Phase Analysis for a family of Stochastic Reaction-Diffusion Equations

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Main Problem

- Consider the following stochastic reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + V(u) + \lambda\sigma(u)\xi, \quad t > 0, x \in \mathbf{T}. \tag{1}$$

where $\mathbf{T} = [-1\,,1)$ (torus), ξ is space-time white noise, $\lambda > 0$ is a fixed constant, and σ is a globally Lipschitz function.

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- (PAM) $\partial_t u(t,x) = \partial_x^2 u(t,x) + \lambda u(t,x)\xi(t,x), \quad t > 0, x \in \mathbf{T} := [-1,1)$ with the periodic boundary condition.
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  $u(t,x) = \int_{\mathbf{T}} p_t(x-y)u_0(y)\,dy + \lambda \int_{(0,t)\times\mathbf{T}} p_{t-s}(x-y)u(s,y)\,\xi(ds\,dy).$

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- Intermittency (Carmona-Molchanov '94): $u(t,x)$ is *fully* intermittent if $\frac{\gamma(k)}{k}$ is strictly increasing for $k \geqslant 1$ where $\gamma(k) := \lim_{t\to\infty} \frac{\log \mathrm{E}|u(t,x)|^k}{t}$.

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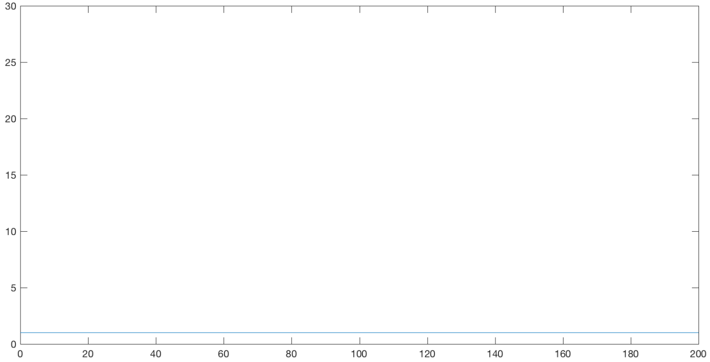
Simulation of PAM ($\partial_t u = \partial_x^2 u + \lambda u \xi$)



Q.) What will happen as $t \to \infty$?

Dissipation of SHE

- Consider the follwoing stochastic heat equation:

$$\partial_t u = \partial_x^2 u + \lambda\sigma(u)\,\xi, \quad t > 0, x \in \mathbf{T} = [-1, 1] \tag{2}$$

with $u_0(x) = 1$ for $x \in \mathbf{T}$.

- $\sigma(u) : \mathbf{R} \to \mathbf{R}$ is a globally Lipschitz function with $0 < \mathrm{L}_\sigma \leqslant \frac{\sigma(u)}{u} \leqslant \mathrm{Lip}_\sigma$ for some constants L_σ and Lip_σ (Foondun-Khoshnevisan '09).

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There exists a constants $c > 0$ such that with probability 1

$$\log \sup_{x \in \mathbf{T}} u(t, x) \leqslant -c\lambda^2 t.$$

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- $u(t,x) = \int_{-1}^{1} p_t(x,y)u_0(y)\,dy + \lambda \int_{(0,t]\times[-1,1]} p_{t-s}(x,y)\sigma(u(s,y))\xi(ds\,dy)$.
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Lemma

Let $X = \{X_t\}_{t\geqslant 0}$ be a continuous $L^2(\mathrm{P})$ martingale and there is a $c > 0$ such that $\langle X\rangle_t \geqslant ct$ for all $t \geqslant 0$, a.s. Then, for all nonrandom constants $\varepsilon, T > 0$,

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Moment estimates via Interpolation

- Let $p_t(x, y)$ be the heat kernel. Then, $p_t(x\,, y) \leqslant 2\max\left(\frac{1}{\sqrt{t}}\,, 1\right)$.
- $\int_{-1}^{1} p_t(x, y) u_0(y)\, dy \lesssim \|u_0\|_{L^\infty} \wedge \frac{\|u_0\|_{L^1}}{\sqrt{t}} \leqslant \|u_0\|_{L^\infty}^{\varepsilon} \left(\frac{\|u_0\|_{L^1}}{\sqrt{t}}\right)^{1-\varepsilon}$.

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Stochastic reaction-diffusion equation

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where $V(x) = x - F(x)$ and $F \in C^2(\mathbf{R}_+)$, $F(0) = 0$, $F' \geqslant 0$ and
  - $\limsup_{x\downarrow 0} F'(x) < 1$ and $\lim_{x\to\infty} F'(x) = \infty$; and
  - There exists a real number $m_0 > 1$ such that $F(x) = O(x^{m_0})$ as $x \to \infty$.
- In this talk, let $V(u) = u - u^\alpha$ for $\alpha > 1$ (e.g. $V(u) = u - u^2$).

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If $\lambda \in (0\,,\lambda_0)$ for some small $\lambda_0 > 0$, then:

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Existence of a nontrivial invariant probability measure

- Let P_t be a Markov semigroup for $u(t)$, i.e., for all Borel sets $\Gamma \subset C_+(\mathbf{T})$ and for every Borel measure ν on $C_+(\mathbf{T})$,

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If $u_0 = \mathbb{1}$ and λ is sufficiently small, then we have

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- We get (4) if we show the following:

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$$\lim_{\varepsilon\downarrow 0} \limsup_{T\to\infty} \frac{1}{T} \mathrm{E}_{\mathbb{1}}\left[\int_0^T \mathbf{1}_{\{\inf_{x\in\mathbf{T}} u(t,x)<\varepsilon\}}\,\mathrm{d}t\right] = 0. \tag{4}$$

In particular, u has an invariant measure μ_+ on $C_+(\mathbf{T})$ such that $\mu_+\{\mathbb{0}\} = 0$.

- We get (4) if we show the following:

$$\mathrm{P}_{\mathbb{1}}\left\{\lim_{\varepsilon\downarrow 0} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{\{\inf_{x\in\mathbf{T}} u(t,x)<\varepsilon\}}\,\mathrm{d}t = 0\right\} = 1.$$

A Random Walk Argument ($\dot{u} = u'' + u - u^2 + \lambda\sigma(u)\xi$)

- Claim: $\mathrm{P}_{\mathbb{1}}\left\{\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{1}_{\{\inf_{x\in\mathbf{T}} u(t,x)<\varepsilon\}}\,\mathrm{d}t = 0\right\} = 1.$
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If $\mathrm{L}_\sigma > 0$, then $\mathrm{P}_{\mathbb{1}}\{\tau_{n+1} < \infty\} = 1$ for all $n \in \mathbf{Z}_+$.

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If λ is small, we have

$$\mathrm{P}(X_{n+1} - X_n = +1 \mid X_n) \geqslant \tfrac{2}{3} \quad \text{for all } n \geqslant 1,$$

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Uniqueness of a nontrivial invariant probability measure

- Recall
$$\partial_t u = \partial_x^2 u + u - F(u) + \lambda\sigma(u)\xi. \tag{6}$$
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- Consider

$$\partial_t u = \partial_x^2 u + \lambda u \xi_1,$$
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where $f(y) := \sqrt{|y| \wedge 1}$ and $g(y) := \sqrt{1 - |f(y)|^2} = \sqrt{1 - (|y| \wedge 1)}$, and ξ_1 and ξ_2 are independent space-time white noises.

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$$\left[\begin{array}{l} \partial_t \Delta = \partial_x^2 \Delta + \lambda \left(\Delta^2 + 2uv \dfrac{f^2(\Delta)}{1 + g(\Delta)} \right)^{1/2} \dot{W}, \\ \text{subject to } \Delta(0) = u_0 - v_0 \geqslant 0, \end{array}\right.$$
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- X_t is a continuous $L^2(\mathrm{P})$ martingale with quadratic variation

$$
\begin{aligned}
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&\geqslant \lambda^2 \inf_{x\in\mathbf{T}} v^2(t,x) \int_{\mathbf{T}} \min\{\Delta(t,x),1\}\,\mathrm{d}x\,\mathrm{d}t \\
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- X_t is a continuous $L^2(\mathrm{P})$ martingale with quadratic variation

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d\langle X\rangle_t &= \lambda^2 \int_{\mathbf{T}} \mathrm{d}x \left[\Delta^2 + 2uv\frac{f^2(\Delta(t\,,x))}{1+g(\Delta(t\,,x))}\right] \mathrm{d}t \\
&\geqslant \lambda^2 \inf_{x\in\mathbf{T}} v^2(t,x) \int_{\mathbf{T}} \min\left\{\Delta(t,x), 1\right\} \mathrm{d}x\,\mathrm{d}t \\
&\geqslant c \int_{\mathbf{T}} \Delta(t,x)\,\mathrm{d}x = X_t\,\mathrm{d}t.
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Anchored monotone (AM) coupling

- Suppose our initial data $u_0 \geqslant 0$ and $v_0 \geqslant 0$ are not comparable.
- We introduce three independent space-time white noises ξ, ξ_1 and ξ_2 and let w denote the solution to the SPDE,

$$\partial_t w = \partial_x^2 w + \lambda w\, \xi, \quad \text{subject to} \quad w_0 = u_0 \vee v_0.$$

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Proposition

Let $A > A_0 > 0$, and $\alpha \in (0\,,1/2)$. Then, for every non-random $u_0 \in C_+(\mathbf{T})$ with $\frac{1}{2}A_0 \leqslant \inf_{x\in\mathbf{T}} u_0(x) \leqslant \|u_0\|_{C^\alpha(\mathbf{T})} \leqslant A$, there exists $t_0 > 0$ and a strictly positive number $\mathbf{p}_{A,A_0}(t_0\,,\alpha\,,\delta)$ such that

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Thank You!