Crossing Probabilities in 2D Critical Lattice Models

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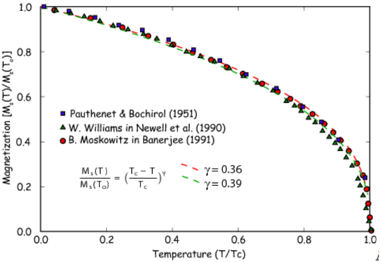
2020.8.26

Ising Model

Pierre and Marie Skłodowska-Curie, 1895

Curie temperature [Pierre Curie, 1895]

Ferromagnet exhibits a phase transition by losing its magnetization when heated above a critical temperature.



Ising Model

Ising Model [Lenz 1920]

A model for ferromagnet, to understand the phase transition.

- $G = (V, E)$ a finite graph
- $\sigma \in \{\ominus, \oplus\}^V$
- $H(\sigma) = -\sum_{x\sim y} \sigma_x \sigma_y$

Ising model is the probability measure of inverse temperature $\beta > 0$:

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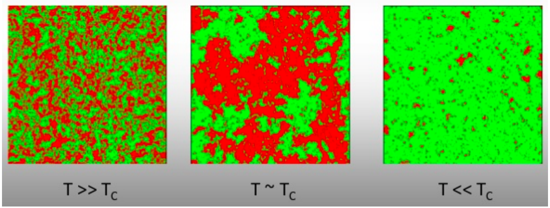

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- $\beta \approx \beta_c$: critical
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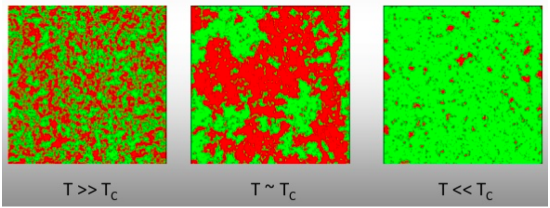

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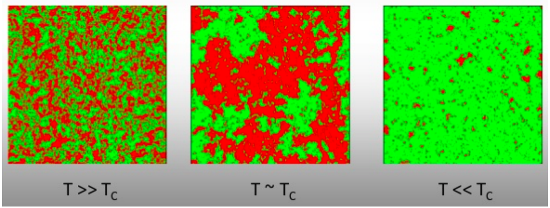

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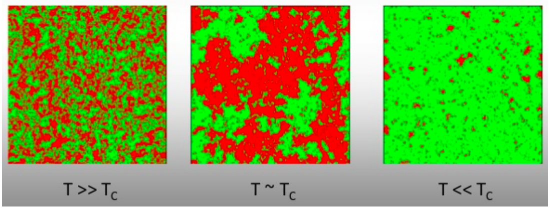

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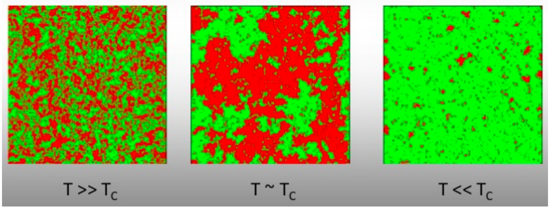

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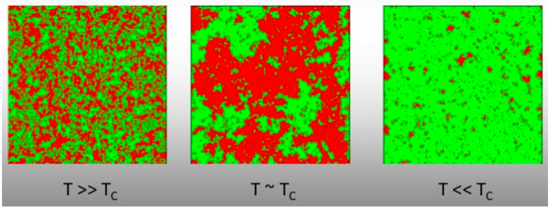

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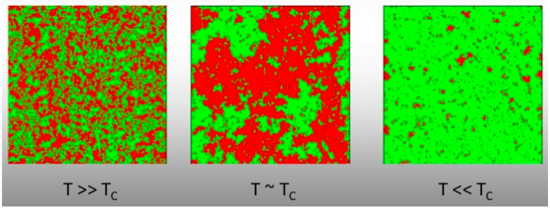

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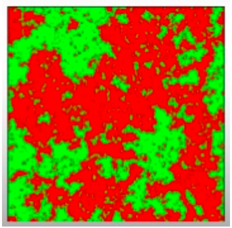
Correlation function

$\mu[\sigma_{z_1}\cdots\sigma_{z_n}] \to \phi(z_1,\ldots,z_n)$.

Schramm Loewner Evolution (SLE)

The law of interfaces is CI.

Conformal Invariance of Interfaces



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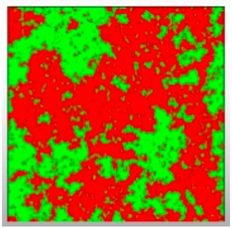

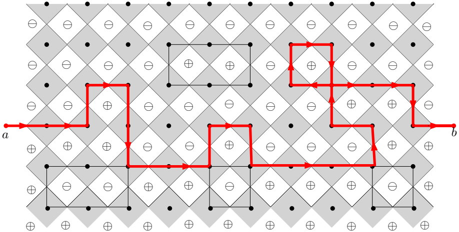

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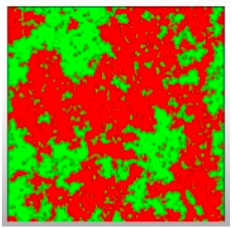

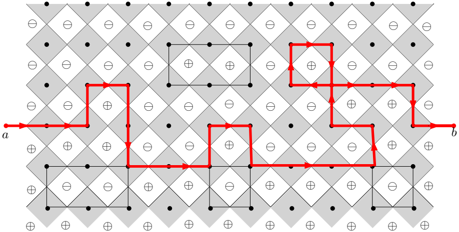

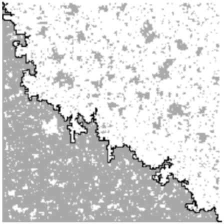

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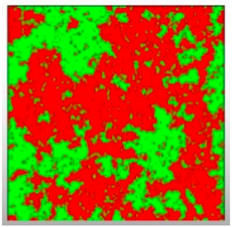
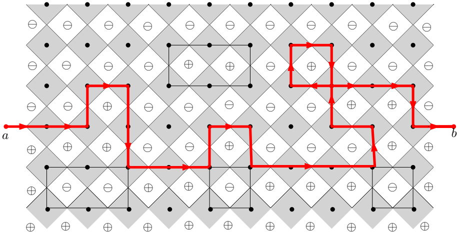
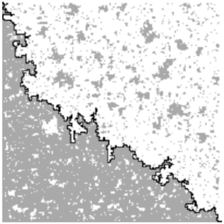




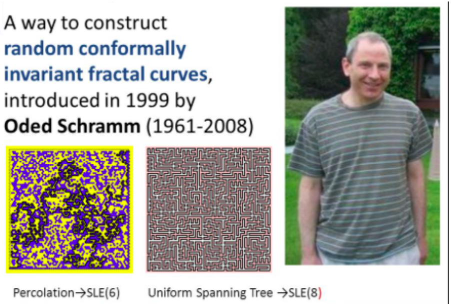


SLE[O. Schramm 1999]

A random fractal curve :

- conformal invariance
- domain Markov property

Classification : SLE(κ), $\kappa > 0$.

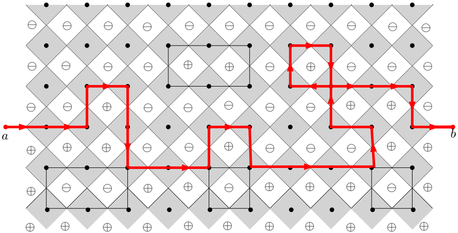


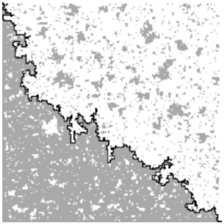
Conformal Invariance in Ising Model

Stanislav Smirnov

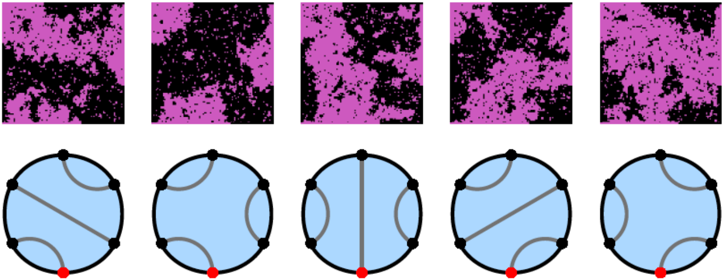
[Chelkak-Smirnov, Invent.Math. '10]

The interface in critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary conditions converges weakly to SLE(3).





Crossing Probabilities of Ising Interfaces



Theorem [Peltola-W. '18]

The connection of Ising interfaces forms a planar link pattern $\mathcal{A}_\delta$.

$$\lim_{\delta\to 0} \mathbb{P}[\mathcal{A}_\delta = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{Ising}(\Omega; x_1, \ldots, x_{2N})},$$

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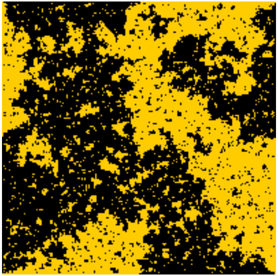
- Conjectured in [Bauer-Bernard-Kytölä, JSP '05].
- Partially solved in [Izyurov, CMP '15].
- Might be related to correlation functions in CFT.

Pure Partition Functions



Pure Partition Functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j-x_i}\partial_j - \frac{(6-\kappa)/\kappa}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1,\ldots,x_{2N}) = 0.$

COV : $\mathcal{Z}(x_1,\ldots,x_{2N}) = \prod_{i=1}^{2N}\varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1),\ldots,\varphi(x_{2N})).$

ASY : $\lim_{x_j,x_{j+1}\to\xi}\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{(x_{j+1}-x_j)^{-2h}} = \mathcal{Z}_{\hat{\alpha}}(x_1,\ldots,x_{j-1},x_{j+2},\ldots,x_{2N})$

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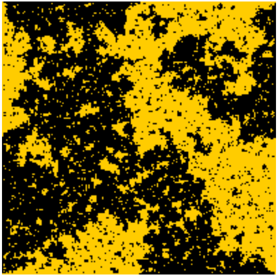

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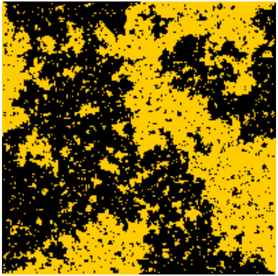
Questions

Existence ? Uniqueness ? Explicit formula ?

Pure Partition Functions



Uniqueness [Flores-Kleban, CMP '15]

If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

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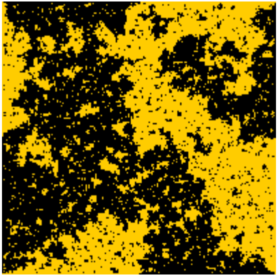

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Existence

- $\kappa \in (0, 8) \setminus \mathbb{Q}$ [Kytöla-Peltola, CMP'16]
- $\kappa \in (0, 4]$ [Peltola-W. CMP'19]
- $\kappa \in (0, 6]$ [W. CMP'20]

- Coulumb gas techniques
- Global multiple SLEs
- Hypergeometric SLE

Pure Partition Functions

Theorem [W. CMP'20]

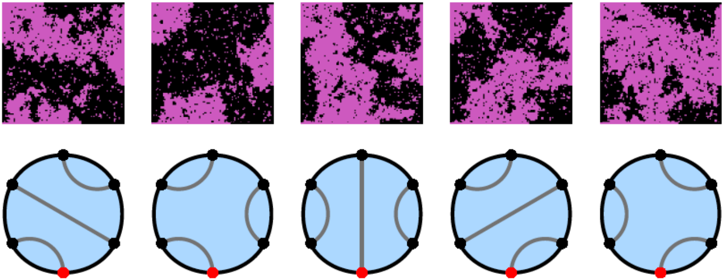
Let $\kappa \in (0, 6]$. There exists a unique collection $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ of smooth functions satisfying the normalization $\mathcal{Z}_\emptyset = 1$ and

PDE, COV, ASY, POS and PLB

the power law bound : for all $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$,

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \le \prod_{j=1}^{N} |x_{b_j} - x_{a_j}|^{-2h}.$$

Crossing Probabilities of Ising Interfaces



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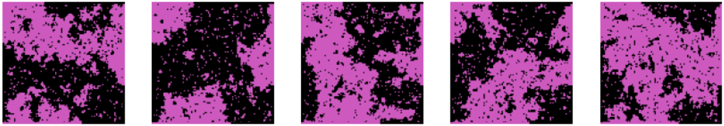
where $\{\mathcal{Z}_\alpha\}$ is the pure partition functions for multiple SLE_3.

Connection Probabilities

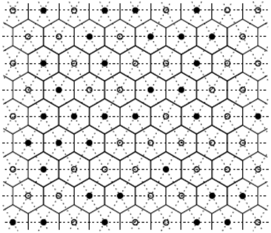


Courtesy to E. Peltola

$$\mathbb{P}[\mathcal{A}=\alpha]=\frac{\mathcal{Z}_\alpha(\Omega;x_1,\ldots,x_{2N})}{\mathcal{Z}^{(N)}(\Omega;x_1,\ldots,x_{2N})},\qquad \mathcal{Z}^{(N)}=\sum_{\alpha\in\mathrm{LP}_N}\mathcal{Z}_\alpha.$$

- LERWs in UST : $\kappa=2$. [Karrila-Kytölä-Peltola, CMP'19]
- Multiple Ising interfaces : $\kappa=3$. [Peltola-W. '18]
- Multiple level lines of GFF : $\kappa=4$. [Peltola-W. CMP'19]
- Multiple percolation interfaces : $\kappa=6$. [Peltola-W. '20+]

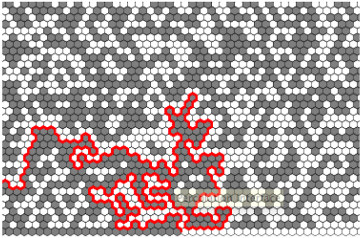
Percolation



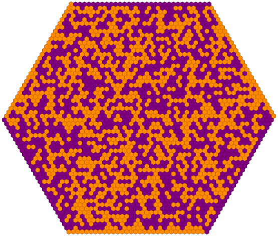
Site percolation on triangular lattice : each site is chosen independently to be black or white with equal probability $1/2$.

Thm. [Smirnov, '01]

The interface of critical site percolation on triangular lattice converges weakly to SLE(6).



Crossing Probabilities in Critical Percolation



- When $N = 2$: Cardy's formula [Smirnov '01]

Theorem [Peltola-W. '20+]

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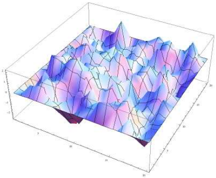
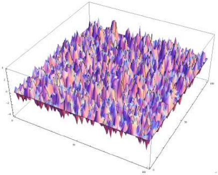
where $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is the pure partition functions for multiple SLE_6.

DGFF (Discrete Gaussian Free Field)





DGFF with mean zero : a measure h on functions $\rho : D \to \mathbb{R}$ and $\rho = 0$ on ∂D with density

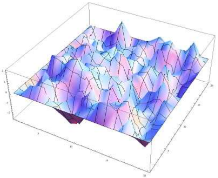
$$\frac{1}{\mathcal{Z}} \exp(-\frac{1}{2} \sum_{x \sim y} (\rho(x) - \rho(y))^2).$$

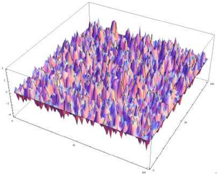
- For each vertex x, $h(x)$ Gaussian r.v.
- Covariance : Green's function for SRW
- Mean value : zero.

DGFF with mean h_∂ : DGFF with mean zero plus a harmonic function h_∂.

- For each vertex x, $h(x)$ Gaussian r.v.
- Covariance : Green's function for SRW
- Mean value : $h_\partial(x)$

GFF (Continuum Gaussian Free Field)





DGFF $\to$ GFF h

- (h, ρ) Gaussian r.v.
- Covariance :

$$cov((h, \rho_1), (h, \rho_2)) = \iint dxdy G_D(x, y)\rho_1(x)\rho_2(y).$$

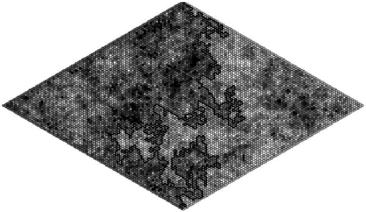
- Mean value : $\mathbb{E}((h, \rho)) = (h_\partial, \rho)$.

GFF is :

- height functions of dimer model, domino tiling...
- starting point for quantum field theory...

Conformal Invariance
Domain Markov Property

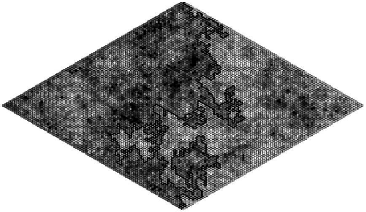
Level lines of DGFF



[Schramm-Sheffield, ACTA'09]

- DGFF with boundary value $+\lambda$ on $\mathbb{R}_+$ and $-\lambda$ on $\mathbb{R}_-$
- γ^δ : the level line of DGFF with height zero
- γ^δ converges in distribution to SLE_4 as δ goes to zero

Level lines of DGFF



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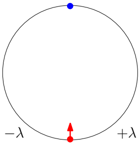
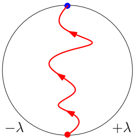
- γ^δ converges in distribution to SLE_4 as δ goes to zero
- $\rightarrow \mathrm{SLE}_4$ is the "level line" of GFF with height zero

Interacting level lines





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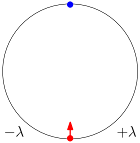
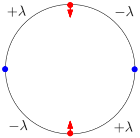
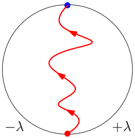
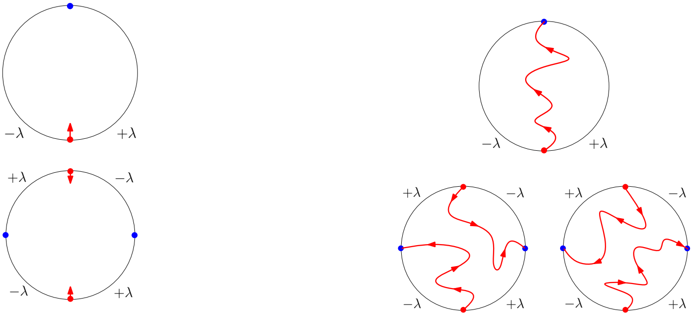






Interacting level lines



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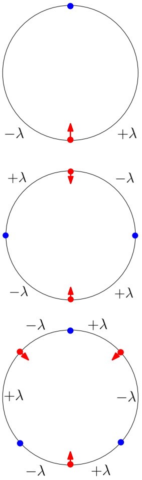
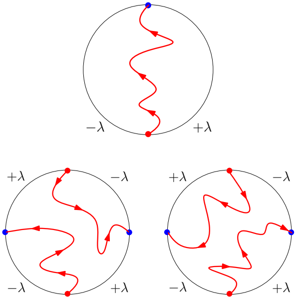
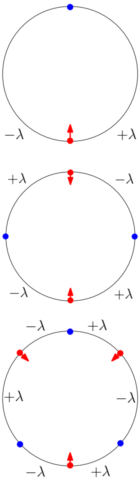
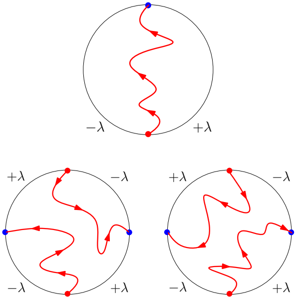
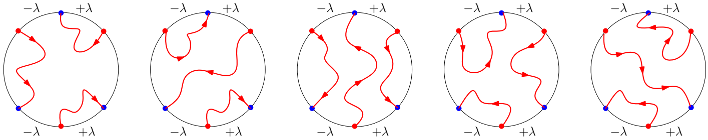




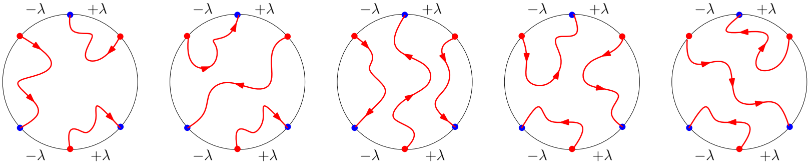
Interacting level lines







Connection Probabilities



- $2N$ marked points
- N level lines
- LP_N : planar link patterns

Theorem [Peltola-W. CMP'19]

The connection of level lines of GFF forms a planar link pattern $\mathcal{A}$: for any $\alpha \in \mathrm{LP}_N$,

$$\mathbb{P}[\mathcal{A} = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\mathrm{GFF}}(\Omega; x_1, \ldots, x_{2N})}, \quad \mathcal{Z}_{\mathrm{GFF}} = \sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha,$$

where $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is the pure partition functions for multiple SLE_4.

Connection Probabilities

Theorem [Kenyon-Wilson, '11]

The collection $\{\mathcal{Z}_\alpha\}$ are explicit when $\kappa = 4$:

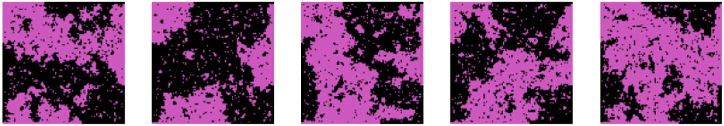
$$\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) = \sum_{\beta \in \mathrm{LP}_N} \mathcal{M}^{-1}_{\alpha,\beta} \mathcal{U}_\beta(x_1, \ldots, x_{2N}),$$

where

$$\mathcal{U}_\beta(x_1, \ldots, x_{2N}) := \prod_{1 \le i < j \le 2N} (x_j - x_i)^{\frac{1}{2}\vartheta_\beta(i,j)},$$

$$\mathcal{M}_{\alpha,\beta} = 1\{\alpha \leftarrow \beta\}, \qquad \mathcal{M}^{-1}_{\alpha,\beta} = (-1)^{|\alpha/\beta|} \# \mathcal{C}(\alpha/\beta) 1\{\alpha \preceq \beta\}.$$

Connection Probabilities



Courtesy to E. Peltola

$$\mathbb{P}[\mathcal{A} = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}^{(N)}(\Omega; x_1, \ldots, x_{2N})}, \quad \mathcal{Z}^{(N)} = \sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha.$$

- LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP'19]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. '18]
- Multiple level lines of GFF : $\kappa = 4$. [Peltola-W. CMP'19]
- Multiple percolation interfaces : $\kappa = 6$. [Peltola-W. '20+]

References Thanks !

- (Peltola-W. CMP'19) E. Peltola, H. Wu.
  Global and Local Multiple SLEs for $\kappa \leq 4$ and Connection Probabilities of Level Lines of GFF.
  Comm. Math. Phys. 366(2) :469-536, 2019
- (W. CMP'20) H. Wu.
  Hypergeometric SLE : Conformal Markov Characterization and Applications. *Comm. Math. Phys. 374(2) : 433-484, 2020*
- (Beffara-Peltola-W. '18) V. Beffara, E. Peltola, H. Wu.
  On the Uniqueness of Global Multiple SLEs.
  arXiv:1801.07699.
- (Peltola-W. '18) E. Peltola, H. Wu.
  Crossing Probabilities of Multiple Ising Interfaces.
  arXiv:1808.09438.

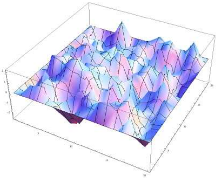
Crossing Probabilities for Level Lines of Gaussian Free Field

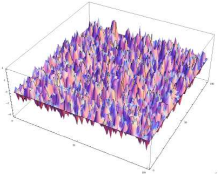
Hao Wu

Yau Mathematical Sciences Center, Tsinghua University, China

2020.8.26

DGFF (Discrete Gaussian Free Field)





DGFF with mean zero : a measure h on functions $\rho : D \to \mathbb{R}$ and $\rho = 0$ on ∂D with density

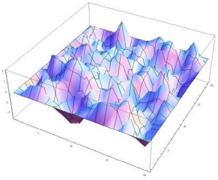
$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (\rho(x) - \rho(y))^2\right).$$

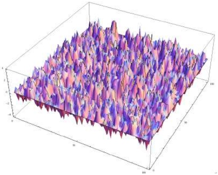
- For each vertex x, $h(x)$ Gaussian r.v.
- Covariance : Green's function for SRW
- Mean value : zero.

DGFF with mean h_∂ : DGFF with mean zero plus a harmonic function h_∂.

- For each vertex x, $h(x)$ Gaussian r.v.
- Covariance : Green's function for SRW
- Mean value : $h_\partial(x)$

GFF (Continuum Gaussian Free Field)





DGFF $\to$ GFF h

- (h, ρ) Gaussian r.v.
- Covariance :

$$cov((h,\rho_1),(h,\rho_2)) = \iint dxdy G_D(x,y)\rho_1(x)\rho_2(y).$$

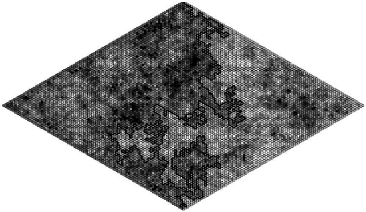
- Mean value : $\mathbb{E}((h,\rho)) = (h_\partial, \rho)$.

GFF is :

- height functions of dimer model, domino tiling...
- starting point for quantum field theory...

Conformal Invariance
Domain Markov Property

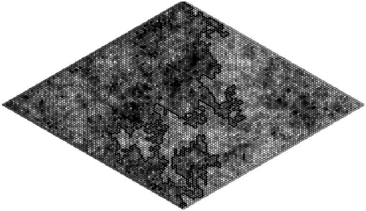
Level lines of DGFF



[Schramm-Sheffield, ACTA'09]

- DGFF with boundary value $+\lambda$ on $\mathbb{R}_+$ and $-\lambda$ on $\mathbb{R}_-$
- γ^δ : the level line of DGFF with height zero
- γ^δ converges in distribution to SLE_4 as δ goes to zero

Level lines of DGFF



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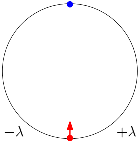
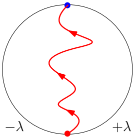
- γ^δ : the level line of DGFF with height zero
- γ^δ converges in distribution to SLE_4 as δ goes to zero
- $\rightarrow$ SLE_4 is the "level line" of GFF with height zero

Interacting level lines





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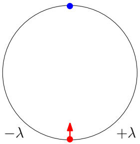

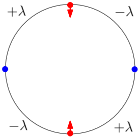

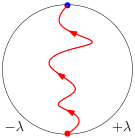

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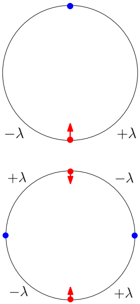
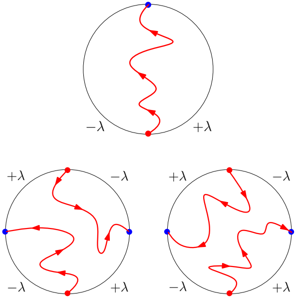
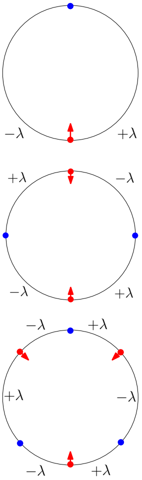
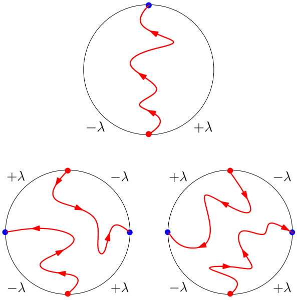




Interacting level lines





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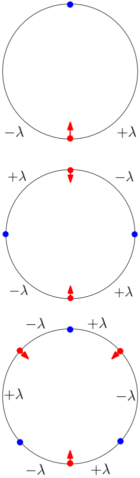
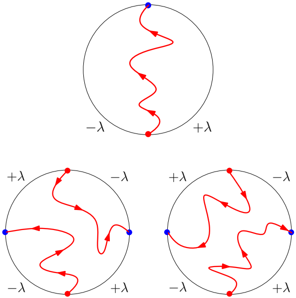
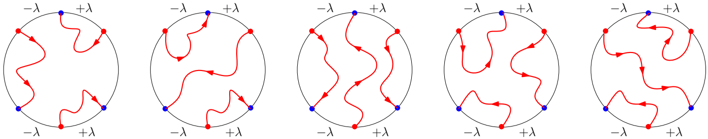
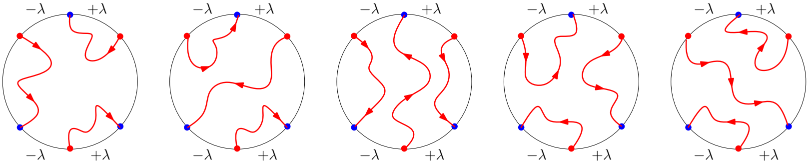






Connection Probabilities



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where $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is the pure partition functions for multiple SLE_4.

Law of the level lines

- Driving function $(W_t, t \geq 0)$: a real-valued continuous function.
- For $z \in \mathbb{H}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

- The swallowing time of z :

$$T_z = \sup \left\{ t \geq 0 : \inf_{s \in [0,t]} |g_s(z) - W_s| > 0 \right\}.$$

- Loewner chain : $(K_t, t \geq 0)$: K_t is the closure of $\{z \in \mathbb{H} : T_z \leq t\}$.

Consequence : g_t is the unique conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ with the normalization : $\lim_{z \to \infty} |g_t(z) - z| = 0$.

Law of the level lines

- Fix $x_1 < \cdots < x_{2N}$ and consider GFF in $\mathbb{H}$ with the following boundary data :

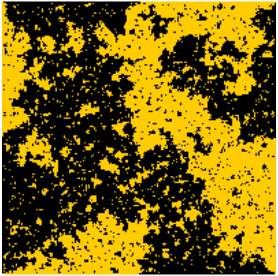
$$\lambda \text{ on } (x_{2j-1}, x_{2j}), \quad \text{and} \quad -\lambda \text{ on } (x_{2j}, x_{2j+1}).$$

- Let η be the level line starting from x_1.

The law of η is $\mathrm{SLE}_4(-2, +2, -2, +2, \ldots, -2)$. Its driving function satisfies the following SDE :

$$dW_t = 2dB_t + \sum_{i=2}^{2N} \frac{2(-1)^i dt}{g_t(x_i) - W_t}.$$

Pure Partition Functions



Pure Partition Functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{(6-\kappa)/\kappa}{(x_j - x_i)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0.$

COV : $\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \ldots, \varphi(x_{2N})).$

ASY : $\lim_{x_j, x_{j+1}\to\xi} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = \mathcal{Z}_{\hat{\alpha}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N})$

Pure Partition Functions

Theorem [W. CMP'20]

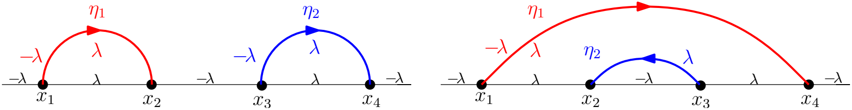
Let $\kappa \in (0, 6]$. There exists a unique collection $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ of smooth functions satisfying the normalization $\mathcal{Z}_\emptyset = 1$ and

PDE, COV, ASY, POS and PLB

the power law bound : for all $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$,

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \leq \prod_{j=1}^{N} |x_{b_j} - x_{a_j}|^{-2h}.$$

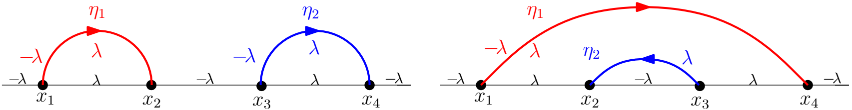
Martingale observable



Denote

$$P_\alpha(\Omega; x_1, \ldots, x_{2N}) = \mathbb{P}[\text{connectivity pattern} = \alpha].$$

Martingale observable



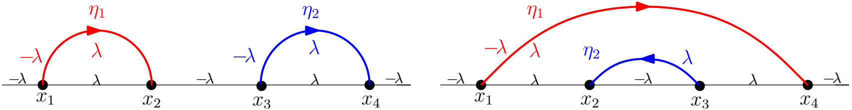
Denote

$$P_\alpha(\Omega; x_1, \ldots, x_{2N}) = \mathbb{P}[\text{connectivity pattern} = \alpha].$$

Define

$$M_t(\mathcal{Z}_\alpha) := \frac{\mathcal{Z}_\alpha(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}{\mathcal{Z}_{\mathrm{GFF}}(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}.$$

Martingale observable



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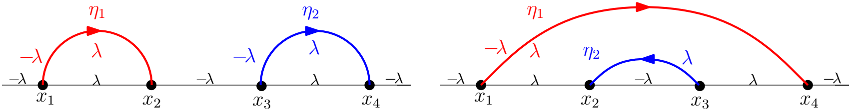
- PDE$\rightarrow$: $M_t(\mathcal{Z}_\alpha)$ is a local martingale.

Martingale observable



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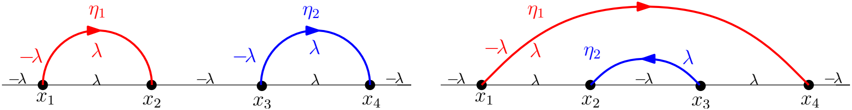
- PDE$\rightarrow$: $M_t(\mathcal{Z}_\alpha)$ is a local martingale.
- POS$\rightarrow$: $0 \le M_t(\mathcal{Z}_\alpha) \le 1$. Thus $M_0(\mathcal{Z}_\alpha) = \mathbb{E}[M_T(\mathcal{Z}_\alpha)]$.

Martingale observable



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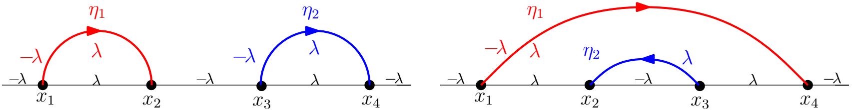
- ASY+PLB → : $M_T(\mathcal{Z}_\alpha) = P_{\alpha/\{1,2\}}(\mathbb{H} \setminus \eta; x_3, \ldots, x_{2N}) 1_{\{\eta(T)=x_2\}}$.

Martingale observable



Denote

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$$M_t(\mathcal{Z}_\alpha) := \frac{\mathcal{Z}_\alpha(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}{\mathcal{Z}_{\mathrm{GFF}}(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}.$$

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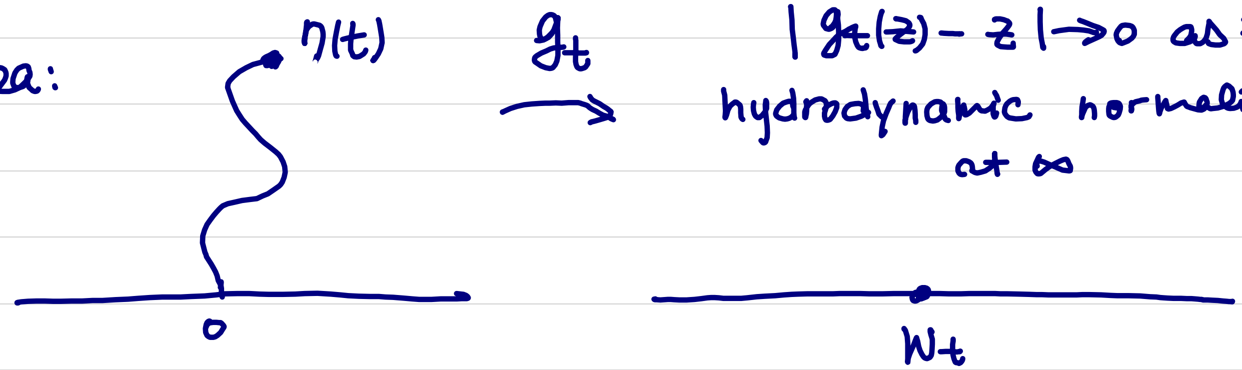
$$\begin{aligned} P_\alpha(\mathbb{H}; x_1, \ldots, x_{2N}) &= \mathbb{E}[P_{\alpha/\{1,2\}}(\mathbb{H} \setminus \eta; x_3, \ldots, x_{2N}) 1_{\{\eta(T)=x_2\}}] \\ &= \mathbb{E}[M_T(\mathcal{Z}_\alpha)] = M_0(\mathcal{Z}_\alpha). \end{aligned}$$

Loewner chain:

idea:



$|g_t(z) - z| \to 0$ as $z \to \infty$
hydrodynamic normalization at ∞

$$\underline{(\eta(t), t \geq 0)} \leftrightarrow \underline{(W_t, t \geq 0)}.$$

time parameter: $g_t(z) = z + \frac{2t}{z} + o\left(\frac{1}{z}\right)$ as $z \to \infty$.

t : half-plane capacity.

Formal:

$(W_t, t \geq 0)$: real-valued continuous.

For $z \in \mathbb{H}$, $\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$, $g_0(z) = z$.

swallowing time $T_z = \sup\{t : \inf_{s \in [0,t]} |g_s(z) - W_s| > 0\}$.

K_t = closure of set $\{z : T_z \leq t\}$.

g_t is the unique conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$.

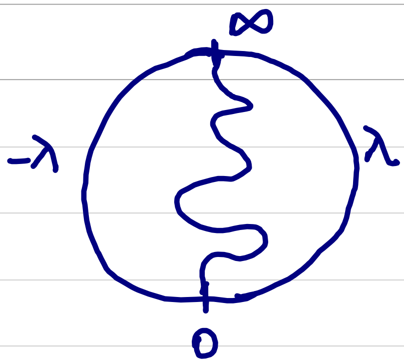
$$|g_t(z) - z| \to 0 \quad \text{as } z \to \infty.$$

$W_t \equiv 0$. $\quad g_t(z) = \sqrt{z^2 + 4t}$

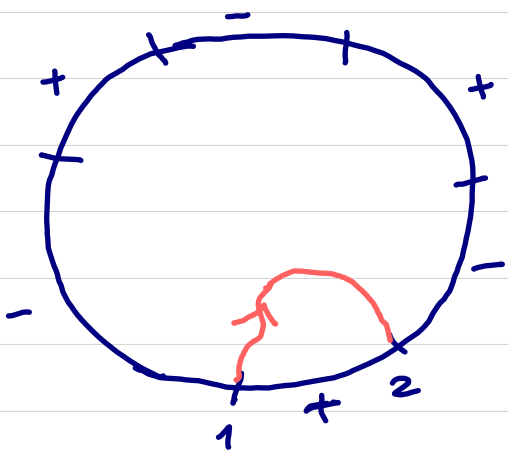
SLE_κ : $\quad W_t = \sqrt{\kappa} \cdot B_t$, $\quad (B_t, t \geq 0)$ 1-d BM.

$\kappa \in (0, 4]$. continuous simple curve.

We focus on $\kappa = 4$. $\quad SLE_4 \sim$ level line of GFF.



$W_t = 2B_t$.



η: $SLE_4(-2, +2, \cdots -2)$

SDE

$$dW_t = 2dB_t + \sum_{i=2}^{2N} \frac{2(-1)^i \, dt}{g_t(x_i) - W_t}.$$

variant of Dyson BM.

$\{ Z_\alpha , \alpha \in LP_N \}$ Z_α : $x_1 < \cdots < x_{2N}$.

PDE. COV.

ASY. $\alpha \in LP_N$. x_j, x_{j+1}.

$(j, j+1) \in \alpha$.

$$\frac{Z_\alpha(x_1, \cdots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} \to Z_{\hat\alpha}(x_1, \cdots, x_3, x_{j+2}, \cdots, x_{2N})$$

$\hat\alpha = \alpha / \{j, j+1\}$.

$(j, j+1) \in \alpha$.

$\to 0$.

$$0 < Z_\alpha(x_1, \cdots, x_{2N}) \leq \prod_{j=1}^{N} |x_{b_j} - x_{a_j}|^{-2h}$$

POS PLB $\alpha = \{\{a_1, b_1\}, \cdots, \{a_N, b_N\}\}$.

Idea: $P_\alpha(\Omega; x_1, \dots, x_{2N}) = \mathbb{P}[\text{connectivity} = \alpha]$.

η: level line starting from x_1.

$M_t^\alpha := \mathbb{P}[\underline{\text{connectivity} = \alpha} \mid \eta[0,t]]$

mart.

$M_0^\alpha = \mathbb{E}[M_T^\alpha]$

$= \mathbb{E}[P_\alpha] = P_\alpha$

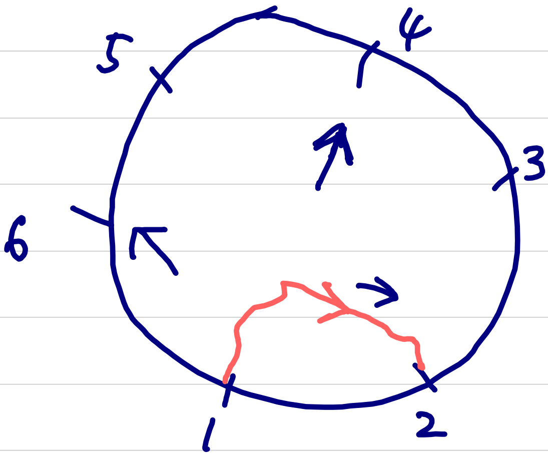
Formal proof:

$$M_t(Z_\alpha) := \frac{Z_\alpha(W_t, g_t(x_2), \dots, g_t(x_{2N}))}{Z_{GFF}(W_t, g_t(x_2), \dots, g_t(x_{2N}))}.$$

PDE → : $M_t(Z_\alpha)$ local mart.

POS → : $0 \le M_t(Z_\alpha) \le 1$. $M_0(Z_\alpha) = \mathbb{E}[M_T(Z_\alpha)]$.

$$M_t(Z_\alpha) = \frac{Z_\alpha(W_t, g_t(x_2), \cdots, g_t(x_{2N}))}{Z_{GFF}(W_t, g_t(x_2), \cdots, g_t(x_{2N}))}$$



$(1,2) \in \alpha$.

case 1. $\eta(t) \to x_2$: $W_t, g_t(x_2) \to \xi$.

$$M_t(Z_\alpha) \to \frac{Z_{\hat\alpha}(g_T(x_3), \cdots, g_T(x_{2N}))}{Z_{GFF}(g_T(x_3), \cdots, g_T(x_{2N}))}.$$

ASY.

$\hat\alpha = \alpha / \{1,2\}$.

case 2. $\eta(t) \to x_{2j}$, $j > 1$.

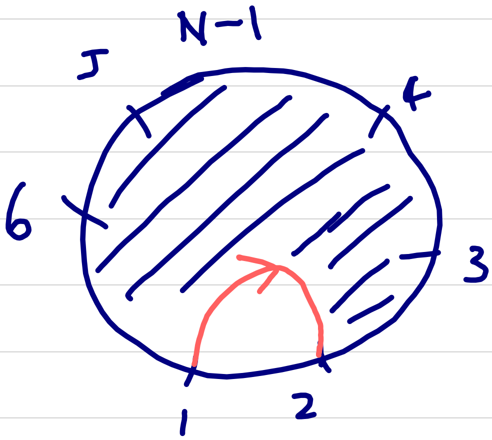
PLB.

$M_t(Z_\alpha) \longrightarrow 0$.

summary. $M_T(Z_\alpha) = \frac{Z_{\hat\alpha}}{Z_{GFF}} \mathbb{1}\{\eta(T) = x_2\}$.

$$M_0(Z_\alpha) = \mathbb{E}[\, M_T(Z_\alpha) \,]$$

$$= \mathbb{E}\left[\frac{Z_{\hat\alpha}}{Z_{GFF}} \mathbb{1}\{\eta(T) = x_2\} \right]$$



$$= \mathbb{E}[\, P_{\hat\alpha} \, \mathbb{1}\{\eta(T) = x_2\} \,]$$

induction hypothesis.

$$= P_\alpha .$$

$$P_\alpha = M_0(Z_\alpha) = Z_\alpha / Z_{GFF} .$$

Summary:

PDE.
POS.
ASY. PLB.

□.