Disordered Monomer-Dimer Model on Cylinder Graphs

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- We will denote the collection of all matchings by $\mathcal{M}$.

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- Statistics of interest are typical number of edges or typical number of unpaired vertices, denoted U.

Monomer Dimer Configurations

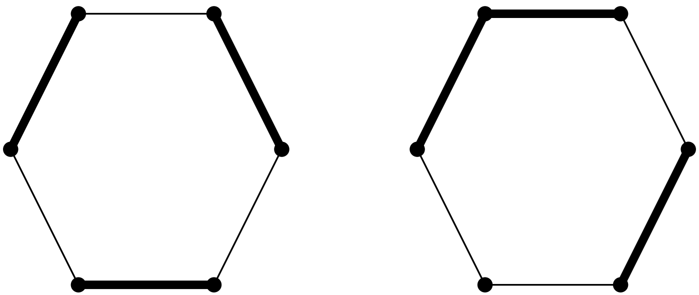

Figure 1: Matching vs. non matching

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- The recurrence can be used to prove the absence of phase transition, as well as exact computation of the partition function in special cases (line graph, complete graph, regular trees).

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- Hielmann-Lieb recursion can be recovered via Gaussian integration by parts.

Absence of Phase Transition

- When convergence of limiting free energy can be established, can be shown that it is an analytic function in the weights. No phase transitions
- Phase transitions can be induced in certain situations:
  - Introduction of imitative potential. Studied by Alberici and Contucci.
  - Monomer weight $\nu_v = -\infty$. Widely studied on planar and other surface graphs. Kasteleyn, Kenyon, etc..
- Key task therefore to establish free energy convergence.

Why are Monomer Dimer Models Studied?

- Statistical physics, either equivalent or related to several models of interest such as Ising Models with external field, Random Assignment Problems, etc..
- Computing the partition function with constant weights is equivalent to the computation of the permanent of a $\{0, 1\}$ valued matrix (#P class). Quick probabilistic algorithms are thus of interest.

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- The partition function and free energy $\log Z_n$ can have non trivial limiting behavior.
- For statistics like U, we have environmental and ensemble contributions to fluctuations, need to distinguish.

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- Alberici and Contucci (2014) also analyse the monomer dimer model on locally tree like graphs, such as the Erdös-Renyi graph, establish exact solution via fixed point argument.
- Methods suited for mean field situation, similar to the cavity method in the study of spin glasses.

Our Context

Definition (Cylinder Graph)

Let $H = (V_H, E_H)$ be a fixed graph with $|V_H| = h$ and G_n be the line graph on n vertices with vertex set $[n]$. A cylinder graph $\mathcal{G}_n$ is given by the graph Cartesian product

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- We will work with i.i.d families $\{\nu\}_{v \in V}$ and $\{\omega_e\}_{e \in E}$, though not necessarily with same distribution.
- We will require $\mathbb{E}|\nu|^{2+\epsilon} + \mathbb{E}|\omega|^{2+\epsilon} < \infty$ for some $\epsilon > 0$.

Notation

- Let $\mathcal{G}_{[k:l]}$ denote the principal subgraph of $\mathcal{G}_n$ generated by the vertices with G_n components in the interval $[k,l]$.
- Let $Z_{[k:l]}$ denote the partition function of the monomer-dimer model on $\mathcal{G}_{[k:l]}$.
- Let $U_{[k,l]}$ denote the number of unpaired vertices of a matching $\mathfrak{m}$ on $\mathcal{G}_n$ contained in the section $\mathcal{G}_{[k:l]}$.

Main Result (1)

Theorem (Dey, K. 2021)

Assume that $\mathrm{E}(|\nu_v|^{2+\varepsilon} + |\omega_e|^{2+\varepsilon})$ is finite for some $\varepsilon > 0$. We have $f \in \mathbb{R}$ and $\sigma_F > 0$ depending only on the distributions of ν and ω such that

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and

$$n^{-1/2} \cdot (\log Z_n - \mathbb{E} \log Z_n) \xrightarrow{(\mathrm{d})} \mathrm{N}(0, \sigma_F^2) \text{ as } n \to \infty.$$

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Then as $n \to \infty$, $\exists u > 0$ and $\sigma > 0$

$$\left(\frac{\theta_n(t) - ntu}{\sqrt{n}} \right)_{t \in [0,1]} \xrightarrow{(d)} (\sigma B_t)_{t \in [0,1]}$$

in probability, in the sense of finite dimensional distributions, B_t is standard Brownian Motion.

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- The new form of the recursion also enables us to write the free energy and the number of unpaired vertices as a sum of i.i.d random variables, with an error term.
- We can show that the error vanishes in all relevant limits.

Splitting the Graph

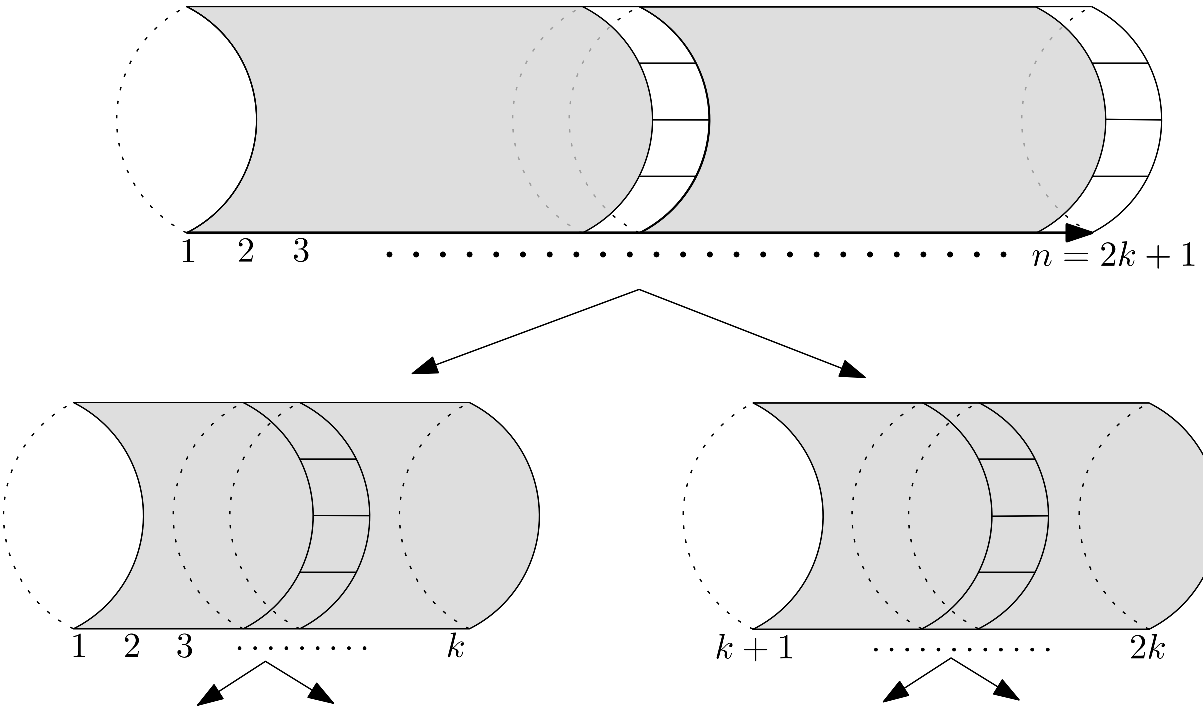

Figure 2: First step of the subdivision

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$$R_{n,k} = \log\left(\sum_{A \subseteq \mathcal{E}} \prod_{i:e_{k,i} \in A} e^{\omega_{k,i} - \nu_{k,i} - \nu_{k+1,i}} \cdot \frac{Z^A_{[1:k]}}{Z_{[1:k]}} \cdot \frac{Z^A_{[k+1:n]}}{Z_{[k+1:n]}}\right)$$

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- Central Limit Theorem follows from the Lyapunov condition.

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- In particular, need to control $\partial_x Z^A_{(.)}/Z_{(.)}$, which is equivalent to bounding $\partial_x \log Z^A_{(.)} - \partial_x \log Z_{(.)}$.

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$$\lambda_i \leqslant \lambda_i^v \leqslant \lambda_{i+1}.$$
- All quantities will be evaluated at $x = 0$.

Interlacing Hierarchy

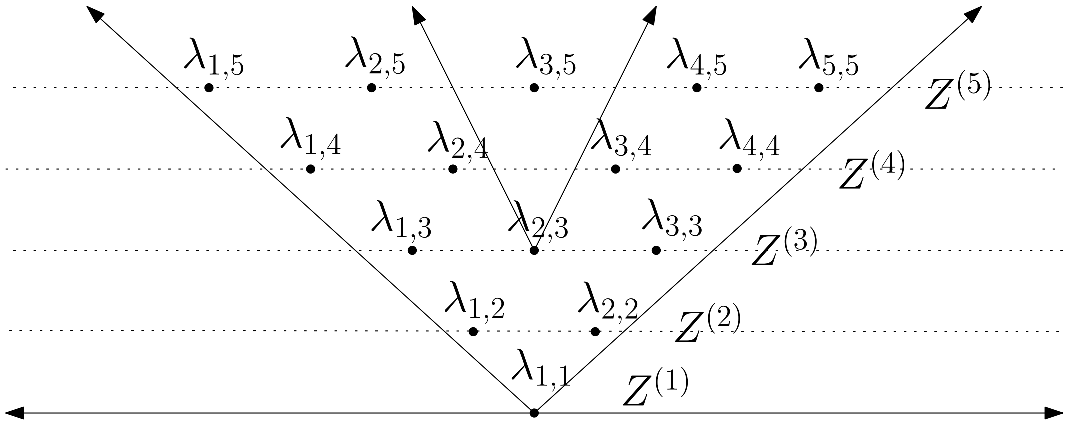

Figure 3: Interlacing shown for the first 5 levels

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- This can be finitely iterated to yield a constant order bound for $\partial_x R_{n,k}$
- Subadditive lemma then applies to $\langle U \rangle_n$, we denote $u := \lim_{n\to\infty} n^{-1} \cdot \langle U \rangle_n$

Empirical Measure of Zeroes

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$$n^{-1} \cdot \log Z_n = \frac{1}{n} \sum_{v \in V} \nu_v + \frac{1}{2} \int_{\mathbb{R}} \log(e^{2x} + \lambda^2) d\rho_n(\lambda)$$

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- Free energy may be written as

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$$n^{-1}\cdot \langle U\rangle_n = \int_{\mathbb{R}}\frac{e^{2x}}{e^{2x}+\lambda^2}d\rho_n(\lambda)$$

Convergence of Cumulants

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- Weak convergence of ρ_n implies convergence of all quenched moments of $n^{-1}U$, we denote limiting variance as σ_Q^2
- The boundedness of the third cumulant of $n^{-1}U$ is particularly useful.

CLT (Quenched)

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- Taylor's theorem shows that $\log M(\xi) = \sigma_Q^2 \xi^2/2$ is the limit.

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- CLT for the free energy can be adapted to establish CLT for $n^{-1/2}\overline{\langle U \rangle}$.

CLT (Joint)

- Let $t \in (0,1)$ and $k = \lfloor tn \rfloor$
- For disjoint sections $[1,k]$ and $[k+1:n]$, $\langle U \rangle_{[1:k]}$ and $\langle U \rangle_{[k+1:n]}$ are independent.

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- Quenched and Annealed fluctuations are independent in the limit.

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- Process level convergence yet to be shown.

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- Convergence of free energy analogous to Thouless formula.

Thanks!