Convex Hulls of Random Point Sets

Joe Yukich

Based on joint work with Pierre Calka (Rouen)

I Introduction

· **Convex hull of a planar point set:** Smallest convex polygon containing the points.

· Newton, 1676

· April 1864, Educational Times, J. J. Sylvester (1814-1897): Place four points uniformly at random inside an **ellipse.** How many vertices does the convex hull have?

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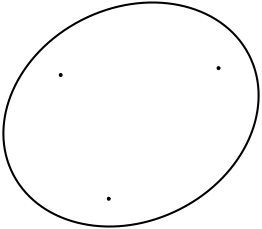
· Three points:



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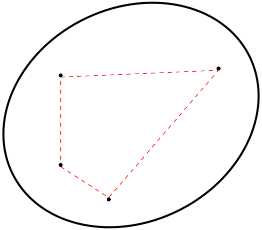


·The fourth point can be either inside or outside the triangle formed by the first three points. If it is inside, then the convex hull has three vertices.

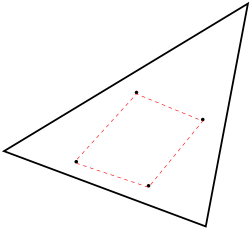
· Otherwise there are potentially four vertices (extreme points)

· Spatial dependencies present a formidable technical obstacle.

I Introduction

· Place four points uniformly at random in a **triangle.**

· The number of vertices in the convex hull is either three or four. What is the expected number?



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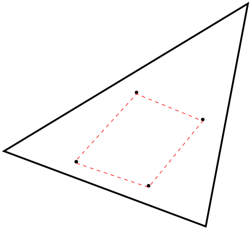


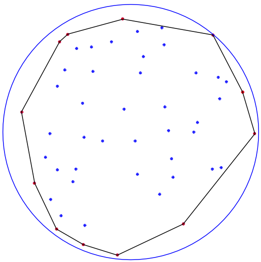
· Expected number of vertices is $\frac{11}{3}$. (Sylvester)

· Answer does not depend on size or shape of triangle.

I Introduction

· General set-up: $X_1, ..., X_n$ i.i.d. uniform points in $K \subset \mathbb{R}^2$, K compact.

· K_n **= convex hull of** $X_1, ..., X_n$**.** This is the smallest convex polygon containing the points.



· $f_0(K_n)$: number of vertices in K_n.

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- $K = \Delta^2 \quad \mathbb{E}\, f_0(K_4) = \frac{11}{3}$ (Sylvester)
- $K = \mathbb{B}^2 \quad \mathbb{E}\, f_0(K_4) = \frac{48\pi^2 - 35}{12\pi^2}$ (Woolhouse)

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- $K = \square \quad \mathbb{E}\, f_0(K_4) = \frac{133}{36}$ (Woolhouse)

  The formula does not depend on the aspect ratio of the rectangle.
- Blaschke, Crofton, Dalla, Efron, Groemer, Herglotz, Larman, Schneider

I Introduction

- What about dimensions larger than 2?
- Let K be a compact convex subset of $\mathbb{R}^d$, $d \geq 3$.
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· The formulas are becoming more complicated as the number of points in the sample increases.

· $K = \Delta^3 \quad \mathbb{E}\,\mathrm{Vol}(K_n) = ?$

Buchta and Reitzner (2001):

Theorem 3. *The expected volume $\mathscr{V}(n)$ of the convex hull of n random points chosen independently and uniformly from a tetrahedron of volume one is given by*

$$
\begin{aligned}
\mathscr{V}(n) = 1 - \frac{2}{n+1} - \frac{3(n-1)n}{4}\left[\frac{1}{(n+1)^3} + \sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\frac{1}{(k+3)^3}\right] \\
-\frac{9(n-1)n}{2} \sum_{\substack{j_1+\cdots+j_5=n-2\\ k_1+k_2+k_3=4\\ j_1,\dots,j_5,k_1,k_2,k_3\geq 0}} \binom{n-2}{j_1,\dots,j_4}\binom{4}{k_1,k_2}2^{k_2}3^{j_2+j_3} \\
\times B(j_2+2j_3+3j_4+3j_5+k_2+2k_3+1, 3j_1+2j_2+j_3+2k_1+k_2+1) \\
\times B(n+1, j_5+k_3+1)B(2j_1+j_2+k_1+1, j_5+2) \\
\times {}_3F_2(j_5+k_3, n+1, 2j_1+j_2+k_1+1; j_5+k_3+n+2, 2j_1+j_2+j_5+k_1+3; 1) \\
+6(n-1)n \sum_{\substack{j_1+\cdots+j_5=n-2\\ l_1+l_2=2\\ l_3+l_4=2\\ j_1,\dots,j_5,l_1,l_2,l_3,l_4\geq 0}} \binom{n-2}{j_1,\dots,j_4}\binom{2}{l_1}\binom{2}{l_3}3^{j_2+j_3} \\
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\end{aligned}
$$

Statistics of random polytopes

- $f_0(K_n) =$ number of vertices of K_n
- $f_k(K_n) =$ number of k-faces of K_n, $k \in \{0, ..., d-1\}$,
- $\mathrm{Vol}(K_n) =$ volume of K_n.
- Efron (1965): $\mathbb{E}\, f_0(K_n) = n(1 - \mathbb{E}\,\mathrm{Vol}(K_{n-1}))$, when $\mathrm{Vol}(K) = 1$.
- Average vertex count $\mathbb{E}\, f_0(K_n)$ yields results about average volumes.

I Introduction

- Efron (1965): $\mathbb{E}\, f_0(K_n) = n(1 - \mathbb{E}\,\mathrm{Vol}(K_{n-1}))$, when $\mathrm{Vol}(K) = 1$.

- Proof.

$$f_0(K_n) = \sum_{k=1}^{n} \mathbf{1}(X_k \notin \mathrm{Conv}(\mathrm{X_1, ..., X_{k-1}, X_{k+1}, ..., X_n}))$$

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- Take expectations.

$$\begin{aligned}
\mathbb{E}\, f_0(K_n) &= n\mathbb{P}(X_n \notin \mathrm{Conv}(\mathrm{X_1, ..., X_{n-1}})) \\
&= n\mathbb{E}\,\mathbb{E}\,(\mathbf{1}(X_n \notin \mathrm{Conv}(\mathrm{X_1, ..., X_{n-1}}))|\mathrm{X_1, ..., X_{n-1}}) \\
&= n\left(1 - \frac{\mathbb{E}\,\mathrm{Vol}(K_{n-1})}{\mathrm{Vol} K}\right). \qquad \square
\end{aligned}$$

I Introduction

Connections with other fields

- computational geometry: approximation of convex sets by random polytopes
- optimization: vertices of convex hull are extreme points of multivariate random samples

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- optimization: vertices of convex hull are extreme points of multivariate random samples
- statistics: data depth, convex hull peeling
- ethology

I Introduction

Simple questions and surprises

· Recall $f_0(K_n) =$ number of vertices of the convex hull of n i.i.d. points uniformly distributed in K.

· Is the expected vertex count monotone in input size? Do we have

$$\mathbb{E}\, f_0(K_n) \leq \mathbb{E}\, f_0(K_{n+1})?$$

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· If $K \subseteq L$ do we have $\mathbb{E}\,\mathrm{Vol}(K_n) \leq \mathbb{E}\,\mathrm{Vol}(L_n)$?

Answer: No (L. Rademacher, 2012).

II Expectation asymptotics

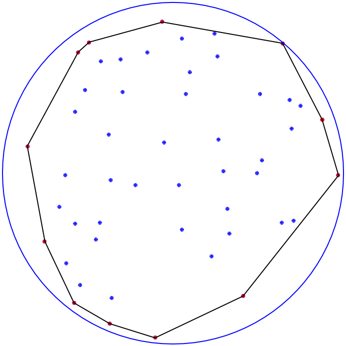
· Difficult to derive explicit formulas for statistics of convex hulls on finite number of i.i.d. points in a body K.

· Investigation has focussed on behavior as input size $n \to \infty$.

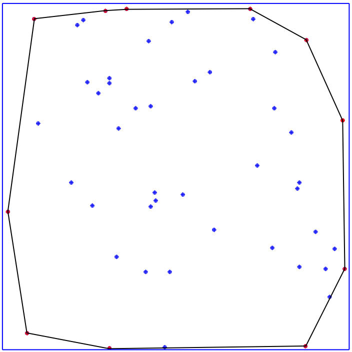
· The shape of ∂K determines the order of magnitude of $\mathbb{E}\, f_0(K_n)$.

II Expectation asymptotics

· The convex hull of 50 i.i.d. uniform points in the unit ball (red points are vertices):

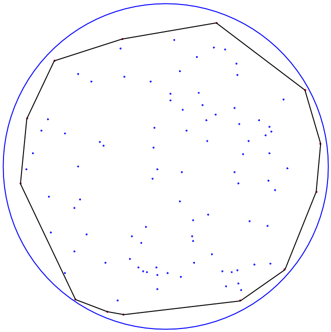


- The convex hull of 50 i.i.d. uniform points in the unit square:



II Expectation asymptotics

- The convex hull of 100 i.i.d. uniform points in the unit ball:



Spatial dependence: whether a point is a vertex of the convex hull depends on locations of other points. There are 13 vertices here.

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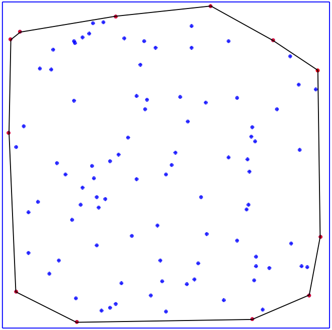

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II Expectation asymptotics

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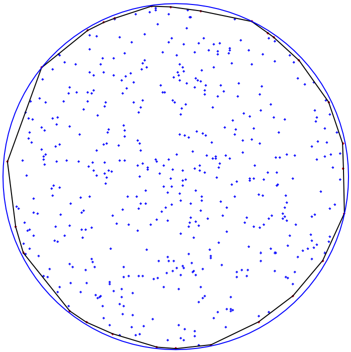

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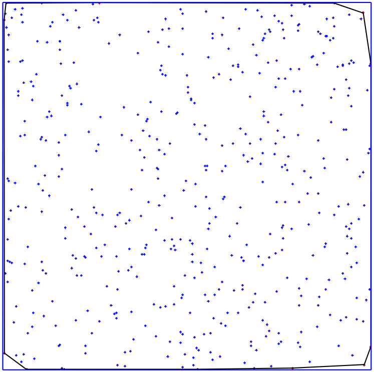


II Expectation asymptotics (d = 2)

· $f_0(K_n) =$ number of vertices of the convex hull K_n of n i.i.d. points uniformly distributed in K.

· $K = \mathbb{B}^2$: Rényi and Sulanke (1963), $X_i, 1 \le i \le n$, i.i.d. uniform in $\mathbb{B}^2$:

$$\lim_{n\to\infty} \frac{\mathbb{E}\, f_0(K_n)}{n^{1/3}} = C.$$

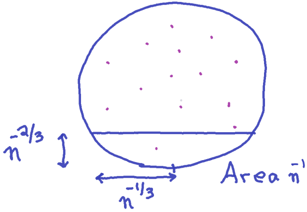
· Why the unusual cube root asymptotics $n^{1/3}$?

· A *cap* of $\mathbb{B}^2$ is the intersection of a half-plane with $\mathbb{B}^2$.

· Every extreme point is contained in a cap which does not contain any other sample point.

II Expectation asymptotics

- $n^{1/3}$ scaling heuristics
- This cap will have area roughly equal to n^{-1} and height $n^{-2/3}$:



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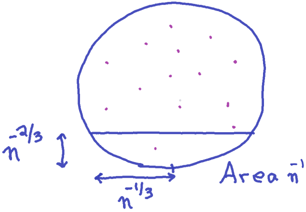

- The expected number of extreme points is the expected number of points in an annulus of width $n^{-2/3}$.

II Expectation asymptotics

· $f_k(K_n) =$ number of k-dimensional faces of the convex hull K_n of n i.i.d. points uniformly distributed in K.

· Rényi and Sulanke (1963), X_i i.i.d. in $K \subset \mathbb{R}^2$, ∂K smooth, $\mathrm{Vol} K = 1$:

$$\lim_{n\to\infty} \frac{\mathbb{E}\, f_0(K_n)}{n^{1/3}} = c \int_{\partial K} \kappa(x)^{1/3} dx.$$

· $\kappa(x)$: curvature at $x \in \partial K$

II Expectation asymptotics

- Dichotomy between smooth K and those which are polytopes.
- Reitzner (2005): $K \subset \mathbb{R}^d, d \geq 2$, ∂K of class C^2,

$$\lim_{n \to \infty} \frac{\mathbb{E}\, f_k(K_n)}{n^{(d-1)/(d+1)}} = c_{d,k} \int_{\partial K} \kappa(x)^{1/(d+1)} dx.$$

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· $\kappa(x)$: Gaussian curvature at $x \in \partial K$ (product of principal curvatures)

· Affine surface area: $\int_{\partial K} \kappa(x)^{1/(d+1)} dx$.

· Reitzner (2005): if $K \subset \mathbb{R}^d$ is a convex polytope with N vertices then

$$\lim_{n \to \infty} \frac{\mathbb{E}\, f_k(K_n)}{(\log n)^{d-1}} = e_{d,k} \cdot N.$$

· Half-spaces defining the convex hull have more long range impact when the points come from a convex polytope.

II Expectation asymptotics

- Rényi and Sulanke (1963-64):

- K_n is convex hull of n i.i.d. standard normal r.v. on $\mathbb{R}^d$:

$$\lim_{n\to\infty} \frac{\mathbb{E}\, f_0(K_n)}{(\log n)^{(d-1)/2}} = C_d.$$

Variance asymptotics

· Question: What are the precise variance asymptotics?

· As is the case with expectations, the correct scaling depends on the geometry, as seen in the next slide.

Variance asymptotics

· $K \subset \mathbb{R}^d, d \geq 2$, ∂K of class C^3, $\mathrm{Vol}(K) = 1$, $k \in \{0, 1, ..., d-1\}$,

$$\lim_{n \to \infty} \frac{\mathrm{Var} f_k(K_n)}{n^{(d-1)/(d+1)}} = V_{k,d} \int_{\partial K} \kappa(x)^{1/(d+1)} dx.$$

· $K \subset \mathbb{R}^d, d \geq 2$, is a simple polytope with N vertices, $\mathrm{Vol}(K) = 1$, $k \in \{0, 1, ..., d-1\}$,

$$\lim_{n \to \infty} \frac{\mathrm{Var} f_k(K_n)}{(\log n)^{(d-1)}} = \nu_{k,d} \cdot N.$$

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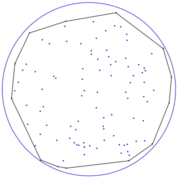
· K_n is Gaussian polytope:

$$\lim_{n \to \infty} \frac{\mathrm{Var} f_k(K_n)}{(\log n)^{(d-1)/2}} = v_{k,d}.$$

· Calka, Schreiber and Y (2013); Calka and Y (2014,2015,2017)

III Fluctuations of convex hull boundaries



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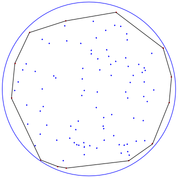

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· The boundary $\partial(nK_n)$ is **a random interface**. As n increases, new points appear, creating new faces which subsume existing faces.

Maximal radial fluctuation: width of minimal annulus which contains $\partial(nK_n)$. It is a measure of maximal local roughness.

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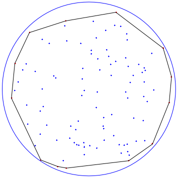

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Maximal longitudinal fluctuation: maximal diameter of faces in $\partial(nK_n)$.

III Fluctuations of convex hull boundaries

· Much is known concerning the fluctuations of dynamic and equilibrium systems.

III Fluctuations of convex hull boundaries

· For a planar object or planar system of linear scale n, there is known or believed to be:

· **Roughness** of order $n^{1/3}$, i.e. radial fluctuations are of this order.

· **Longitudinal correlation length** of order $n^{2/3}$, which may be interpreted as the typical length of an edge in the convex hull of the extreme points of the considered object.

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· Planar models exhibiting $\frac{1}{3}, \frac{2}{3}$ scaling are ubiquitous. The corner growth model is one example.

· The **convex hull boundary** is another example.

III Fluctuations of convex hull boundaries

· **Maximal radial (transversal) fluctuation** $MRF(nK_n)$: width of minimal annulus which contains $\partial(nK_n)$.

· **Maximal longitudinal fluctuation** $MLF(nK_n)$: maximal diameter of faces in $\partial(nK_n)$.

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· $d = 2$: We obtain order of growth of fluctuations, up to logarithmic precision.

$$\frac{MRF(nK_n)}{n^{1/3}(\log n)^{2/3}} = \Theta(1) \quad a.s., \qquad \frac{MLF(nK_n)}{n^{2/3}(\log n)^{1/3}} = \Theta(1) \quad a.s.$$

· The convex hull boundary exhibits $\frac{1}{3}, \frac{2}{3}$ **scaling**. Is this a coincidence?

· $d \geq 3$: We also obtain order of growth of fluctuations, up to logarithmic precision.

· **Maximal radial fluctuation** $MRF(nK_n)$: width of minimal annulus which contains $\partial(nK_n)$.

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III Fluctuations of convex hull boundaries, general case $\mathbb{B}^d$

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· **Maximal longitudinal fluctuation** $MLF(nK_n)$: maximal diameter of faces in $\partial(nK_n)$.

· We obtain order of growth of fluctuations, up to logarithmic precision:

$$MRF(nK_n) = \Theta(n^{\chi(d)}(\log n)^{2/(d+1)}) \quad a.s.,$$

$$MLF(nK_n) = \Theta(n^{\xi(d)}(\log n)^{1/(d+1)}) \quad a.s.,$$

where $\chi(d) := \frac{d-1}{d+1}$, $\xi(d) := \frac{d}{d+1}$ satisfy the relation

$$\chi = 2\xi - 1.$$

III Fluctuations of convex hull boundaries; $\mathbb{B}^d$

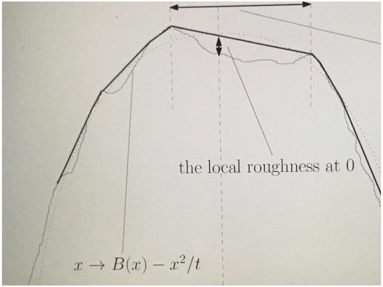
· Another model where one observes similar fluctuations:

· Brownian motion with parabolic drift:

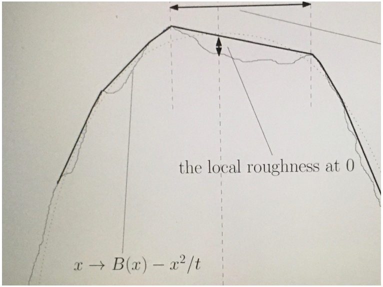
$$x \mapsto B(x) - \frac{x^2}{t}.$$

· How does Brownian motion with parabolic drift fluctuate with respect to its convex hull?

III Fluctuations of convex hull boundaries; $\mathbb{B}^d$



$\cdot$ Hammond: The facet length L of the convex hull may be identified as the horizontal scale above which parabolic curvature is dominant and below which Brownian fluctuation dominates.

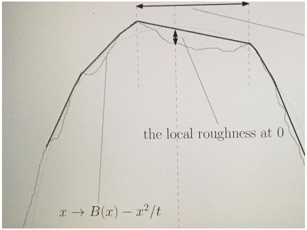


· Hammond: The facet length L of the convex hull may be identified as the horizontal scale above which parabolic curvature is dominant and below which Brownian fluctuation dominates.

· The scale of L may be identified by equating the two effects: Brownian fluctuation $\sim$ parabolic curvature, i.e.

$$L^{1/2} \sim \frac{L^2}{t}, \quad \text{i.e.,} \quad L \sim t^{2/3}.$$

III Fluctuations of convex hull boundaries; $\mathbb{B}^d$



· The facet length of the convex hull scales like $t^{2/3}$.

· The inward deviation (local roughness) of the BM from its convex hull scales like $t^{1/3}$.

III Fluctuations of convex hull boundaries; $\mathbb{B}^d$

Four characteristics of the KPZ universality class:

- an exponent of $2/3$ on the spatial scale
- an exponent of $1/3$ for the scale of height
- interfaces which are *locally Brownian*
- as well as *globally parabolic.*

· The above Brownian model has all four of these characteristics.
· The boundary of the convex hull of an i.i.d. sample appears to have only three of these characteristics.

· **Maximal radial fluctuation** $MRF(nK_n)$: width of minimal annulus which contains $\partial(nK_n)$.

· **Maximal longitudinal fluctuation** $MLF(nK_n)$: maximal diameter of faces in $\partial(nK_n)$.

· The order of growth of fluctuations are a corollary to a more precise growth result.

III Fluctuations of convex hull boundaries; $\mathbb{B}^d$

· Given i.i.d. random variables $X_1, ..., X_n$, it is of interest to compute the distribution of maximum, $M_n := \max_{i \leq n} X_i$. One limit distribution is

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{M_n - a_n}{b_n} \leq t \right) = e^{-e^{-\frac{t-a}{b}}}, \quad t \in (-\infty, \infty),$$

the **Gumbel distribution**.

III Fluctuations of convex hull boundaries; $\mathbb{B}^d$

· Given i.i.d. random variables $X_1, ..., X_n$, it is of interest to compute the distribution of maximum, $M_n := \max_{i \le n} X_i$. One limit distribution is

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the **Gumbel distribution**.

· It is difficult to obtain the distribution of the maximum of random variables with spatial correlations.

· The diameters of faces of the convex hull are spatially correlated. So are the radial distances between each face and the boundary of $\mathbb{B}^d$.

III Fluctuations of convex hull boundaries

· The maximal radial fluctuation $MRF(nK_n)$ satisfies a Gumbel extreme value result, i.e., as $n \to \infty$

$$\mathbb{P}\left(c_0\left(\frac{MRF(nK_n)}{n^{(d-1)/(d+1)}}\right)^{\frac{d+1}{2}} - c_1 \log n - c_2 \log(\log n) - c_3 \leq t\right)$$
$$\to e^{-e^{-t}},\ t \in (-\infty, \infty).$$

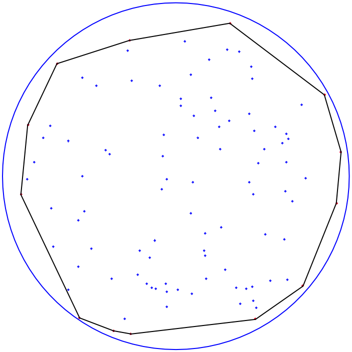
· The maximal diameter of the faces $MLF(nK_n)$ also satisfies a Gumbel extreme value result, but with different scaling.

IV Scaling limit of convex hull boundaries

- What is the scaling limit of the boundary of the convex hull?
- Let's start with the case of i.i.d. uniform points in the unit ball.

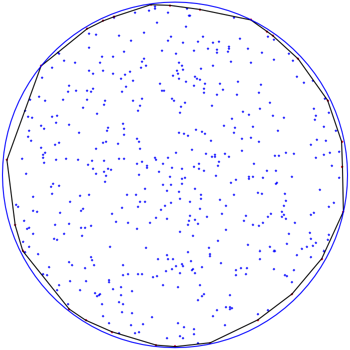
IV Scaling limit of convex hulls

· The convex hull of 100 points which are i.i.d. uniform in the unit ball:



IV Scaling limit of convex hulls

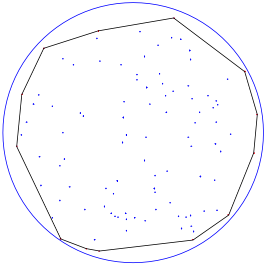
· The convex hull of 500 points which are i.i.d. uniform in the unit ball:



What is the scaling limit of the convex hull boundary?

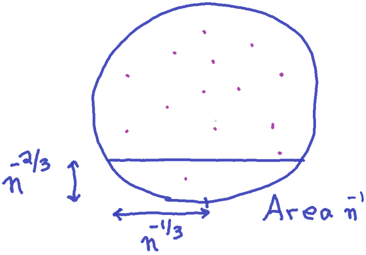
IV Scaling limits of convex hulls in unit ball

- $\mathcal{X}_n$: i.i.d. uniform point set in unit ball of cardinality n.



IV Scaling limits of convex hulls in unit ball

· Parabolic geometry: $y \sim x^2/2$ near the south pole.



IV Scaling limits of convex hulls in unit ball

· The geometry of the unit ball near the boundary is 'parabolic'.

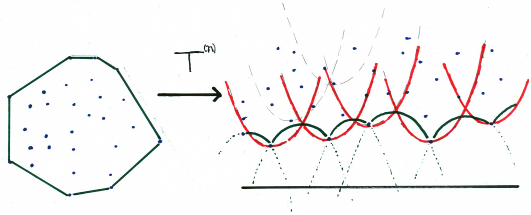
· Thus any reasonable transformation of the unit ball into rectangular coordinates should be such that its scaling in radial direction is the square of scaling in angular direction.

IV Scaling limits of convex hulls in unit ball

· Given an i.i.d. point set $\mathcal{X}_n$ of size n, we assert that there is a transform

$$T^{(n)} : \mathbb{B}^d \to \mathbb{R}^{d-1} \times \mathbb{R}^+$$

which produces the following picture:



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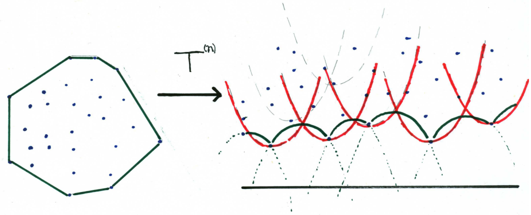
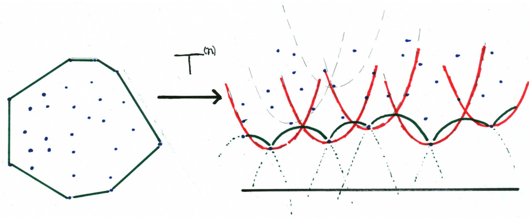


· $T^{(n)}$ carries the boundary of the convex hull to the festoon of inverted down quasi-paraboloids, which are translates of $y = \frac{-x^2}{2} + o(1)$.

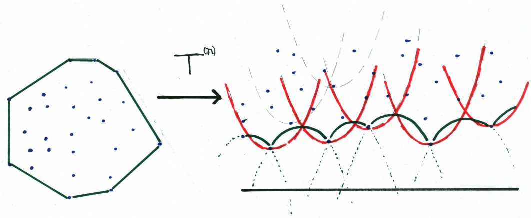
· What about the up quasi-paraboloids?

IV Scaling limits of convex hulls in unit ball



$T^{(n)}$ carries the boundary of the convex hull to the festoon of inverted down quasi-paraboloids. What about the up quasi-paraboloids?

$\mathcal{X}_n$: i.i.d. point set in unit ball of cardinality n.

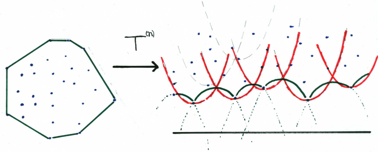


$T^{(n)}$ carries the boundary of the convex hull to the festoon of inverted down quasi-paraboloids. What about the up quasi-paraboloids?

$\mathcal{X}_n$: i.i.d. point set in unit ball of cardinality n.

Definition. $w_0 \in T^{(n)}(\mathcal{X}_n)$ is extreme iff the up quasi-paraboloid in $\mathbb{R}^{d-1} \times \mathbb{R}^+$ with apex at w_0 is not covered by the union of the up quasi-paraboloids with apices at $T^{(n)}(\mathcal{X}_n) \setminus w_0$.

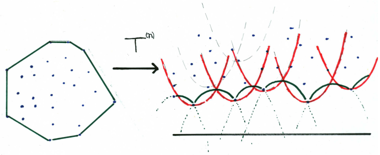
Key idea: a point in the left-hand figure is extreme iff it is carried to an extreme point in the right-hand figure.



Properties of the transform $T^{(n)} : \mathbb{B}^d \to \mathbb{R}^{d-1} \times \mathbb{R}^+$

(i) $T^{(n)}$ sends extreme pts to extreme pts

IV Scaling limits of convex hulls in unit ball

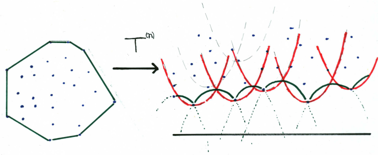


Properties of the transform $T^{(n)} : \mathbb{B}^d \to \mathbb{R}^{d-1} \times \mathbb{R}^+$

(i) $T^{(n)}$ sends extreme pts to extreme pts

(ii) As $n \to \infty$, $T^{(n)}$ sends uniform samples of size n in $\mathbb{B}^d$ to a rate one Poisson point process on $\mathbb{R}^{d-1} \times \mathbb{R}^+$

IV Scaling limits of convex hulls in unit ball



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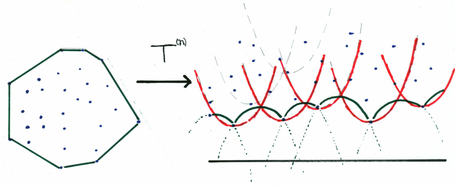
(ii) As $n \to \infty$, $T^{(n)}$ sends uniform samples of size n in $\mathbb{B}^d$ to a rate one Poisson point process on $\mathbb{R}^{d-1} \times \mathbb{R}^+$

(iii) As $n \to \infty$, $T^{(n)}$ sends the boundary of convex hull into the inverted festoon of down-paraboloids (translates of $y = -|x|^2/2$).

(iv) $T^{(n)}$ sends $\mathbb{S}^{d-1}$ to a subset of $\mathbb{R}^{d-1}$ having area $n^{(d-1)/(d+1)}$.

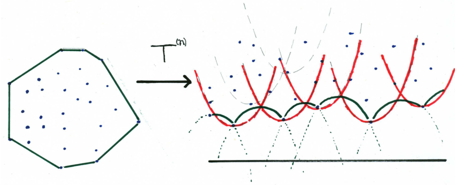
IV Scaling limits of convex hulls in unit ball



Advantages to studying re-scaled picture:

(i) spatial dependencies are easier to localize in re-scaled picture

(ii) the space correlations decay exponentially fast wrt distance. This leads to asymptotic independence and a CLT for the number of extreme points.



Advantages to studying re-scaled picture:

(i) spatial dependencies are easier to localize in re-scaled picture

(ii) the space correlations decay exponentially fast wrt distance. This leads to asymptotic independence and a CLT for the number of extreme points.

(iii) the **scaling limit** of the convex hull boundary is the festoon of inverted down-paraboloids.

IV Scaling limits of convex hulls: scaling transform $T^{(n)}$

- What is the transformation $T^{(n)} : \mathbb{B}^d \mapsto \mathbb{R}^{d-1} \times \mathbb{R}^+$ which does the job?
- For $d = 2$ we require this transformation:

$$(r, \theta) \mapsto (n^{1/3}\theta, n^{2/3}(1 - r)).$$

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- For $d = 2$ we require this transformation:

$$(r, \theta) \mapsto (n^{1/3}\theta, n^{2/3}(1 - r)).$$

- For $d > 2$: use the inverse of the exponential map.

IV Scaling limits of convex hulls: scaling transform $T^{(n)}$

· For $d > 2$: use the inverse of the exponential map to measure the angle with respect to north pole.

· Tan: tangent space to $\mathbb{S}^{d-1}$ at the north pole $u_0 = (0, 0, ..., 1)$.

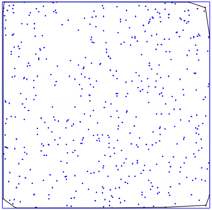
· *Exponential map* $\exp : \mathrm{Tan} \to \mathbb{S}^{d-1}$ maps a vector $v \in \mathrm{Tan}$ to the point $u \in \mathbb{S}^{d-1}$ lying at the end of the geodesic of length $|v|$ starting at u_0 and having direction v.

· Use scaling transform $T^{(n)} : \mathbb{B}^d \mapsto \mathbb{R}^{d-1} \times \mathbb{R}^+$:

$$T^{(n)}(x) := \left(n^{\frac{1}{d+1}} \exp^{-1}(\frac{x}{|x|}), n^{\frac{2}{d+1}}(1 - |x|) \right), \ x \in \mathbb{B}^d \setminus \{\mathbf{0}\}.$$

· $T^{(n)}$ carries the sphere $\mathbb{S}^{d-1}$ into a subset of $\mathbb{R}^{d-1}$ of area $n^{(d-1)/(d+1)}$.

· We are given an i.i.d. sample of points in the unit cube.

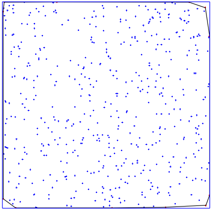


· We can likewise find a transform which:

(i) sends the points in $[0,1]^d$ to points in the upper half-space $\mathbb{R}^{d-1} \times \mathbb{R}^+$,

IV Scaling limits of convex hulls in polytopes

· We are given an i.i.d. sample of points in the unit cube.



· We can likewise find a transform which:

(i) sends the points in $[0,1]^d$ to points in the upper half-space $\mathbb{R}^{d-1} \times \mathbb{R}^+$,

(ii) sends the boundary of the convex hull to a limit shape similar to the parabolic festoon.

IV Scaling limits of convex hulls of Gaussian samples

· We are given an i.i.d. Gaussian sample $X_1, ..., X_n$ in $\mathbb{R}^d$ with standard normal density

$$(2\pi)^{-d/2} \exp(-\frac{|x|^2}{2}), \ x \in \mathbb{R}^d.$$

· We can find a transform $T^{(n)}$ which:

(i) sends the Gaussian sample to points in the product space $\mathbb{R}^{d-1} \times \mathbb{R}$,

IV Scaling limits of convex hulls of Gaussian samples

· We are given an i.i.d. Gaussian sample $X_1, ..., X_n$ in $\mathbb{R}^d$ with standard normal density

$$(2\pi)^{-d/2} \exp(-\frac{|x|^2}{2}), \ x \in \mathbb{R}^d.$$

· We can find a transform $T^{(n)}$ which:

(i) sends the Gaussian sample to points in the product space $\mathbb{R}^{d-1} \times \mathbb{R}$,

(ii) as $n \to \infty$, sends the boundary of the convex hull of the Gaussian sample to a limit parabolic festoon suspended on a Poisson point process $\mathcal{P}$ with density proportional to the exponentiated height h above $\mathbb{R}^{d-1}$, namely the density is

$$d\mathcal{P}((v,h)) = e^h dh dv, \quad (v,h) \in \mathbb{R}^{d-1} \times \mathbb{R}.$$

IV Scaling limits of convex hulls of Gaussian samples

· We are given an i.i.d. Gaussian sample $X_1, ..., X_n$ in $\mathbb{R}^d$ with standard normal density

$$(2\pi)^{-d/2} \exp(-\frac{|x|^2}{2}), \ x \in \mathbb{R}^d.$$

· We find a good transform by first scaling the Gaussian points so that nearly all of them are inside the unit ball. In $d = 2$, we re-scale by R_n, where

$$R_n := \sqrt{2 \log n - \log(2 \cdot (2\pi)^2 \cdot \log n)}.$$

IV Scaling limits of convex hulls of Gaussian samples

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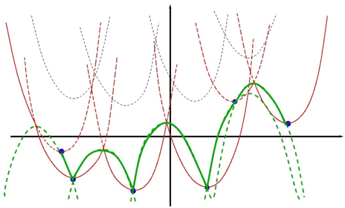
What is the transformation $T^{(n)} : \mathbb{R}^d \mapsto \mathbb{R}^{d-1} \times \mathbb{R}$ which does the job?

· For $d = 2$ we require this transformation:

$$(r, \theta) \mapsto \left(R_n \theta, R_n^2 (1 - \frac{r}{R_n}) \right).$$

IV Scaling limits of convex hulls of Gaussian samples

· Scaling limit of the convex hull of a Gaussian sample of size n, $n \to \infty$:



· Scaling limit (green festoon) consists of translates of the down parabola $y = \frac{-|x|^2}{2}$. It is suspended on points of the limit Poisson point process $\mathcal{P}$ with density

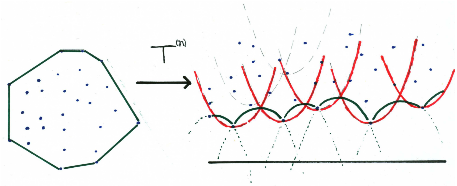
$$d\mathcal{P}((v,h)) = e^h dh dv, \quad (v,h) \in \mathbb{R}^{d-1} \times \mathbb{R}.$$

· Blue points correspond to the limiting extreme points (vertices) of the convex hull.

V Proof ideas

· We focus on proof techniques for k-face functional of the convex hull of an i.i.d. uniform sample in the unit ball.

· Some of the ideas can be used for convex hull of i.i.d. samples in hypercubes and for Gaussian samples.

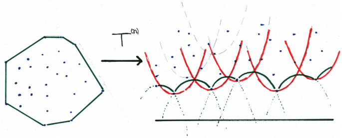


Properties of the transform $T^{(n)} : \mathbb{B}^d \to \mathbb{R}^{d-1} \times \mathbb{R}^+$

(i) As $n \to \infty$, $T^{(n)}$ sends the boundary of convex hull into the inverted festoon of down-paraboloids suspended on the points of a rate one Poisson point process on $\mathbb{R}^{d-1} \times \mathbb{R}^+$.

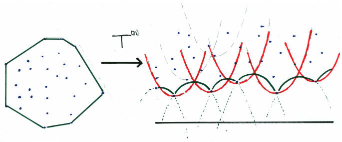
(ii) $T^{(n)}$ transforms the $(d-1)$-dimensional faces of the convex hull K_n into subsets of inverted quasi down-paraboloids, all of which are translates of $y = -\frac{|x|^2}{2} + o(1)$.

V Proof ideas



(iii) The vertices of K_n , being the intersection of d hyperplanes of dimension $d-1$, are transformed by $T^{(n)}$ into the intersection of d inverted quasi down-paraboloids. Extreme points mapped to extreme points and k-faces mapped to k-faces.

V Proof ideas

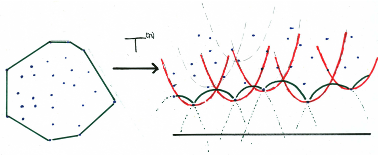


(iii) The vertices of K_n , being the intersection of d hyperplanes of dimension $d-1$, are transformed by $T^{(n)}$ into the intersection of d inverted quasi down-paraboloids. Extreme points mapped to extreme points and k-faces mapped to k-faces.

· If $w \in \mathcal{X}^{(n)}$ is extreme, then for all $k \in \{0, 1, ..., d-1\}$ we put

$$\xi_k^{(n)}(w, \mathcal{X}^{(n)}) = \frac{1}{k+1} (\text{card(k-faces in festoon which contain w)})$$

V Proof ideas



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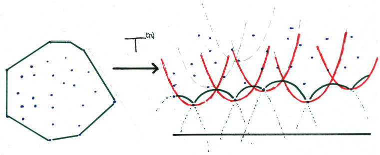
- If $w \in \mathcal{X}^{(n)}$ is not extreme, then set $\xi_k^{(n)}(w, \mathcal{X}^{(n)}) = 0$.

V Proof ideas



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$$\xi_k^{(n)}(w, \mathcal{X}^{(n)}) = \frac{1}{k+1} (\text{card(k-faces in festoon which contain w)})$$

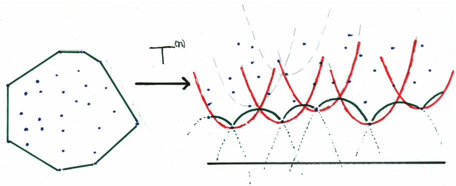
· If $w \in \mathcal{X}^{(n)}$ is not extreme, then set $\xi_k^{(n)}(w, \mathcal{X}^{(n)}) = 0$.

· The number of k-faces in the festoon is given by

$$\sum_{w \in \mathcal{X}^{(n)}} \xi_k^{(n)}(w, \mathcal{X}^{(n)})$$

and this coincides with the number of k-faces in the convex hull K_n.

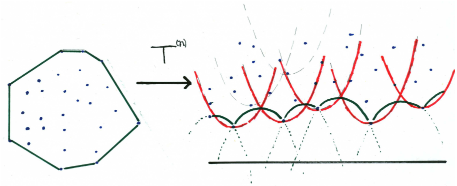
V Proof ideas



· We claim that the geometry in the right hand picture localizes. In other words, the extremality status of a transformed point in right hand picture is determined by 'local data', i.e., data in a cylindrical neighborhood of that point.

· We make this rigorous as follows.

V Proof ideas



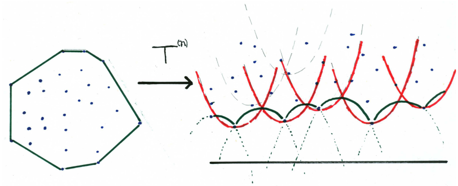
· For $w \in \mathbb{R}^{d-1} \times \mathbb{R}^+$ and $r > 0$, let $C_r(w)$ be the cylinder of radius r with vertical axis through w.

Stabilization radius for $\xi_k^{(n)}$: given $w \in \mathcal{X}^{(n)}$, there is $R := R(w, \mathcal{X}^{(n)})$ such that for all $k \in \{0, 1, ..., d-1\}$

$$\xi_k^{(n)}(w, \mathcal{X}^{(n)} \cap C_R(w)) = \xi_k^{(n)}(w, (\mathcal{X}^{(n)} \cap C_R(w)) \cup \mathcal{A})$$

for any point set $\mathcal{A} \subset C_R(w)^c$.

V Proof ideas



· For $w \in \mathbb{R}^{d-1} \times \mathbb{R}^{+}$ and $r > 0$, let $C_r(w)$ be the cylinder of radius r with vertical axis through w.

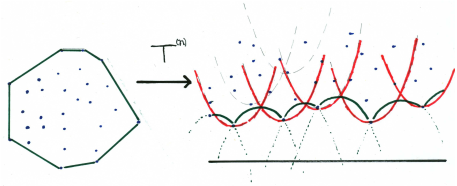
Stabilization radius for $\xi_k^{(n)}$: given $w \in \mathcal{X}^{(n)}$, there is $R := R(w, \mathcal{X}^{(n)})$ such that for all $k \in \{0, 1, ..., d-1\}$

$$\xi_k^{(n)}(w, \mathcal{X}^{(n)} \cap C_R(w)) = \xi_k^{(n)}(w, (\mathcal{X}^{(n)} \cap C_R(w)) \cup \mathcal{A})$$

for any point set $\mathcal{A} \subset C_R(w)^c$.

· Moreover, the random variable R has an exponentially decaying tail, i.e., the scores $\xi_k^{(n)}(w, \mathcal{X}^{(n)})$ are exponentially stabilizing. This is a **non-trivial but essential observation.**

V Proof ideas

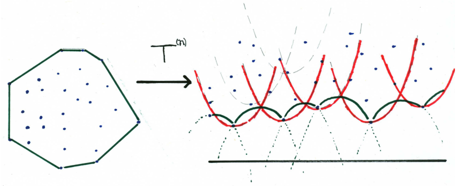


· The number of k-faces in the festoon is given by

$$\sum_{w \in \mathcal{X}^{(n)}} \xi_k^{(n)}(w, \mathcal{X}^{(n)})$$

and this coincides with the number of k-faces in the original convex hull.

V Proof ideas



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· Thus we have represented the k-face statistic as a sum of scores which are exponentially stabilizing. Heuristically, since the spatial dependence of the scores is well controlled, the sum of the scores should behave like a sum of i.i.d random variables.

· Large literature on the limit theory for sums of stabilizing scores.

VI Further results: variance asymptotics

· The number of k-faces in the festoon is given by

$$\sum_{w \in \mathcal{X}^{(n)}} \xi_k^{(n)}(w, \mathcal{X}^{(n)})$$

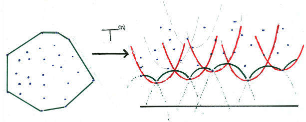
and this coincides with the number of k-faces in the convex hull.

· Weak law of large numbers: K the unit ball, $d \geq 2$,

$$\lim_{n \to \infty} \frac{\mathbb{E}\, f_k(K_n)}{n^{(d-1)/(d+1)}} = c_{d,k}.$$

· Variance asymptotics?

- Define a limit k-face functional as follows.



- If $w \in \mathcal{X}^{(n)}$ is extreme, then for all $k \in \{0, 1, ..., d-1\}$ we recall

$$\xi_k^{(n)}(w, \mathcal{X}^{(n)}) = \frac{1}{k+1} (\text{card(k-faces in festoon which contain w)})$$

VI Further results: variance asymptotics

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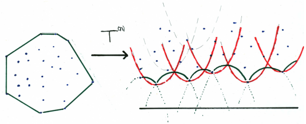

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$$\xi_k^{(n)}(w, \mathcal{X}^{(n)}) = \frac{1}{k+1} (\text{card(k-faces in festoon which contain w)})$$

· Denote by $\xi_k^{(\infty)}(\cdot, \mathcal{P})$ the limit k-face functional operating on the limit PPP $\mathcal{P}$.

VI Further results: variance asymptotics

· A generic point w in $\mathbb{R}^{d-1} \times \mathbb{R}^+$ is given by $w = (v, h)$.

· $\xi_k^{(\infty)}$ is the limit k-face functional operating on the limit PPP $\mathcal{P}$.

Definition. For all $w_1, w_2 \in \mathbb{R}^{d-1} \times \mathbb{R}^+$ and all $k \in \{0, 1, ..., d-1\}$, put

$$c^{\xi_k^{(\infty)}}(w_1, w_2) := \mathbb{E}\, \xi_k^{(\infty)}(w_1, \mathcal{P} \cup \{w_2\}) \xi_k^{(\infty)}(w_2, \mathcal{P} \cup \{w_1\}) - \mathbb{E}\, \xi_k^{(\infty)}(w_1, \mathcal{P}) \mathbb{E}\, \xi_k^{(\infty)}(w_2, \mathcal{P}),$$

$$\sigma^2(\xi_k^{(\infty)}) := \int_0^\infty \mathbb{E}\, \xi_k^{(\infty)}((\mathbf{0}, h_1), \mathcal{P})^2 \mathrm{d}h_1 + \int_0^\infty \int_{\mathbb{R}^{d-1}} \int_0^\infty c^{\xi_k^{(\infty)}}((\mathbf{0}, h_1), (v_2, h_2)) \mathrm{d}h_1 \mathrm{d}v_2 \mathrm{d}h_2.$$

The triple integral is finite because of asymptotic de-correlation and because $\xi_k^{(\infty)}$ decays exponentially fast in height.

VI Further results: variance asymptotics

- The number of k-faces in the festoon is given by

$$\sum_{w \in \mathcal{X}^{(n)}} \xi_k^{(n)}(w, \mathcal{X}^{(n)})$$

and this coincides with the numer of k-faces in the convex hull K_n.

- **Variance asymptotics:** If K the unit ball, $d \geq 2$, then

$$\lim_{n \to \infty} \frac{\mathrm{Var} f_k(K_n)}{n^{(d-1)/(d+1)}} = \sigma^2(\xi_k^{(\infty)}) \in (0, \infty).$$

- Schreiber, Calka, Y.

VI Further results: CLT

· Rates of asymptotic normality for k-face functional.

Theorem (LSY, 2019): For all $k \in \{0, 1, ..., d-1\}$ we have

$$\sup_t \left| \mathbb{P}\left(\frac{f_k(K_n) - \mathbb{E}\, f_k(K_n)}{\sqrt{\mathrm{Var} f_k(K_n)}} \leq t \right) - \mathbb{P}(\mathcal{N} \leq t) \right| = O\left(\frac{1}{\sqrt{\mathrm{Var} f_k(K_n)}} \right),$$

where $\mathcal{N}$ denotes a mean zero normal random variable with variance one.

VI Further results: CLT

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where $\mathcal{N}$ denotes a mean zero normal random variable with variance one.

· RHS is of order

$$O\left(\frac{1}{n^{(d-1)/2(d+1)}} \right).$$

VI Further results: CLT

· What is the advantage to re-scaling and using the transform $T^{(n)}$?

· If we are only interested in CLT, then re-scaling via $T^{(n)}$ is not required in order to see that the scores localize.

· But if we want second order results (variance asymptotics), then re-scaling is needed. Likewise, re-scaling needed in order to find scaling limit of boundary of convex hull (green festoon).

VI Further results: Brownian limits

· *Exponential map* $\exp : \mathrm{Tan} \to \mathbb{S}^{d-1}$ maps a vector $v \in \mathrm{Tan}$ to the point $u \in \mathbb{S}^{d-1}$ lying at the end of the geodesic of length $|v|$ starting at north pole and having direction v.

· **Def.** For any σ^2 let B^{σ^2} be the Brownian sheet with variance coefficient σ^2 on the injectivity region of the exponential map. That is to say B^{σ^2} is the mean zero continuous Gaussian path process indexed by $\mathbb{R}^{d-1}$ with

$$\mathrm{cov}(\mathrm{B}^{\sigma^2}(\mathrm{v}), \mathrm{B}^{\sigma^2}(\mathrm{w})) = \sigma^2 \cdot \sigma_{\mathrm{d}-1}(\exp([\mathbf{0}, \mathrm{v}] \cap [\mathbf{0}, \mathrm{w}])),$$

where σ_{d-1} is the $(d-1)$-dimensional surface measure on $\mathbb{S}^{d-1}$.

VI Further results: Brownian limits

· Let $V_n(v), v \in \mathbb{R}^{d-1}$, be the volume between ∂K_n and $\mathbb{S}^{d-1} \cap \exp([\mathbf{0}, v])$. There is a σ^2 such that in $\mathcal{C}(\mathbb{R}^{d-1})$

$$n^{(d+3)/2(d+1)}(V_n(\cdot) - \mathbb{E}\, V_n(\cdot)) \xrightarrow{\mathcal{D}} B^{\sigma^2},\ n \to \infty.$$

· The variance scales like $n^{-(d+3)/(d+1)}$.

· There are $\sim n^{(d-1)/(d+1)}$ facets, each facet contributes a defect volume having variance equal to the square of the product of facet area $n^{-(d-1)/(d+1)}$ and defect height $n^{-2/(d+1)}$.

VI Further results: Convex hull peeling

· The process of sorting a set of sample points $\mathcal{X} \subset \mathbb{R}^d$ into convex layers is called convex hull peeling or convex hull ordering.

· **Convex peeling of $\mathcal{X} \subset \mathbb{R}^d$:** Consider the nested sequence of closed convex sets defined by $L_1(\mathcal{X}) = \mathrm{conv}(\mathcal{X})$ and

$$L_2 = \mathrm{conv}(\mathcal{X} \cap \mathrm{int}(L_1(\mathcal{X})))$$

$$L_{n+1} = \mathrm{conv}(\mathcal{X} \cap \mathrm{int}(L_n(\mathcal{X}))),$$

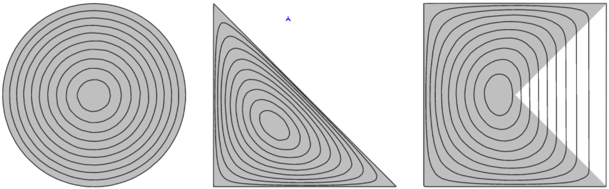
where $\mathrm{conv}(\mathcal{X})$ is the convex hull of $\mathcal{X}$ and $\mathrm{int}(L)$ is the interior of L.

· Note $L_{n+1} \subset L_n$ for all $n = 1, 2, \ldots$ The difference $L_n \setminus L_{n+1}$ is called a 'peel'.

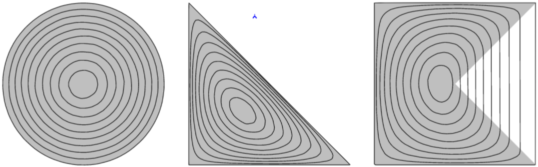
VI Further results: Convex hull peeling

· This peeling procedure will eventually exhaust the entire dataset.

· The first 10 peels of data sets of 10^5 i.i.d. uniform points in three different planar regions:



VI Further results: Convex hull peeling



· The scaling transform which we used to study the first layer could also be used to study the second, third, etc. layers.

· This transform gives expectation asymptotics, variance asymptotics, central limit theorems and scaling limits for the number of points in the kth layer.

VI Further results: Convex hull peeling

· **Convex hull peeling depth:**

$$h_{\mathcal{X}}(x) = \sum_{n \geq 1} \mathbf{1}(x \in \mathrm{int}(L_n(\mathcal{X}))).$$

· What is the shape of $h_{\mathcal{X}}$ for random finite sets $\mathcal{X} \subset \mathbb{R}^d$?

· Dalal (2004): There is a constant $C > 0$ such that, if $\mathcal{X}_n \subset \mathbb{R}^d$ consists of n points chosen independently and uniformly at random from the unit ball, then

$$C^{-1} n^{2/(d+1)} \leq \mathbb{E}\left[\max h_{\mathcal{X}_n}\right] \leq C n^{2/(d+1)}.$$

VI Further results: Convex hull peeling

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· The re-scaled height functions $n^{-2/(d+1)} h_{\mathcal{X}_n}$ converge almost surely:

$$n^{-2/(d+1)} h_{\mathcal{X}_n}(x) \to \alpha(\frac{d+1}{2d})(1 - |x|^{2d/(d+1)}).$$

· Calder and Smart (2020): When the underlying points in $\mathcal{X}_n \subset \mathbb{R}^d$ have a density f then the rescaled height functions $n^{-2/(d+1)} h_{\mathcal{X}_n}$ converge to the solution of a PDE involving f.

THANK YOU