \eta_s \sim \mathcal{N}(0, s).$$

So F is a random superposition of waves w/ frequencies $\leftrightarrow$ supp(μ) and random amplitude, phase $\leftrightarrow$ mass assigned by μ.

More generally, for any spectral measure μ, F can be written as follows. Let $\{f_n\}_{n\in\mathbb{N}}$ be an ONB for $L^2_{\text{symm}}(\mu)$, then

$$F(t) = \xi_1 \widehat{f_1}(t) + \xi_2 \widehat{f_2}(t) + \cdots, \text{ w/ } \xi_n \overset{iid}{\sim} \mathcal{N}(0, 1),$$

where $\widehat{f}(t) := \int_{\mathbb{R}} e^{-its} f(s) d\mu(s)$.

Spectral measure $\mu \overset{?}{\longleftrightarrow}$ Zero count $\mathcal{N}_T$

In principle, μ contains all the information about F, and hence its zero count $\mathcal{N}_T$. The following are some properties of μ which influence $\mathcal{N}_T$.

Moments of μ: Define $C_m := \int_{\mathbb{R}} |x|^m d\mu(x)$.

Borel–TIS $\Rightarrow$ Tail of $\|F^{(n-1)}\|_{[0,n]} \ll$Tail of $\mathcal{N}(C_n, C_n^2)$.

Higher derivatives of $f \longleftrightarrow$ Oscillations of $f \longleftrightarrow$ Zero count of f.

$\therefore$ Higher moments of $\mu \longleftrightarrow$ Zero count of F.

Absolutely continuous part $\mu_{\mathbf{ac}}$ of μ:

Typically overcrowding in an interval $I \Rightarrow \|F\|_{L^\infty(I)} \ll 1$.

$\mu_{\mathrm{ac}} \neq 0 \Rightarrow \|F\|_{L^\infty(I)} \ll 1$ ($\sim$ small ball estimates), and hence overcrowding, is very unlikely .

Existence of a smooth density $\Rightarrow$ the covariance k decays and the zero count in well separated intervals is *quasi-independent*.

Since the zero count in a large interval is the sum of zero counts in smaller intervals, we can regard $\mathcal{N}_T$ (approximately) as a sum of $M-$dependent r.v.

Statistics of $\mathcal{N}_T$: Known results

	Work of	Conditions on μ or k	Result
Moment conditions	Kac–Rice	$C_2 < \infty$	$\mathbb{E}[\mathcal{N}_T] = (\sqrt{C_2}/\pi)T$
	Beljaev	$C_{2m} < \infty$	$\mathbb{E}[\mathcal{N}_T^m] < \infty$
	Nualart Wschebor	$C_p < \infty$, for some $p > 2m$	somewhat explicit bound for $\mathbb{E}[\mathcal{N}_T^m]$
	LP	$C_m < \infty, \forall m$ & $\mu_{ac} \neq 0$	Overcrowding estimates
Mixing+ moment conditions	Cuzik	$k, k'' \in L^2(\mathbb{R})$ Geman condition	$\mathrm{Var}\mathcal{N}_T \asymp T$ CLT
	BDFZ	$\mathrm{Supp}(\mu)$ compact $\int_{\mathbb{R}} \lvert k(x)\rvert dx < \infty$	$\mathbb{P}(\lvert\mathcal{N}_T - \mathbb{E}[\mathcal{N}_T]\rvert > \eta T) \lesssim e^{-c_\eta T}$

BDFZ: Basu, Dembo, Feldheim, Zeitouni.

Our results

(i) Overcrowding estimates

Theorem

If F is a centered SGP on $\mathbb{R}$ with spectral measure μ. Then,

$$\begin{array}{ccc} \mu \text{ has finite moments} & \Longrightarrow & \text{Overcrowding estimates} \\ + \, \mu_{ac} \neq 0 & & \text{in terms of moments } C_n. \end{array}$$

Growth of C_n	**Example of μ**	**Constraints on T and n**	$\log \mathbb{P}(\mathcal{N}_T \geq n)$
$C_n \leq A^n$, for $A > 0$	Any μ with $\mathrm{supp}(\mu) \subseteq [-A, A]$	$n \geq CT$, for $C \gg 1$	$\asymp -n^2 \log(\frac{n}{T})$
$C_n \leq n^{\alpha n}$, for $\alpha \in (0,1)$	$\mu \sim \mathcal{N}(0,1)$, with $\alpha = 1/2$	$n \geq T^{1/\kappa}$, for $\kappa \in (0, 1-\alpha)$	$\asymp -n^2 \log n$
	$\mu \sim \mathcal{N}(0,1)$	$n \geq T^{2+\epsilon}$	

Two results of interest to us ...

Exponential concentration: If $\mathrm{supp}(\mu)$ is compact & $\int_{\mathbb{R}} |k(x)| dx < \infty$,

$$\mathbb{P}(\mathcal{N}_T > (\tfrac{\alpha}{\pi} + \epsilon)T) \leq e^{-C_\epsilon T}, \ \forall \epsilon > 0.$$

Overcrowding estimates: If $\mathrm{supp}(\mu)$ is compact and $\mu_{ac} \neq 0$, then

$$e^{-CT^2} \leq \mathbb{P}(\mathcal{N}_T > (\tfrac{\alpha}{\pi} + \epsilon)T) \leq e^{-cT^2}, \ \forall \epsilon \gg 1.$$

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Question: Either estimate is not sharp, or $\mathbb{P}(\mathcal{N}_T \geq \eta T)$ undergoes a transition in its behaviour as η increases. So which one is it?

Ans: If μ has *sufficient* mass around points $\pm b \in \mathbb{R}$, we can show that

$$\mathbb{P}(\mathcal{N}_T \geq \tfrac{b}{\pi} T) \geq e^{-cT},$$

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Question: How does this transition occur?

Ans: There is a sharp transition at $\eta_c = A/\pi$, where A is the edge of the spectrum.

(ii) Transition in the overcrowding probability

Theorem (w/ Naomi Feldheim & Ohad Feldheim)

Suppose $A > 0$ *is the smallest number such that* $\mathrm{supp}(\mu) \subseteq [-A, A]$, *and* $\mu_{ac} \neq 0$. *Then*

$$(1)\quad \mathbb{P}\left(\mathcal{N}_T \geq \frac{A}{\pi} T + \varepsilon T\right) \leq \exp(-C_\varepsilon T^2).$$

$$(2)\quad \mathbb{P}\left(\mathcal{N}_T \geq \frac{A}{\pi} T - \epsilon T\right) \geq \exp(-c_\epsilon T).$$

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In (1), we can let $\varepsilon \in [b_1\sqrt{\log T}/T, b_2]$; and $C_\varepsilon = \varepsilon^4$. Hence for $3/4 \leq \beta < 1$, we have

$$\mathbb{P}\left(\mathcal{N}_T \geq \frac{A}{\pi}T + T^\beta\right) \leq \exp(-T^{4\beta-2}).$$

(iii) Transition in the undercrowding probability

Theorem (w/ Naomi Feldheim & Ohad Feldheim)

Let $0 < B < A$, and let B be the largest and A the smallest number such that $supp(\mu) \subseteq [-A, -B] \cup [B, A]$. If $\mu_{ac} \neq 0$, then

$$\mathbb{P}\left(\mathcal{N}_T \leq \frac{B}{\pi} T - \varepsilon T\right) \leq \exp(-C_\epsilon T^2),$$

$$\mathbb{P}\left(\mathcal{N}_T \leq \frac{B}{\pi} T + \varepsilon T\right) \geq \exp(-c_\epsilon T).$$

Hence for the process F (as in the above theorem) to imitate a sine or cosine function with frequency smaller than B, it is very difficult.

Some heuristics

Transition: Why A/π?

Higher derivatives of F:

Borel–TIS $\Rightarrow \|F^{(n)}\|_{L^\infty[0,n]} \lesssim A^n$, with a very high probability.

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Formally: Since F is $\mathbb{R}$-analytic, extend $F : \mathbb{C} \to \mathbb{C}$ to be $\mathbb{C}$-analytic, a.s.

- F is an entire function of exponential type at most A,
- F belongs to the *Cartwright's class* $\mathcal{C}$.

Cartwright's class $\mathcal{C}$

Cartwright's class $\mathcal{C}$ consists of entire functions $f : \mathbb{C} \to \mathbb{C}$ such that:

- f is of exponential type, i.e., $\exists\ \sigma, B > 0$ s.t. $|f(z)| \le Be^{\sigma|z|}$,
- Slow growth on the $\mathbb{R}$ axis:

$$\int_{\mathbb{R}} \frac{\log^+ |f(t)|}{1+t^2} dt < \infty.$$

Ex: When $|f(t)| \le \exp(|t|^{0.9})$ on $\mathbb{R}$ (or more generally, whenever $|f(t)| \ll \exp(|t|)$), then this condition holds.

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Prototypical example: $f(z) = \sin(Az)$ and $f(z) = \cos(Az)$.

Speciality of $\mathcal{C}$: asymptotic behaviour of $|f|$ & their zeros

- Zeros of functions $f \in \mathcal{C}$ have been studied extensively and their zero set has been shown to be very *regular*.

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What makes it possible to study zeros of functions in $\mathcal{C}$ is the following asymptotic behaviour:

• If $f \in \mathcal{C}$ is of exponential type A, then we have the following on $\mathbb{C}$, except on a small exceptional set:

$$\text{for } z = (x, y),\ \log|f(z)| = A|y| + o(|z|).$$

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<u>Ex:</u> Consider $f(z) = \sin(Az)$, then

$$\sin(Az) = \frac{e^{iAz} - e^{-iAz}}{2i} = \frac{e^{-Ay}e^{iAx} - e^{Ay}e^{-iAx}}{2i},$$

$$\therefore \log|\sin(Az)| = A|y| + O(1).$$

Asymptotics of $\log|f|$ + Jensen's formula $\overset{?}{\rightsquigarrow}$ Zero count of f

A simple calculation: Asymptotics of $\log|f|$ along with Jensen's formula gives useful information about the zero count of f. Jensen's formula gives:

$$\begin{aligned}\int_0^r \frac{n_f(t)}{t}dt &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta - \log|f(0)|,\\ &= \frac{1}{\pi}\int_0^{\pi} Ar\sin\theta\ d\theta + o(|r|) - \log|f(0)|,\\ &= \frac{A}{\pi}\cdot 2r + \text{rem}.\end{aligned}$$

Here $n_f(t) = \sharp$ zeros of f in the disc $D(0,t) \subseteq \mathbb{C}$.

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Asymptotic zero count [Levinson–Cartwright]: If $f \in \mathcal{C}$ of exponential type A, then

$$\frac{n_f(R)}{R} \longrightarrow \frac{2A}{\pi} \quad \text{as } R \to \infty.$$

Global regularity vs. Local variability

Global regularity [Levinson–Cartwright]: If $f \in \mathcal{C}$ is of exponential type A, then its asymptotic real zero density is at most A/π.

$$\frac{\sharp\{\text{zeros of } f \text{ in } [-R,R]\}}{R} \le \frac{n_f(R)}{R} \xrightarrow{R\to\infty} \frac{2A}{\pi}.$$

We will show that a.s. $F \in \mathcal{C}$ and it is of exponential type at most A. Hence the above statement holds a.s. for F.

On the other hand, we have:

Local variability: If $\mu_{\mathrm{ac}} \neq 0$, then on any compact interval $I = [-M, M]$, up to a cosine factor, F can *imitate* any continuous function. That is, there is $b > 0$ such that for any continuous g on I, we have with a positive probability:

$$F(x) \approx \cos(bx) \cdot g(x) \text{ on } I.$$

Sketch of the proof

A.s. F is of exp. type $\leq A$, and $F \in \mathcal{C}$

- $F^{(n)}(0) \sim \mathcal{N}(0, C_{2n})$, where $C_k := \int_{[-A,A]} |x|^k d\mu(x)$. Note that:
  - $C_k \leq A^k$, and hence $\mathbb{P}(|F^{(n)}(0)| \geq nA^n) \leq e^{-n^2}$,
  - Writing a Taylor series expansion for F around 0 gives

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  $$F(z) = \sum_{n \geq 0} \frac{F^{(n)}(0)}{n!} z^n, \text{ and hence } |F(z)| \lesssim e^{(A+\varepsilon)|z|}.$$
- It follows from Borel–TIS that F has very slow growth on $\mathbb{R}$: $|F(t)| \lesssim \sqrt{1 + \log |t|}$ and hence satisfies
$$\int_{\mathbb{R}} \frac{\log^+ |F(t)|}{1 + t^2} dt < \infty.$$

A bound for $\log|f|$ when $f \in \mathcal{C}$

A Phragmén–Lindelöf result: If $f : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type A and $|f| \leq M$ on $\mathbb{R}$, then for $z = (x, y)$ we have

$$|f(z)| \leq Me^{A|y|}, \text{ and hence } \log|f(z)| \leq A|y| + \log M.$$

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Analogous result: Let f be an entire function of exponential type A and $\int_{\mathbb{R}} \frac{\log^+ |f(t)|}{1+t^2} dt < \infty$. (That is, $f \in \mathcal{C}$ and is of exponential type A), then

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Hence the following holds a.s.:

$$\log|F(z)| \leq A|y| + |y| \int_{\mathbb{R}} \frac{\log^+|F(t)|}{|t-z|^2} dt,$$

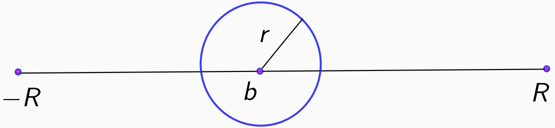
and w.p. $\geq (1 - e^{-R^2})$, term is $O(\log R)$, $\forall |z| \leq R$.

$\log|F(z)| \leq A|y| + O(\log|z|)$ + Jensen's formula $\overset{?}{\rightsquigarrow}$ zero count

- w.p. $\geq (1 - e^{-R^2})$: $\forall b \in [-R, R]$, $\forall r \leq R$:

$$\log|F(b + re^{i\theta})| \leq Ar|\sin\theta| + \log R,$$

$$\frac{1}{2\pi}\int_{[0,2\pi]} \log|F(b + re^{i\theta})| d\theta \leq 2r\frac{A}{\pi} + \log R.$$



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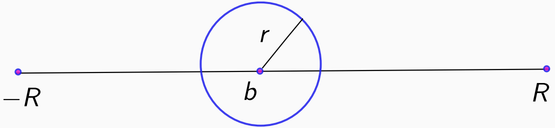

- Hence, by Jensen's formula:

$$\therefore \int_0^r \frac{n_F(t;b)}{t} dt = \frac{1}{2\pi}\int_{[0,2\pi]} \log|F(b + re^{i\theta})| d\theta - \log|F(b)|,$$
$$\leq 2r\frac{A}{\pi} + \log R - \log|F(b)|.$$

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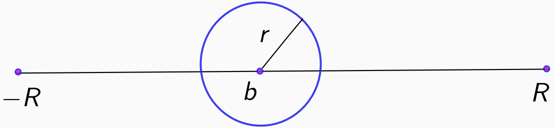

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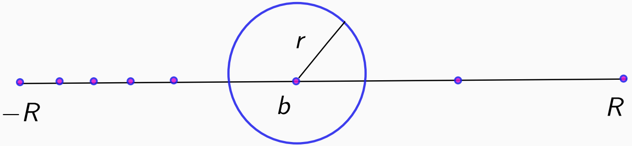
$$\le 2r\frac{A}{\pi} + \log R - \log|F(b)|.$$

- **Q:** We have a lot of *local* information about the zero count, how to get information about $\mathcal{N}_{[-R,R]}$ from here?

W.p. $\geq (1 - e^{-R^2})$:

If we forget the $\log R - \log |F(b)|$ term, and let $b \in \mathbb{Z} \cap [-R, R]$, we get

$$\int_0^r \frac{n_F(t; b)}{t} dt \leq 2r\frac{A}{\pi}.$$



A rough calculation: Local zero count info $\overset{?}{\rightsquigarrow}$ Total zero count

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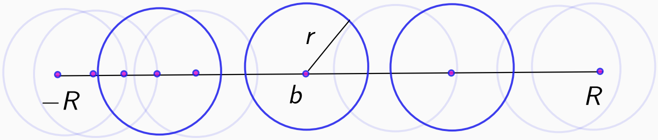
If we forget the $\log R - \log |F(b)|$ term, and let $b \in \mathbb{Z} \cap [-R, R]$, we get

$$\sum_{b \in \mathbb{Z} \cap [-R,R]} \int_0^r \frac{n_F(t;b)}{t} dt \leq \sum_{b \in \mathbb{Z} \cap [-R,R]} 2r\frac{A}{\pi}.$$



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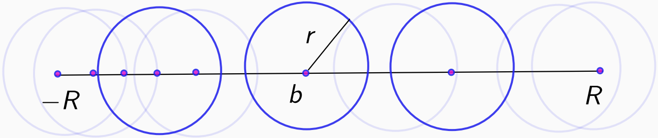

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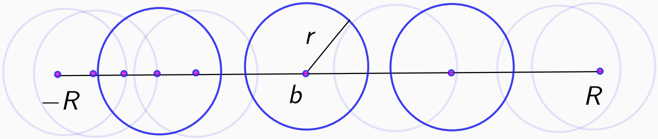

Here $\mathcal{Z} = \{z \in [-R, R] : F(z) = 0\}$.

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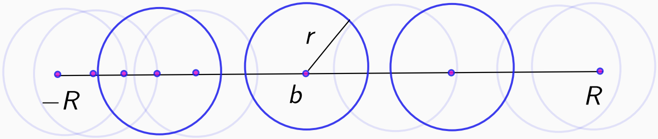

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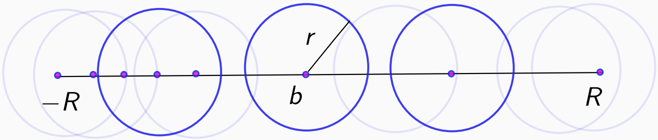

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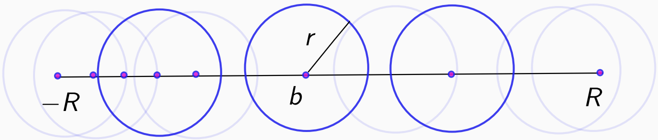
$$\therefore \mathcal{N}_{[-R,R]}(F) \leq \frac{A}{\pi} \cdot 2R.$$



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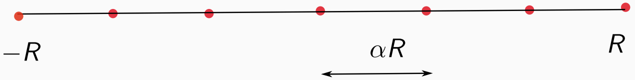

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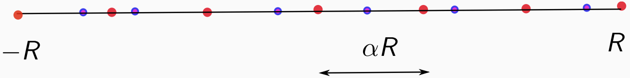

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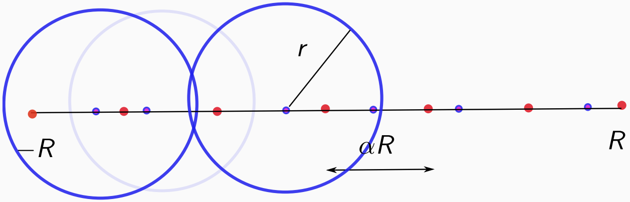


Good point is where $|F| \geq e^{-\varepsilon^2 R}$; $\alpha = \varepsilon^2$; $C_\epsilon = \epsilon^4$, $r = \varepsilon R$.

Conclusion: $\mathbb{P}(\mathcal{N}_R \geq \frac{A}{\pi} R + \varepsilon R) \leq \exp(-C_\varepsilon R^2)$.

An event on which the $\mathcal{N}_T \geq \frac{A}{\pi} T$

- Assume that $d\mu(x) = f(x)dx + d\mu_s(x)$, where $f \geq c > 0$ near $\pm A$. Say $c = 1$, for simplicity. Define $f_1 \in L^2_{\text{symm}}(\mu)$ by

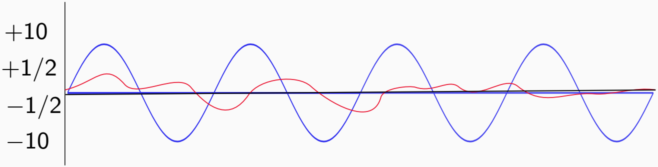
$$f_1 := \frac{1}{\sqrt{2\delta}} \mathbb{1}_{[-A,-A+\delta]\cup[A-\delta,A]},$$

$$\text{then, } \widehat{f_1}(x) \simeq C\sqrt{\delta} \cdot \cos\left((A - \tfrac{\delta}{2})x\right).$$

- We write F as the following, with $\delta = 1/T$:

$$F(x) = \xi_1 \widehat{f_1} \oplus G(x), \text{ w/ } G \text{ is a centered GP, } \xi_1 \sim \mathcal{N}(0,1),$$

$$\simeq \xi_1 \frac{C}{\sqrt{T}} \cos\left((A - \tfrac{1}{2T})x\right) \oplus G(x).$$



$$|\xi_1| \gtrsim \sqrt{T} \ \& \ \|G\|_{L^\infty[0,T]} \leq \tfrac{1}{2} \Rightarrow \mathcal{N}_T \geq \tfrac{A}{\pi} T.$$

Conclusion

- Since ξ_1 and G are independent, we have

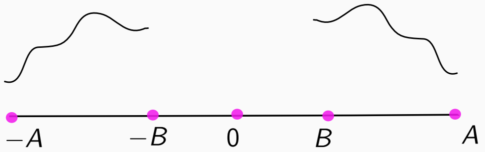
$$\begin{aligned}\mathbb{P}(|\xi_1| \gtrsim \sqrt{T} \ \&\ \|G\|_{L^\infty[0,T]} \le \tfrac{1}{2}) &= \mathbb{P}(|\xi_1| \gtrsim \sqrt{T}) \cdot \mathbb{P}(\|G\|_{L^\infty[0,T]} \le \tfrac{1}{2}),\\ &\ge e^{-T} \cdot e^{-T} = e^{-cT}.\end{aligned}$$

- Hence we have

$$\mathbb{P}\left(\mathcal{N}_T \ge \frac{A}{\pi} T\right) \ge e^{-cT}.$$

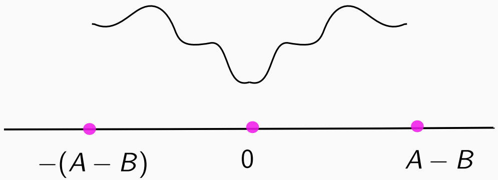
Phase transition in the undercrowding probability

Assume that $\mathrm{supp}(\mu) \subseteq [-A, -B] \cup [A, B]$ and μ assigns non-trivial mass near $\pm A$ and $\pm B$.



Measure μ

Consider $\tilde{\mu}$ and the corresponding centered SGP $\widetilde{F}$.



Measure $\widetilde{\mu}$

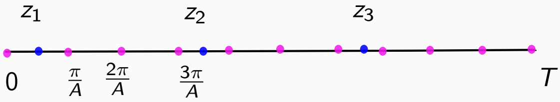
Phase transition in the undercrowding probability

- Suppose $\{f_0, f_1, \ldots\}$ is an ONB for $L^2_{\text{symm}}(\mu)$, using this we get $\{\widetilde{f}_0, \widetilde{f}_1, \ldots\}$ which is an ONB for $L^2_{\text{symm}}(\widetilde{\mu})$.
- Using these bases to get a series representation for F and $\widetilde{F}$, we get

$$\widetilde{F}(x) = \cos(Ax)F(x) - \sin(Ax)H(x),$$

where H is some centered GP. Note that $\forall k \in \mathbb{Z}$, we have

$$\widetilde{F}(\tfrac{k\pi}{A}) = (-1)^k F(\tfrac{k\pi}{A}).$$



Lattice $\frac{\pi}{A}\mathbb{Z}$ & zeros of F

- Let $z_1 < z_2 < \cdots < z_n$ be the zeros of F in $[0, T]$. Between z_i and z_{i+1}, F does not change sign. Hence $\widetilde{F}$ has $\lceil (z_{i+1} - z_i)A/\pi \rceil$ many zeros in (z_i, z_{i+1}).

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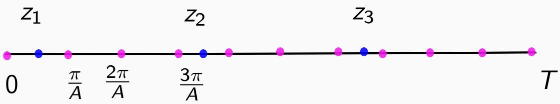

Lattice $\frac{\pi}{A}\mathbb{Z}$ & zeros of F

• Between z_i and z_{i+1}, F does not change sign. Hence $\widetilde{F}$ has $\lceil (z_{i+1} - z_i)A/\pi \rceil$ many zeros in (z_i, z_{i+1}). Thus

$$\mathcal{N}_T(\widetilde{F}) + \mathcal{N}_T(F) \geq \frac{AT}{\pi}.$$

• Recall that the edge of the support of $\widetilde{\mu}$ is $\pm(A - B)$.

From our analysis of the OC event, *w.h.p.* $\mathcal{N}_T(\widetilde{F}) \leq (A - B + \epsilon)T/\pi$,

$$\text{hence } \textit{w.h.p. } \mathcal{N}_T(F) \geq \frac{(B - \epsilon)T}{\pi}.$$

More questions ...

Pertaining to transition: the finer details of how the transition occurs?

More generally: With merely some weak assumptions on μ, it is possible to obtain a lot of information about the zero count. So probably strong assumptions (like existence and smoothness of density) are not essential to get exponential concentration of zeros?

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Thank You!