Large deviations for random hives and the spectrum of the sum of two random matrices

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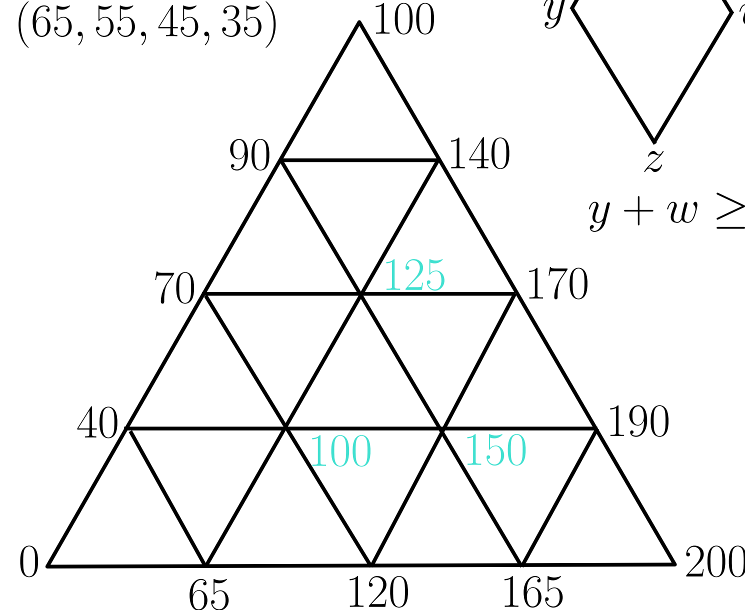
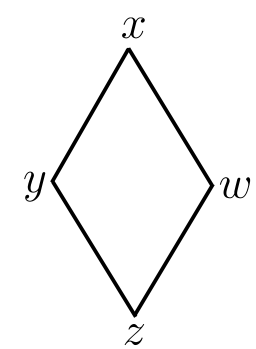
Hives

$\lambda = (40, 30, 20, 10)$

$\mu = (40, 30, 20, 10)$

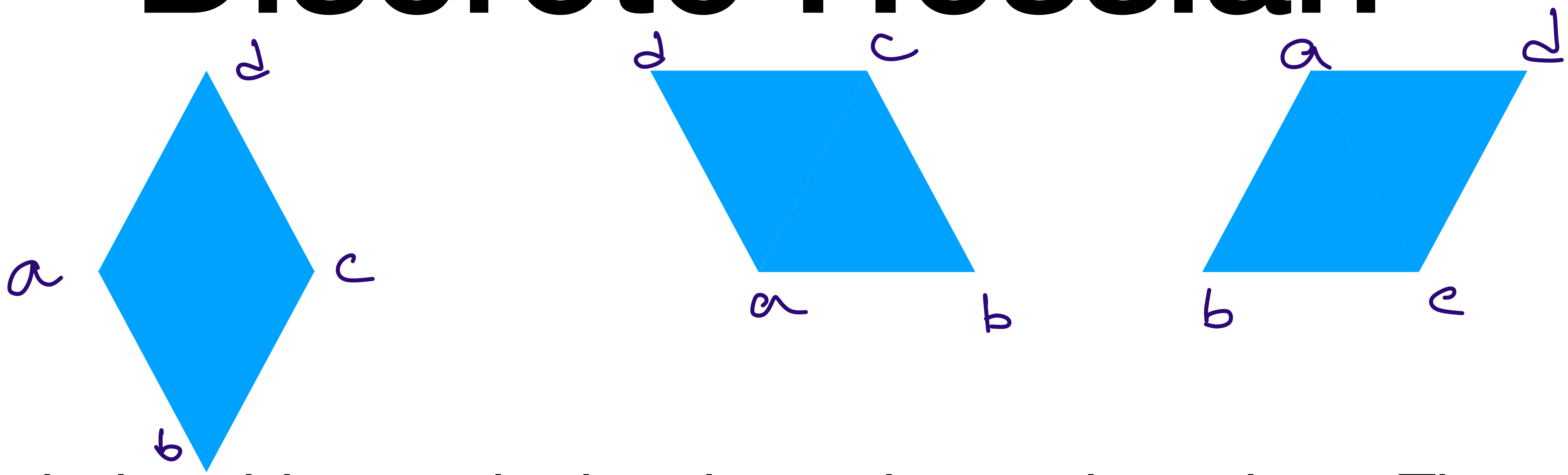
$\nu = (65, 55, 45, 35)$





$y + w \geq x + z$

Discrete Hessian

- 

- Unit rhombi come in the above three orientations. The discrete Hessian is a map from the set of unit rhombi to the reals.

- For a given unit rhombus, with the above labels (denoting values that the hive takes), the Hessian evaluated on the corresponding rhombus, is $b + d - a - c$ in each case.

Connection to representation theory

$\lambda = (40, 30, 20, 10)$
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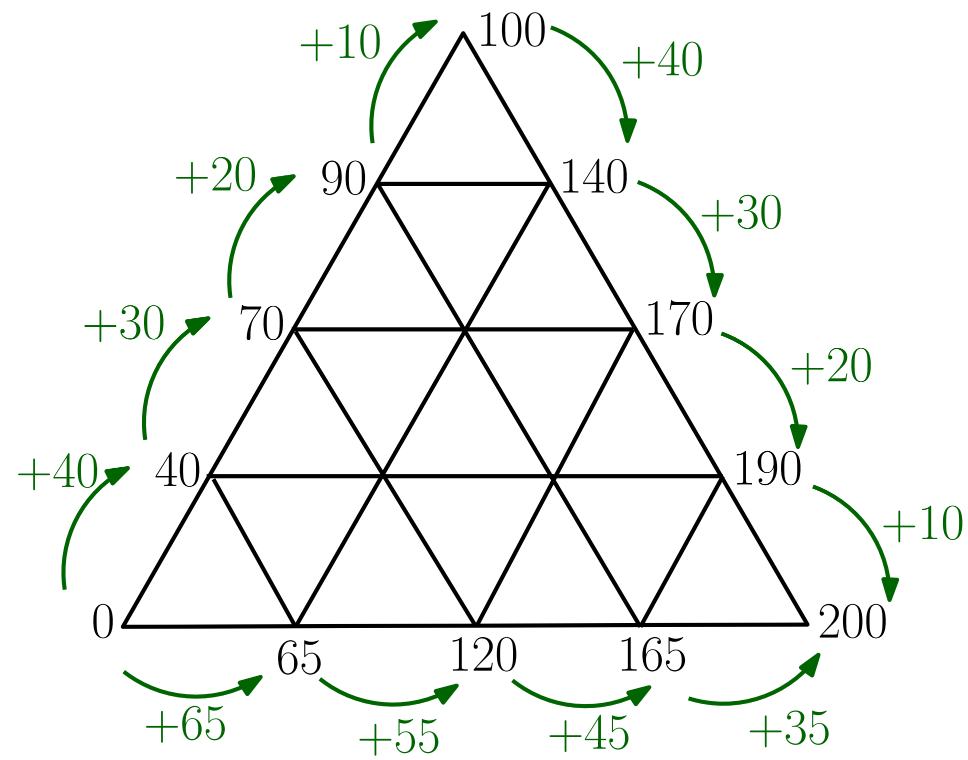


$$V_\lambda \otimes V_\mu \equiv \oplus_\nu \, c^{\nu}_{\lambda\mu} V_\nu$$

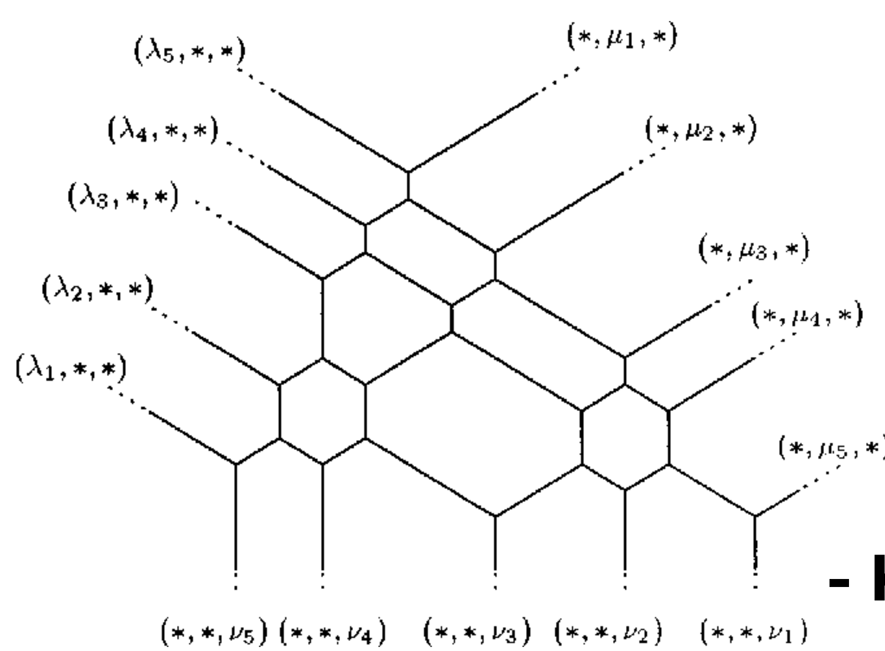
Connection to representation theory

- The number of integer valued concave functions (also called hives) on a equilateral lattice on an equilateral triangle of side n with fixed boundary conditions count Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$, which are Clebsch-Gordon coefficients corresponding to the group $GL_n(\mathbb{C})$ (Knutson-Tao 1999).

$$V_\lambda \otimes V_\mu \equiv \oplus_\nu \, c^{\nu}_{\lambda\mu} V_\nu$$

Connection to statistical physics

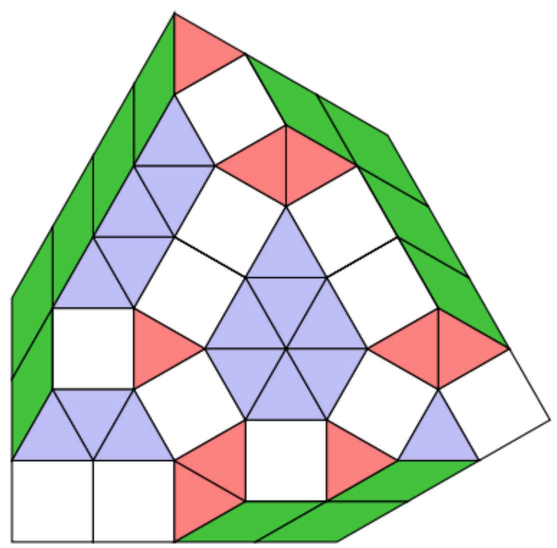
- Height functions of random tilings.
- The gradients of a random concave function measured at the centers of the unit triangles form the vertices of a random hexagonal tiling of the plane.

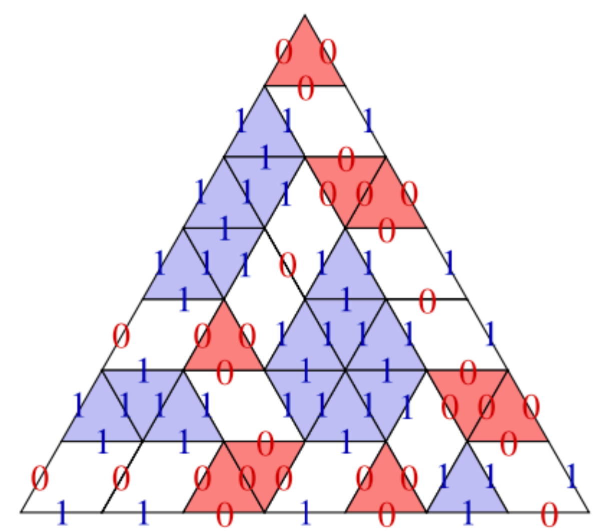


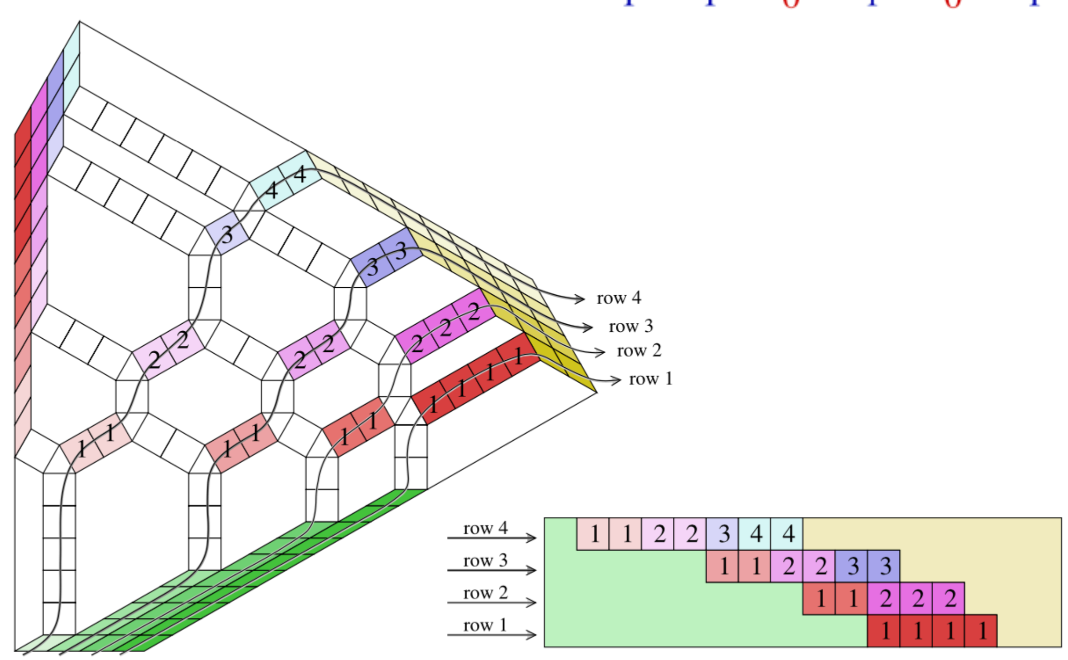
- Knutson-Tao 1999

Littlewood-Richardson coefficients and mosaics

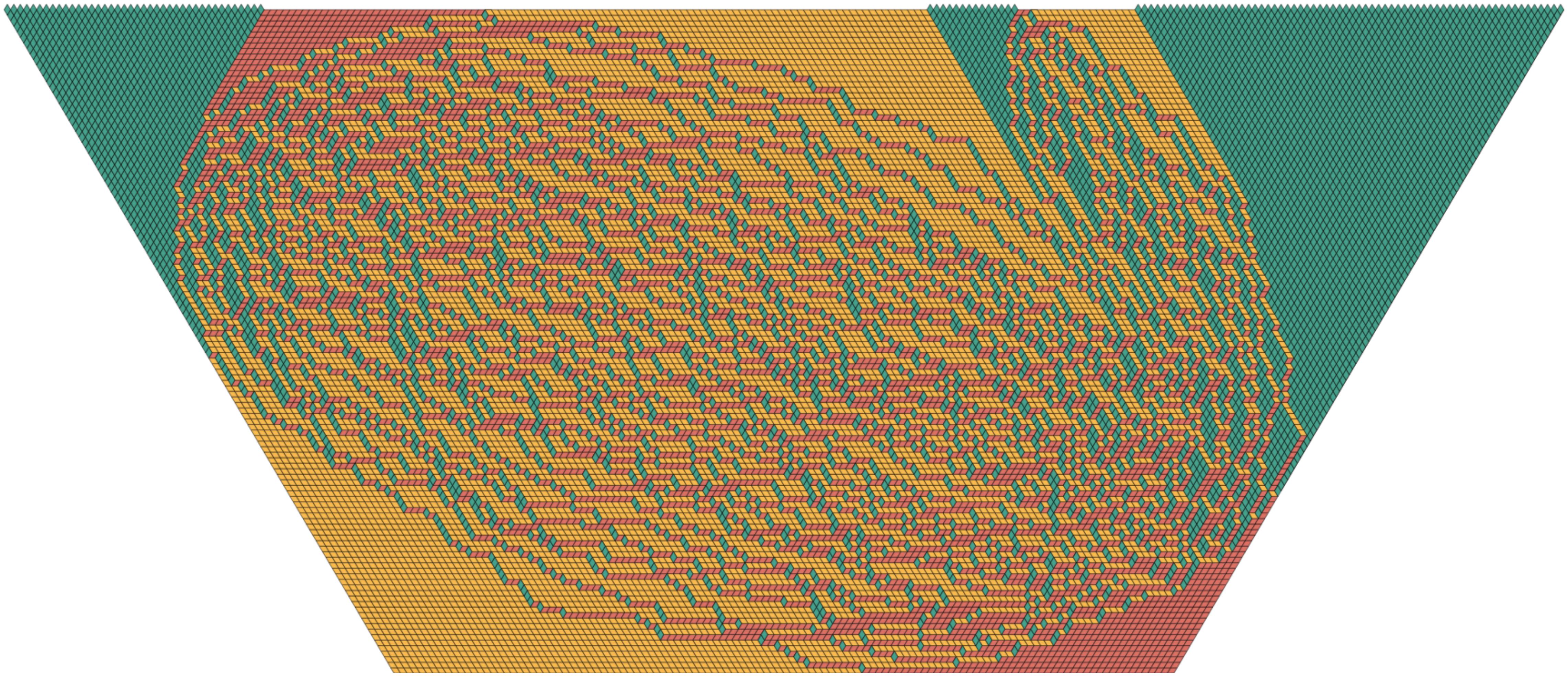
Kevin Purbhoo - Puzzles, Tableaux and mosaics







Lozenge tilings

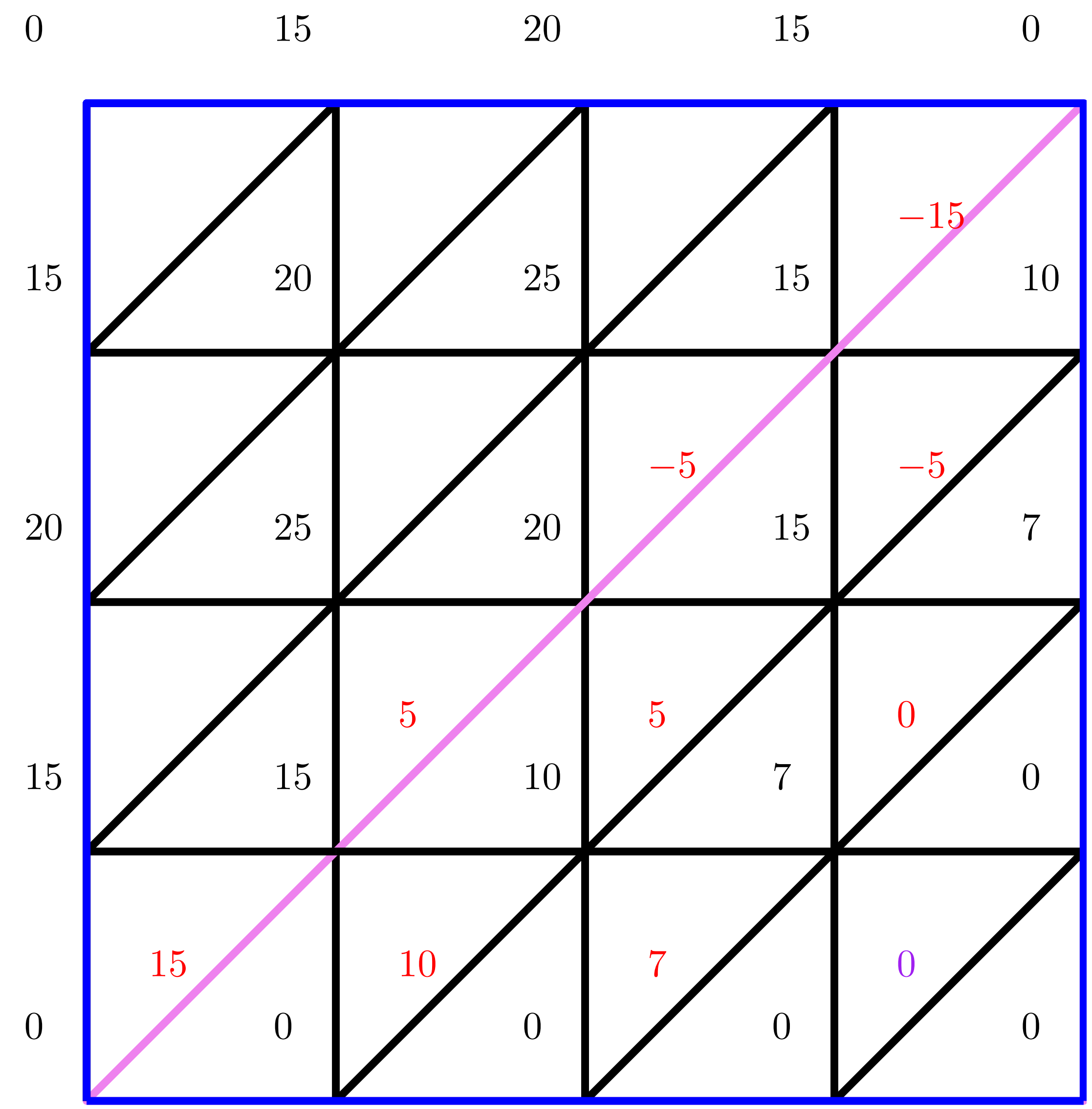


Courtesy L. Petrov

Connection to random matrices

- The volume of the polytope $P^{\nu_n}_{\lambda_n \mu_n}$ of all real hives is equal, up to known multiplicative factors involving Vandermonde determinants, to the probability density of obtaining a Hermitian matrix with spectrum ν_n when two independent Haar random Hermitian matrices with spectra λ_n, μ_n are added (Knutson-Tao 2003).

Spectrum of the sum of two random matrices



Density of the Marginals of random augmented hives along the diagonal give the density of the spectrum of the sum of two random matrices

Large deviation principle for the spectrum of the sum of two independent Haar random matrices.

- Let $\{\mathbb{P}_n\}$ be the sequence of Borel probability measures on $L^\infty([0,1]),$ supported on the set of Lipschitz concave functions that are zero on the endpoints, defined by piecewise linear extensions of $\nu_n = \mathrm{spec}(X_n + Y_n),$ where X_n and Y_n are independent and Haar random with $\mathrm{spec}(X_n) = \lambda_n$ and $\mathrm{spec}(Y_n) = \mu_n$.

Large deviation principle for the spectrum of the sum of two independent Haar random matrices.

- Here λ, μ are strongly decreasing, bounded on $[0,1]$ and integrate to 0.
- $\lambda_n(i) = n\int_{\frac{i-1}{n}}^{i/n} \lambda.$ $\qquad \mu_n(i) = n\int_{\frac{i-1}{n}}^{i/n} \mu.$
- $\sigma(s) = -\log \mathbf{f}(s).$
- $\mathrm{V}(\lambda) := \exp\left(\int_{T\setminus\{(t,t)|t\in[0,1]\}} \log\left(\frac{\lambda(x)-\lambda(y)}{x-y}\right) dxdy\right).$
- Let the rate function I at γ such that $D\gamma = \nu$ be given by
- $$I(\gamma) := \log\left(\frac{V(\lambda)V(\mu)}{V(\nu)V(\tau)}\right) + \inf_{h\in H(\lambda,\mu;\nu)} \int_T \sigma((-1)(\nabla^2 h)_{ac})\mathrm{Leb}_2(dx).$$

Large deviation principle for the spectrum of the sum of two independent Haar random matrices.

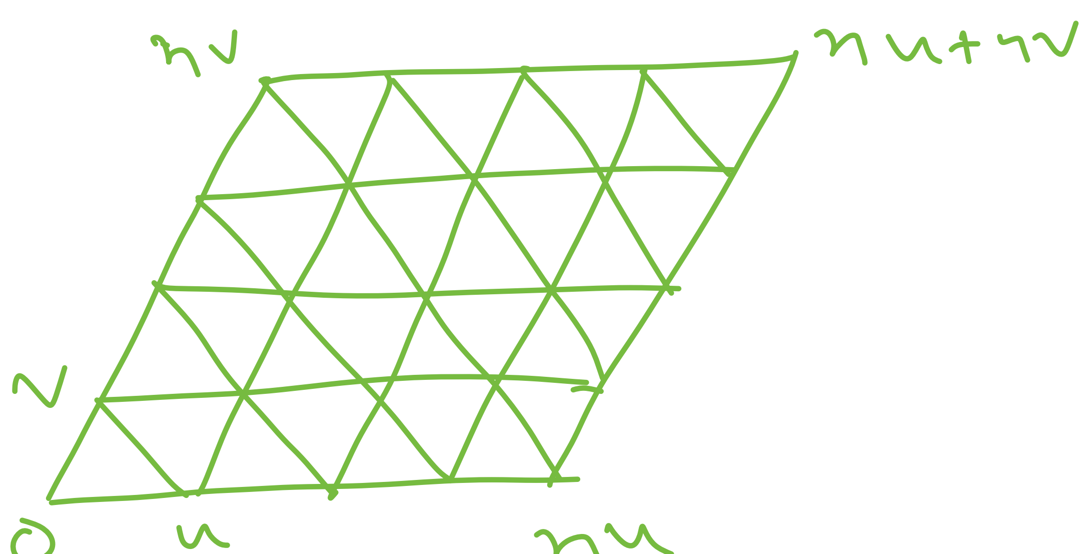
- Main Theorem:

- Let $a_n = n^2/2$. For each Borel measurable set $E \subset L^\infty([0,1])$,

$$-\inf_{\gamma \in E^\circ} I(\gamma) \le \liminf_{n\to\infty} a_n^{-1} \log(\mathbb{P}_n(E)) \le \limsup_{n\to\infty} a_n^{-1} \log(\mathbb{P}_n(E)) \le -\inf_{\gamma \in \overline{E}} I(\gamma).$$

Periodic Hessians

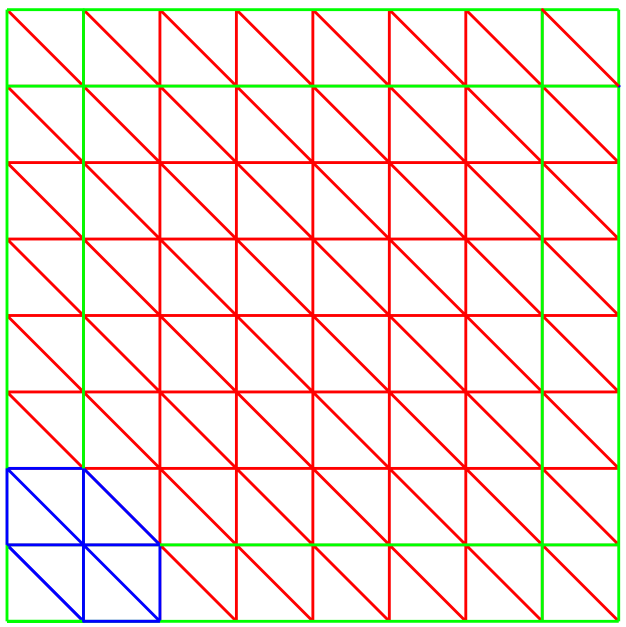
- Consider the polytope $P_n(s)$ of mean zero functions on a discrete torus obtained from a fundamental domain of the equilateral lattice generated by unit vectors u and v in two dimensional euclidean space, of functions, whose discrete Hessians are dominated by $s = (s_0, s_1, s_2)$. This is a polytope of dimension $n^2 - 1$.



Volume of $P_n(s)$

- Suppose $s = (s_0, s_1, s_2)$ and that $2 = s_0 \leq s_1 \leq s_2$.
- We see that any mean zero function on the discrete torus taking values in $[-1/2, 1/2]$ belongs to $P_n(s)$, which therefore contains a central section of the unit cube. By a result of J. Vaaler, any central section of the unit cube has volume at least 1. Therefore $|P_n(s)| \geq 1$.

- The marginals of log-concave densities are log-concave, as a consequence of the Prekopa-Leindler inequality.



- Using differential entropy, we show that $\limsup_{n\to\infty} |P_n(s)|^{\frac{1}{n^2-1}} \le 2e.$

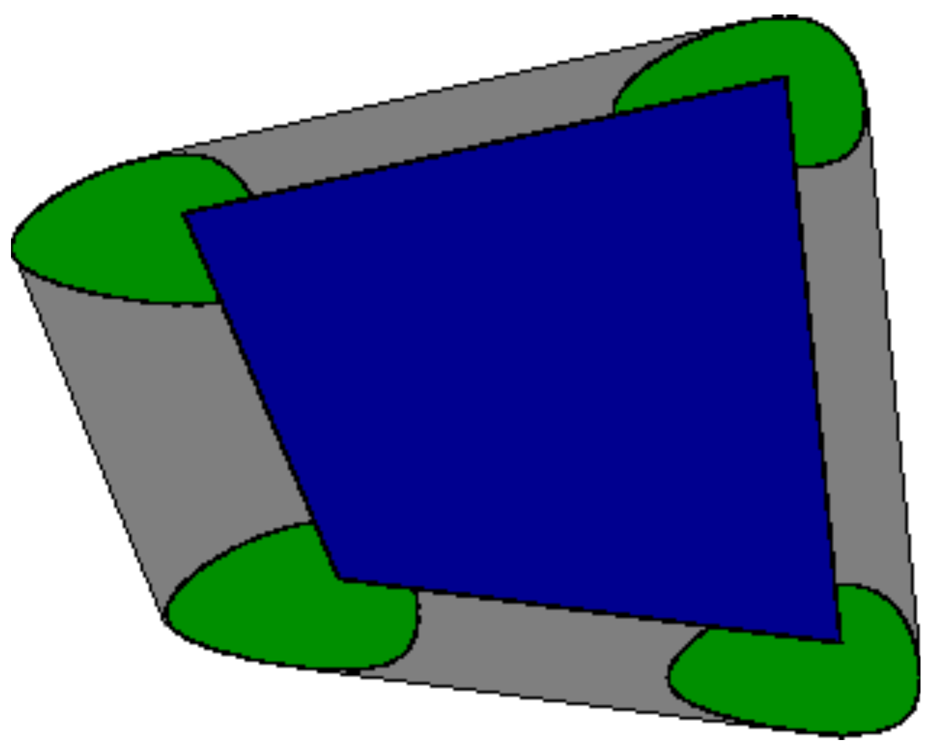
Volume of $P_n(s)$

- We thus have
$$1 \leq \liminf_{n\to\infty} |P_n(s)|^{\frac{1}{n^2-1}} \leq \limsup_{n\to\infty} |P_n(s)|^{\frac{1}{n^2-1}} \leq 2e.$$

- We show that as a function of n, $|P_n(s)|^{\frac{1}{n^2-1}}$ is "approximately" monotonically increasing, and this tells us that $\lim_{n\to\infty} |P_n(s)|^{\frac{1}{n^2-1}} \in [1,2e]$.

Convex geometry

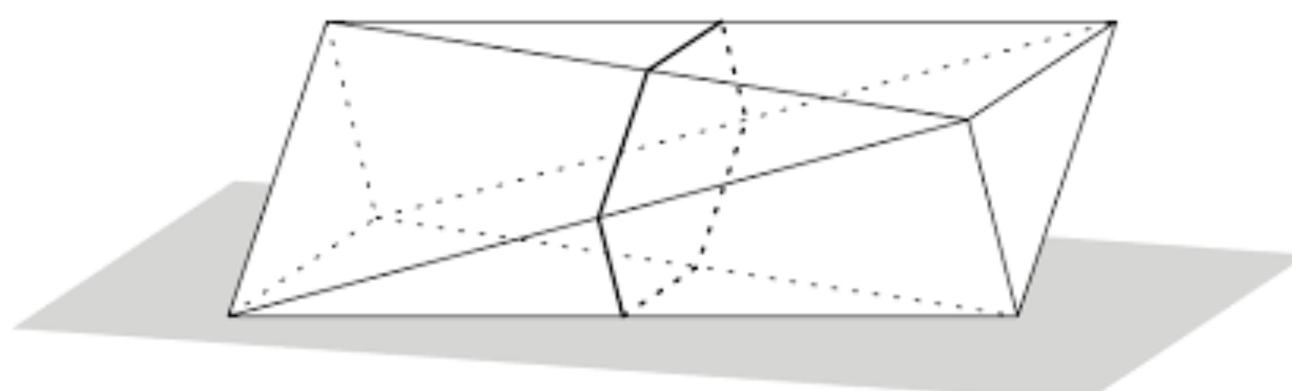
- Let K and L be compact convex subsets of $\mathbb{R}^m$, where $m \geq 1$. The Brunn-Minkowski inequality states that $|K+L|^{\frac{1}{m}} \geq |K|^{\frac{1}{m}} + |L|^{\frac{1}{m}}$.



Minkowski Sum

In the periodic case

- By the Brunn-Minkowski Inequality, the volumes of parallel d-dimensional sections of a convex body raised to d^{-1} define a concave function.



- Consequently, $\mathbf{f}_n(s) := |P_n(s)|^{\frac{1}{n^2-1}}$ is a concave function of s.

- Proof idea:
- Partition into small

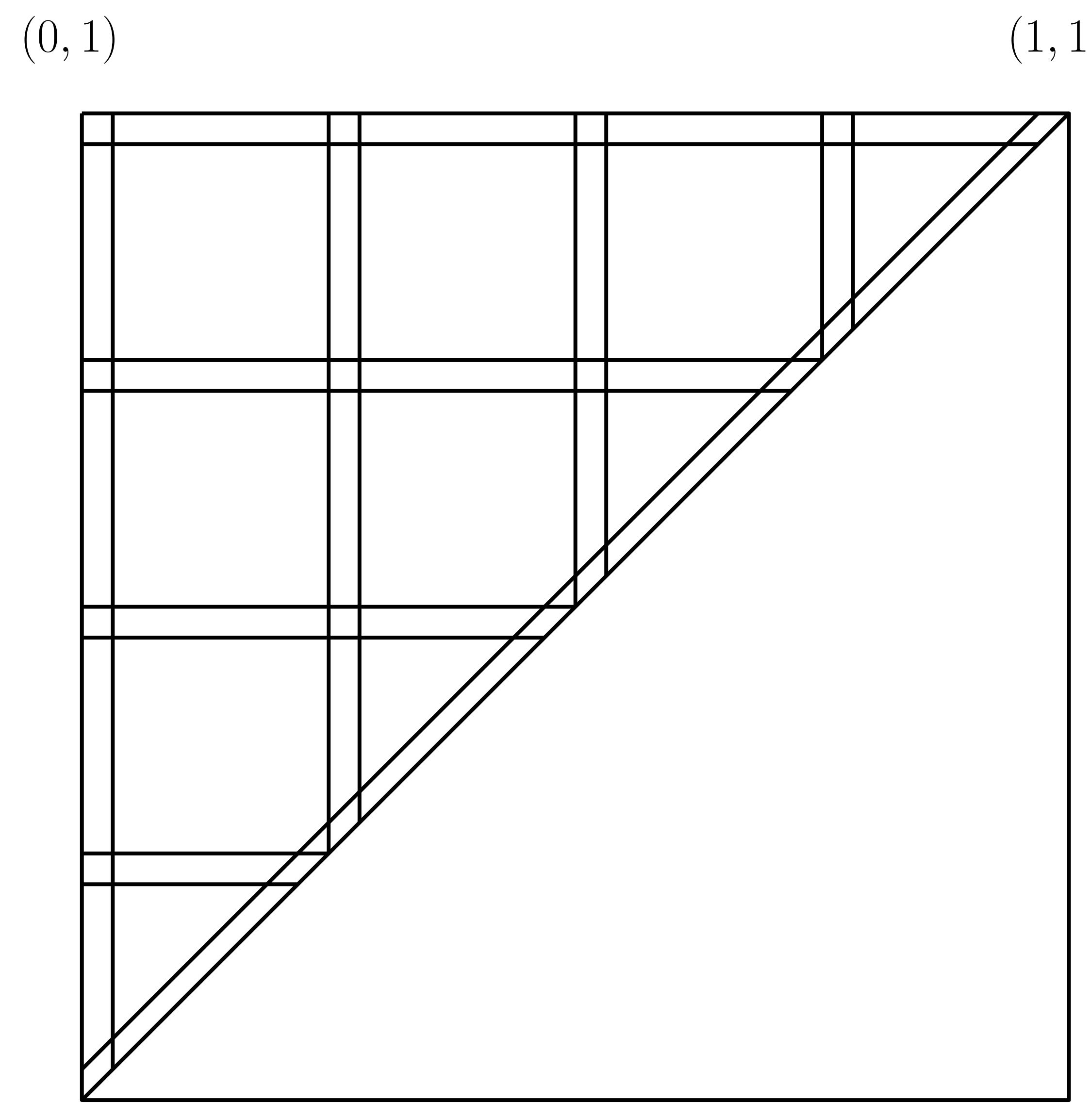
squares $\kappa \in \mathcal{D}_a$

and triangles separated

by four or five layer

boundaries. Fix the values of

the hive on the boundaries.



Lemma relating the discrete to the continuous

Lemma:

Let $|\kappa| := \mathrm{Leb}_2(\kappa)$, and κ^o be the interior of κ. For any fixed

$$\mathrm{h}_* \in \bigcup_{\nu' \in \overline{B}_{\mathcal{I}}(\nu,\epsilon)} H(\lambda, \mu; \nu'),$$

such that

$$\inf_T \min_{0 \le i \le 2} ((-1) D_i h_*)_{ac} > 0,$$

Then,

$$\lim_{a \to \infty} \exp\left(- \sum_{\kappa \in \mathcal{D}_a} |\kappa| \sigma \left((-1)|\kappa|^{-1} \int_\kappa \nabla^2 h_*(dx) \right) \right) = \exp\left(- \int_T \sigma((-1)(\nabla^2 h_*(x))_{ac}) \mathrm{Leb}_2(dx) \right).$$

Proof

- Let $\mathcal{F}_a \subset \mathcal{F}$ be the $\sigma-$field generated by all possible events κ where κ belongs to the set $\mathcal{D}_a$ of all dyadic triangles or dyadic squares of side length 2^{-a} contained in T, the κ being half-open.

- Let $\tilde{X}(\omega) = (-\nabla^2 h_*)_{ac}(\omega)$ and $\tilde{X}_a = \mathbb{E}[\tilde{X}|\mathcal{F}_a]$.

- Let $\tilde{Y}_a(\omega) = (2|\kappa|)^{-1}(-\nabla^2 h_*)_{sing}(\kappa),.$

Proof

- Since σ is convex, this implies that
- $\mathrm{E}[\sigma(\tilde{X}_{a+1} + \tilde{Y}_{a+1})|\mathcal{F}_a] \geq \sigma(\tilde{X}_a + \tilde{Y}_a)$, and so $\{\sigma(\tilde{X}_a + \tilde{Y}_a)\}_{a\geq 1}$ is a submartingale. Similarly $\{\sigma(\tilde{X}_a)\}_{a\geq 1}$ is a submartingale.
- A collection of random variables $X_i, i \in I$, is said to be uniformly integrable if

$$\lim_{M\to\infty}\left(\sup_{I\in I}\mathbb{E}(|X_i|\mathbf{1}_{|X_i|>M})\right) = 0.$$

Proof

- For a submartingale, the following are equivalent.

1. It is uniformly integrable.
2. It converges almost surely and in L^1.

- The logarithmic growth of -σ gives uniform integrability via a criterion of de la Vallée Poussin, which together with an application of the Lebesgue differentiation theorem, gives us that as $a \to \infty$, $\sigma(\tilde{X}_a + \tilde{Y}_a)$ converges almost surely and in L^1 to $\sigma(\tilde{X})$. Thus,
- $$\lim_{a\to\infty} \exp\left(-\sum_{\kappa\in\mathcal{D}_a} |\kappa|\sigma\left((-1)|\kappa|^{-1}\int_\kappa \nabla^2 h_*(dx)\right)\right) = \exp\left(-\int_T \sigma((-1)(\nabla^2 h_*(x))_{ac})\mathrm{Leb}_2(dx)\right).$$

Upper bound

- One dyadic square

with a four layer

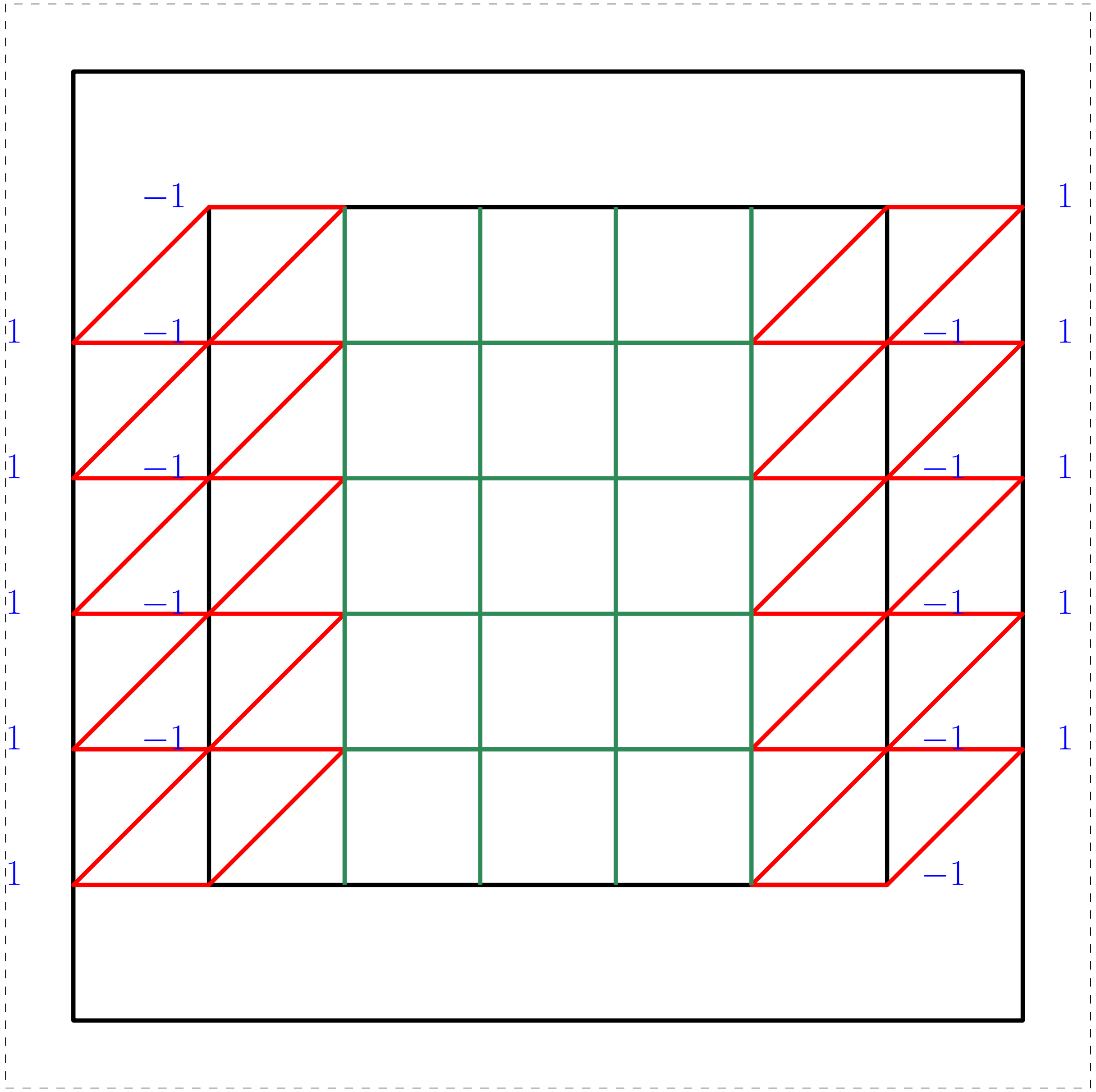
boundary.

Use Brunn-Minkowski

for upper bound on

volume of the relevant

Polytope as follows.



Upper bound

- Lemma:

Let t be a function from the rhombi of $\mathbb{T}_{n_1}$ to the nonnegative reals.

Let $Q_{n_1}(t)$ be the polytope consisting of the set of all functions

$g : V(\mathbb{T}_{n_1}) \to \mathbb{R}$

such that g is zero mean and $\nabla^2(g)(e) \leq t(e)$.

Suppose

$$\sum_{e \in E_i(\mathbb{T}_{n_1})} t(e) = n^2 s_i$$

.

Then, $\boxed{|Q_{n_1}(t)| \leq |P_{n_1}(s)|.}$

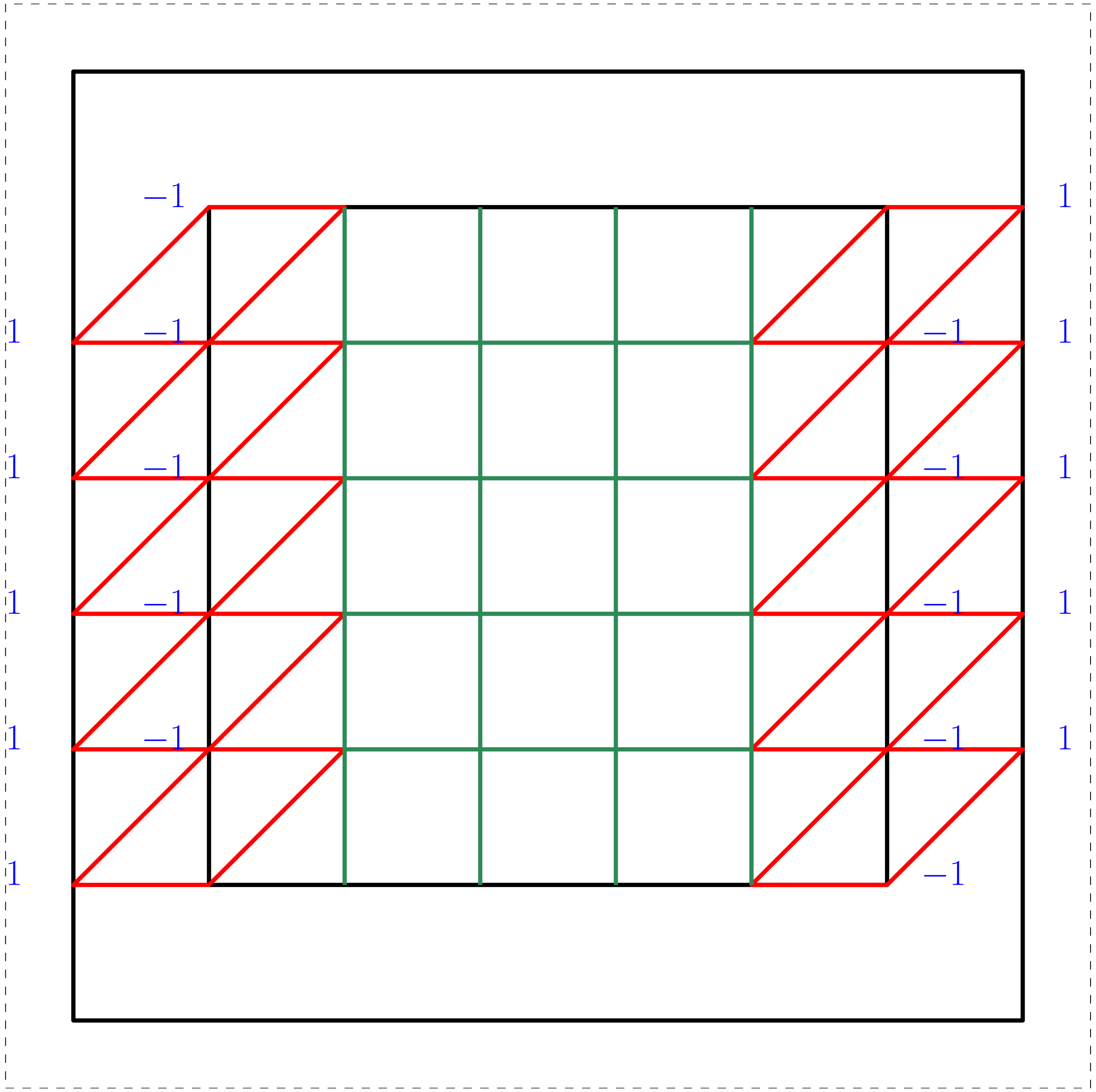
Upper bound

Let s be the average

Hessian of any extension

to the interior. Then,

$\boxed{|Q_{n_1}(t)| \leq |P_{n_1}(s)|}.$

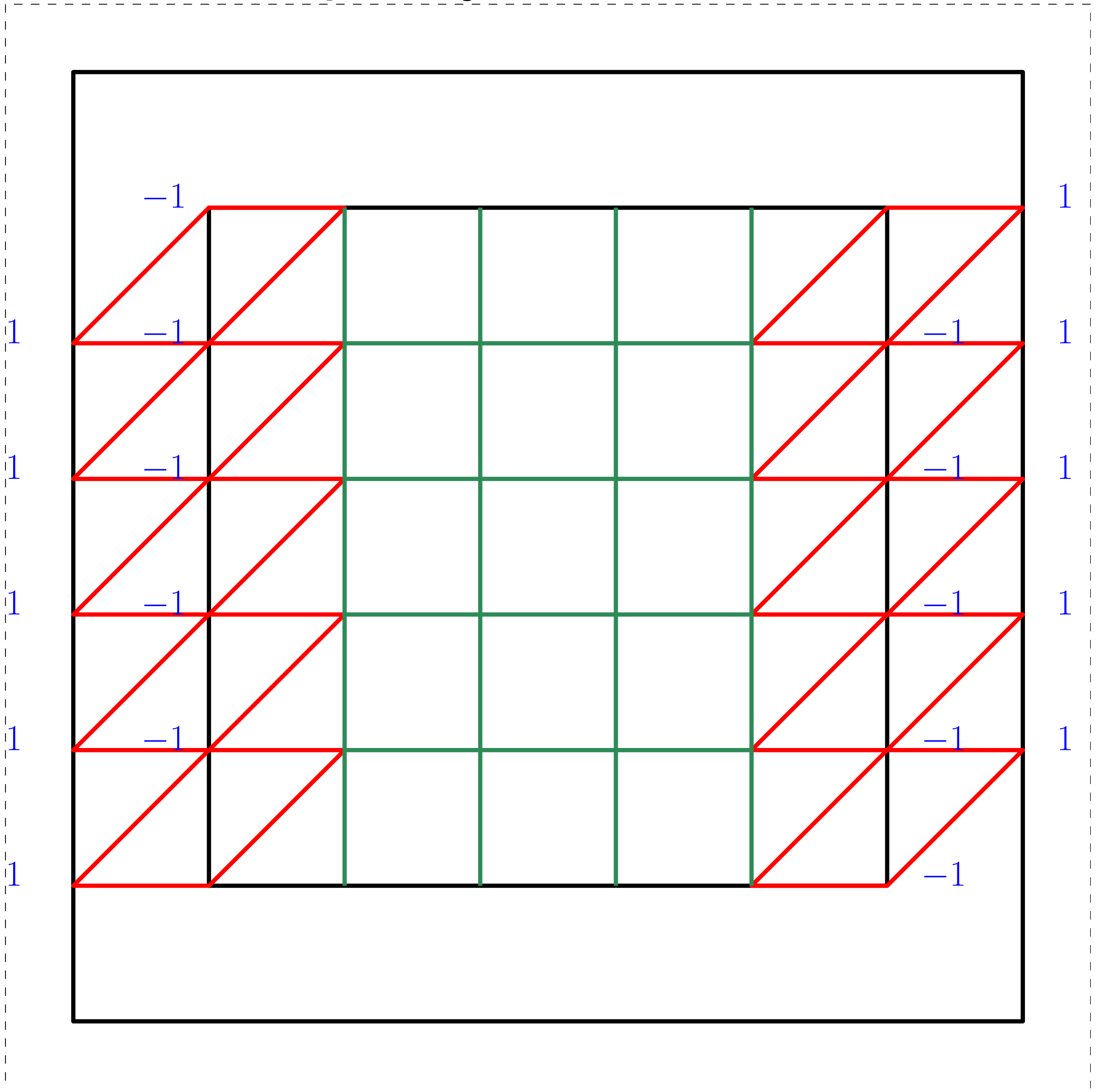


Lower bound

- The marginals of log-concave densities are log-concave, as a consequence of the Prekopa-Leindler inequality.

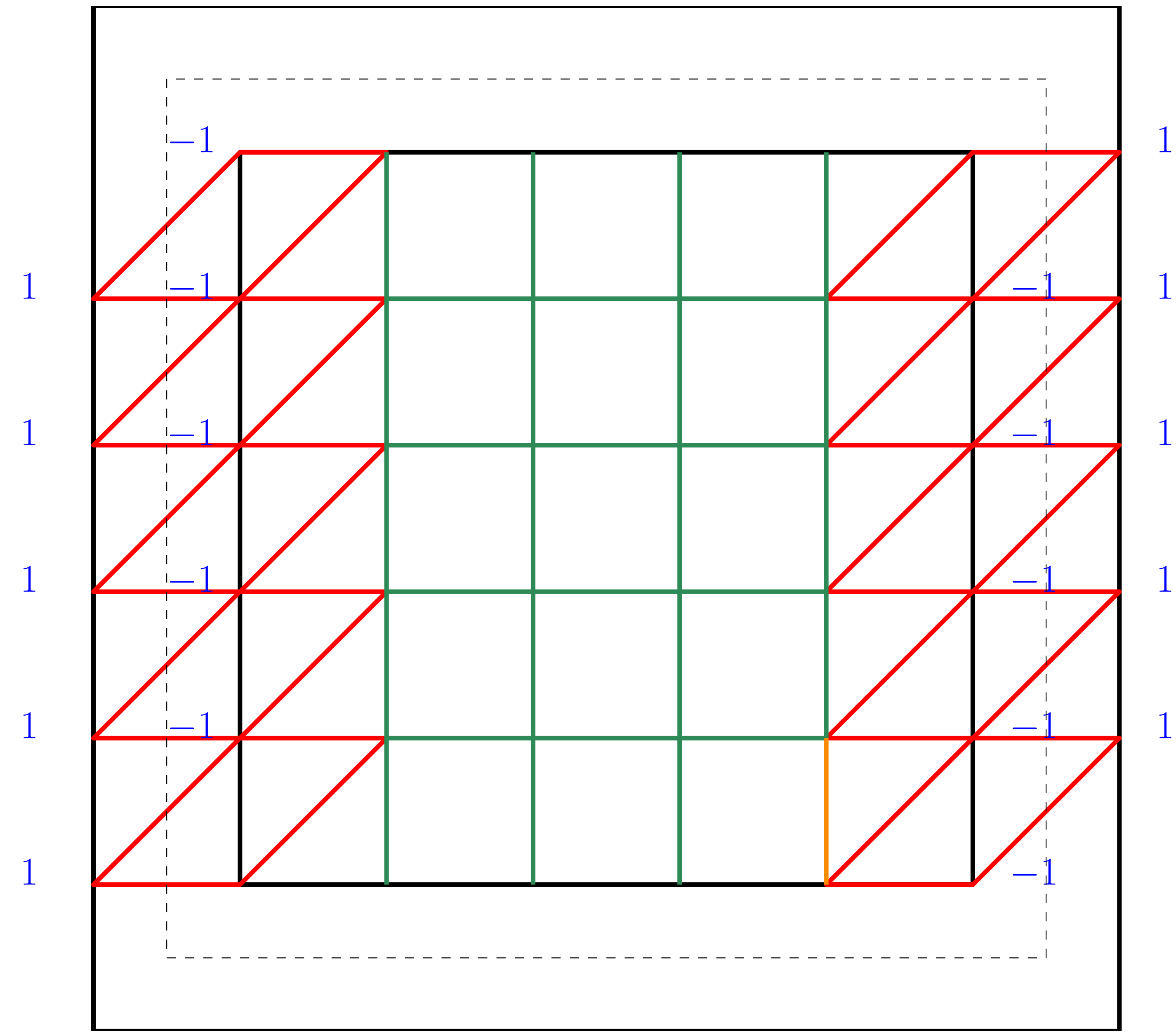
- Fradelizi's Theorem:

The value of a logconcave density on $\mathbb{R}^n$ at its mean is at least $\exp(-n)$ times the density at any other point.



Lower bound

- Assume the hive is C^3 after smoothing. Embed a large volume section of $P_{n_1}(s)$ into the relevant polytope.



Definitions

- $$\mathrm{V}(\lambda) := \exp\left(\int_{T\setminus\{(t,t)|t\in[0,1]\}} \log\left(\frac{\lambda(x)-\lambda(y)}{x-y}\right) dxdy\right).$$
- $$\mathcal{J}(h) := V(\nu')\exp\left(-\int_T \sigma((-1)(\nabla^2 h)_{ac}(x))\mathrm{Leb}_2(dx)\right)$$
- For

$h' \in H(\lambda,\mu;\nu')$,

- Let
- $$\boxed{\mathrm{I}_1(h') := -\log\left(\frac{\mathcal{J}(h')}{V(\lambda)V(\mu)}\right)}.$$

Lower bound in terms of C^2 hives

- Given a C^0 hive $h \in H(\lambda, \mu; \nu)$, let $L_n(h)$ denote the point in $H_n(\lambda_n, \mu_n; \nu_n)$ whose ij^{th} coordinate is given by $(\mathrm{L}_n(h))_{ij} := n^2 h\left(\frac{n+1-i}{n}, \frac{j}{n}\right)$, for all $i, j \in [n+1]$ such that $i + j \leq n + 2.$

- Lemma:

- For any $\epsilon > 0$ and $h_* \in H(\lambda, \mu; \nu)$,

$$\liminf_{n\to\infty} \left(\frac{2}{n^2}\right) \log \mathbb{P}_n \left[h_n \in B_\infty^{\binom{n+1}{2}} \left(L_n(h_*), n^2\epsilon\right)\right] \geq - \inf_{\substack{h' \in B_\infty(h_*, \epsilon) \cap H(\lambda,\mu) \\ h' - \tilde{h} \in H \\ \tilde{h} \in C^2(T)}} I_1(\tilde{h}).$$

Upper bound in terms of C^0 hives

- Lemma:

- For any $\epsilon > 0$ and $h_* \in H(\lambda, \mu; \nu)$,

$$\limsup_{n \to \infty} \left(\frac{2}{n^2}\right) \log \mathbb{P}_n \left[h_n \in B_\infty^{\binom{n+1}{2}} \left(L_n(h_*), n^2 \epsilon\right) \right] \leq - \inf_{h' \in B_\infty(h_*, \epsilon) \cap H(\lambda, \mu)} I_1(h').$$

Equality when λ and μ are C^1

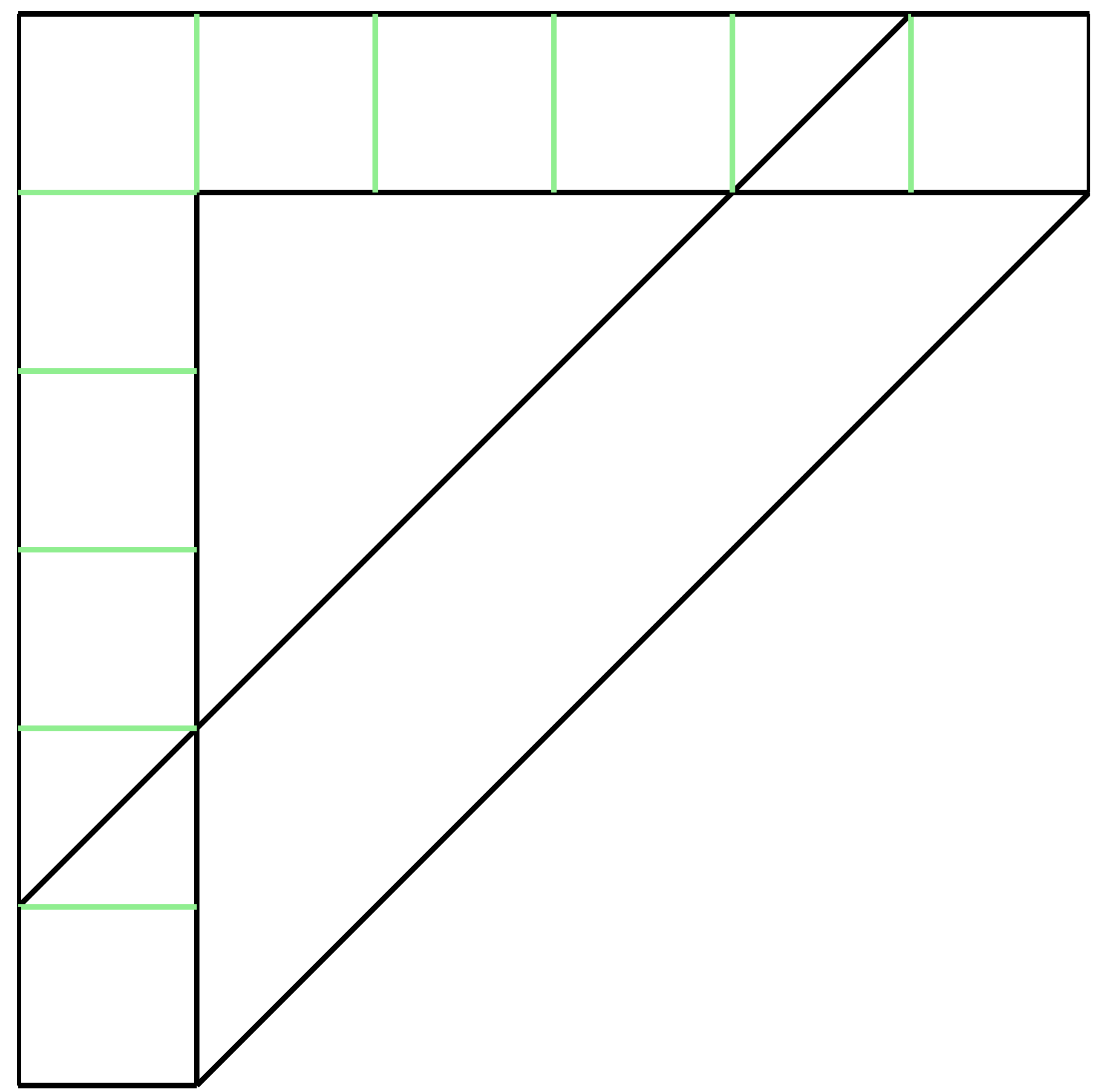
- Lemma:
- Suppose λ and μ are C^1.
- For any $\epsilon > 0$ and $h_* \in H(\lambda, \mu; \nu)$,
- $$\lim_{n\to\infty} \left(\frac{2}{n^2}\right) \log \mathbb{P}_n \left[h_n \in B_\infty^{\binom{n+1}{2}} \left(L_n(h_*), n^2\epsilon \right) \right] = - \inf_{h' \in B_\infty(h_*,\epsilon) \cap H(\lambda,\mu)} I_1(h').$$

Large deviation principle for hives when λ and μ are C^1

- Theorem:
- Suppose λ and μ are C^1.
- Let $a_n = n^2/2$. For each Borel measurable set $E \subset L^\infty(T)$,
- - $\inf_{h \in E^o} I_1(h) \leq \liminf_{n \to \infty} a_n^{-1} \log(\mathbb{P}_n(E)) \leq \limsup_{n \to \infty} a_n^{-1} \log(\mathbb{P}_n(E)) \leq -\inf_{h \in \overline{E}} I_1(h)$

Proof idea

We take a C^0 hive that minimises the rate function and alter it very slightly to obtain a C^2 hive with a very similar rate function.



Large deviation principle for the spectrum of the sum of two independent Haar random matrices.

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- $$I(\gamma) := \log\left(\frac{V(\lambda)V(\mu)}{V(\nu)V(\tau)}\right) + \inf_{h \in H(\lambda,\mu;\nu)} \int_T \sigma((-1)(\nabla^2 h)_{ac})\mathrm{Leb}_2(dx).$$

Large deviation principle for the spectrum of the sum of two independent Haar random matrices.

- Theorem:

- Let $a_n = n^2/2$. For each Borel measurable set $E \subset L^\infty([0,1])$,

$$-\inf_{\gamma \in E^\circ} I(\gamma) \le \liminf_{n\to\infty} a_n^{-1} \log(\mathbb{P}_n(E)) \le \limsup_{n\to\infty} a_n^{-1} \log(\mathbb{P}_n(E)) \le -\inf_{\gamma \in \overline{E}} I(\gamma).$$

Open Problems

$$\mathcal{J}(h) := V(\nu') \exp\left(- \int_T \sigma((-1)(\nabla^2 h)_{ac}(x)) \mathrm{Leb}_2(dx) \right)$$

$$\boxed{\mathrm{I}_1(h') := -\log\left(\frac{\mathcal{J}(h')}{V(\lambda)V(\mu)} \right).}$$

- Lemma:
- For any $\epsilon > 0$ and $h_* \in H(\lambda, \mu; \nu)$,

$$\limsup_{n\to\infty} \left(\tfrac{2}{n^2}\right) \log \mathbb{P}_n \left[h_n \in B_\infty^{\binom{n+1}{2}} \left(L_n(h_*), n^2\epsilon \right) \right] \leq - \inf_{h' \in B_\infty(h_*,\epsilon) \cap H(\lambda,\mu)} I_1(h').$$

Question: Can the above inequality be replaced by an equality?

Open problem: Exact computation of surface tension

- Ultimately, we would like a closed form expression for $\mathbf{f}(s)$ using which we would like to write down a PDE for the scaling limit of a random augmented hive with given boundary conditions, along the lines of Cohn-Kenyon-Propp, “A variational principle for domino tilings.”

- Such a result would shed light on tilings, random matrices, and asymptotic representation theory.

Thank you for your attention!