Excursion probabilities for Gaussian processes and fields

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- Often: $\mathbf{X}$ stationary and/or Gaussian.
- $u \in \mathbb{R}$: a level;

  the excursion set of $\mathbf{X}$ above the level u:

$$A_u = \left\{\mathbf{t} \in \mathbb{R}^d : X(\mathbf{t}) \geq u\right\}.$$

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- The supremum functional: for a compact K:

$$A_u \cap K \neq \emptyset \ \text{ if and only if } \ \sup_{\mathbf{t} \in K} X(\mathbf{t}) \geq u \,.$$

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   what is the probability that they belong to the same connected component of the excursion set?

2. given that a sphere belongs to the excursion set A_u,
   what is the probability that anywhere inside the ball the field is below ru, $0 < r \leq 1$?

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- $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^d$, $\mathbf{a} \neq \mathbf{b}$. A path between $\mathbf{a}$ and $\mathbf{b}$: a continuous map $\xi : [0,1] \to \mathbb{R}^d$ with $\xi(0) = \mathbf{a}$, $\xi(1) = \mathbf{b}$.

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- $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^d$, $\mathbf{a} \neq \mathbf{b}$. A path between $\mathbf{a}$ and $\mathbf{b}$: a continuous map $\xi : [0,1] \to \mathbb{R}^d$ with $\xi(0) = \mathbf{a}$, $\xi(1) = \mathbf{b}$.
- $\mathcal{P}(\mathbf{a},\mathbf{b})$: the collection of such paths. Estimate

$$P\left(\exists\, \xi \in \mathcal{P}(\mathbf{a},\mathbf{b}) : \, X(\xi(v)) > u, \; 0 \leq v \leq 1 \;\middle|\; X(\mathbf{a}) > u, \, X(\mathbf{b}) > u\right).$$

The non-trivial part of the problem: estimate the probability

$$\Psi_{\mathbf{a},\mathbf{b}}(u) := P\left(\exists\ \xi \in \mathcal{P}(\mathbf{a},\mathbf{b}) :\ X(\xi(v)) > u,\ 0 \le v \le 1\right).$$

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If the domain of a random field is $T \subset \mathbb{R}^d$, and $\mathbf{a}$, $\mathbf{b}$ in T:

$$\Psi_{\mathbf{a},\mathbf{b}}(u) = P\left(\exists \xi \in \mathcal{P}(\mathbf{a},\mathbf{b}):\ \xi(v) \in T \text{ and } X(\xi(v)) > u,\ 0 \le v \le 1\right).$$

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An open set:

$$A \equiv A_{\mathbf{a},\mathbf{b}} := \left\{ \boldsymbol{\omega} \in C_0(\mathbb{R}^d) : \ \exists\, \xi \in \mathcal{P}(\mathbf{a},\mathbf{b}),\ \omega(\xi(v)) > 1,\ 0 \le v \le 1 \right\}$$

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- Use the Gaussian large deviations theory: Deutschel and Stroock (1989).

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1. Start with the space of finite linear combinations $\sum_{j=1}^{k} a_j X(\mathbf{t}_j)$ $a_j \in \mathbb{R}$, $\mathbf{t}_j \in \mathbb{R}^d$ for $j = 1, \ldots, k$, $k = 1, 2, \ldots$.

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3. Identify $\mathcal{L}$ with $\mathcal{H}$ via the injection $\mathcal{L} \to C(\mathbb{R}^d)$:

$$H \to w_H = \Big(E\big(X(\mathbf{t})H\big),\ \mathbf{t} \in \mathbb{R}^d\Big)$$

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4. The resulting norm:

$$\|w_H\|_{\mathcal{H}}^2 = E(H^2)\,.$$

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The injection $L^2(F_{\mathbf{X}}) \to C(\mathbb{R}^d)$:

$$h \to S(h) = \left(\int_{\mathbb{R}^d} e^{i(\mathbf{t},\mathbf{x})}\, \bar{h}(\mathbf{x})\, F_{\mathbf{X}}(d\mathbf{x}),\ \mathbf{t} \in \mathbb{R}^d \right) .$$

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$$\|S(h)\|_{\mathcal{H}}^2 = \|h\|_{L^2(F_{\mathbf{X}})}^2 \ = \ \int_{\mathbb{R}^d} \|h(x)\|^2\, F_{\mathbf{X}}(d\mathbf{x})\,.$$

Theorem 1

Let $\mathbf{X} = (X(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d)$ be a continuous stationary Gaussian random field, with covariance function satisfying

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with

$$\mathcal{C}_{\mathbf{X}}(\mathbf{a}) := \inf \Bigg\{ \int_{\mathbb{R}^d} \|h(\mathbf{x})\|^2 \, F_{\mathbf{X}}(d\mathbf{x}) : \text{ for some } \xi \in \mathcal{P}(\mathbf{0}, \mathbf{a}) \\ \int_{\mathbb{R}^d} e^{i(\xi(v),\mathbf{x})} \, \overline{h}(\mathbf{x}) \, F_{\mathbf{X}}(d\mathbf{x}) \geq 1,\ 0 \leq v \leq 1 \Bigg\}\,.$$

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Theorem 2 For a continuous stationary Gaussian random field $\mathbf{X}$,

$$\mathcal{C}_{\mathbf{X}}(\mathbf{a}) = \left[\sup_{\xi \in \mathcal{P}(\mathbf{0},\mathbf{a})} \min_{\mu \in M_1^+([0,1])} \int_0^1 \int_0^1 R_{\mathbf{X}}\big(\xi(u) - \xi(v)\big)\, \mu(du)\, \mu(dv) \right]^{-1}.$$

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An optimal path is a path of *maximal R_X capacity*.

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The set $\mathcal{W}_\xi \subseteq M_1^+([0,1])$ of optimal measures: a weakly compact convex subset of $M_1^+([0,1])$.

Suppose the primary feasible set in

$$\inf\Big\{E(H^2):\ H\in\mathcal{L},\ E\big[X(\xi(v))H\big]\geq 1,\, 0\leq v\leq 1\Big\}$$

is non-empty.

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Then for every $\mu \in \mathcal{W}_\xi$:

$$\mu\left(\left\{0 \leq v \leq 1:\ E\big[X(\xi(v))H_\xi\big] > 1\right\}\right) = 0$$

for the unique primal optimal solution $H_\xi \in \mathcal{L}$.

If the primal feasible set is non-empty then, for every $\varepsilon > 0$:

$$P\left(\sup_{0\le v\le 1}\left|\frac{1}{u}X(\xi(v)) - x_\xi(v)\right| \ge \varepsilon \middle| X(\xi(v)) > u,\, 0\le v\le 1\right) \to 0$$

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Here

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(i) For every $\mu \in \mathcal{W}_\xi$:

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with probability 1.

(ii) A probability measure $\mu \in M_1^+([0,1])$ is a measure of minimal energy if and only if

$$\min_{0 \le v \le 1} \int_0^1 R_{\mathbf{X}}\big(\xi(u), \xi(v)\big)\, \mu(du)$$

$$= \int_0^1 \int_0^1 R_{\mathbf{X}}\big(\xi(u_1), \xi(u_2)\big)\, \mu(du_1)\, \mu(du_2) > 0\,.$$

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- The support of any measure of minimal energy is not 'large'? Not always true!

- If $\mathbf{X}$ is stationary, and the spectral measure is of the full support, the image of any $\mu \in \mathcal{W}_\xi$ on the path ξ is unique.

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The limiting shapes x_a?

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$$x_a(t) = \frac{R_{\mathbf{X}}(t) + R_{\mathbf{X}}(a - t)}{R_{\mathbf{X}}(0) + R_{\mathbf{X}}(a)}, \qquad 0 \leq t \leq a\,.$$

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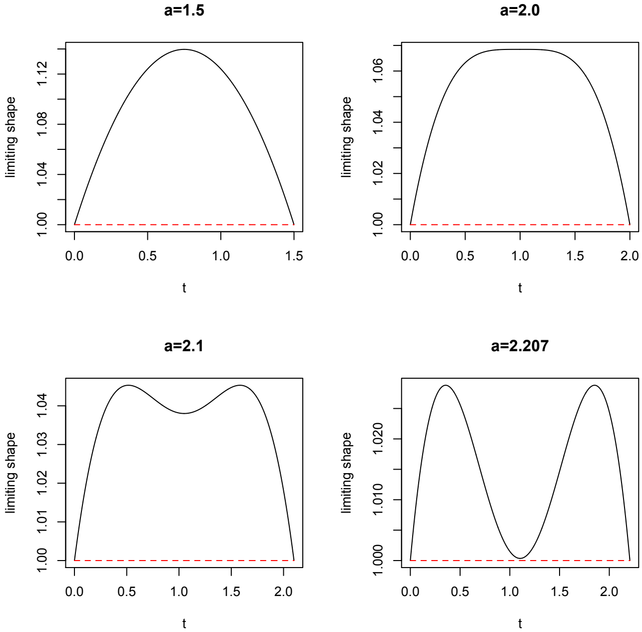
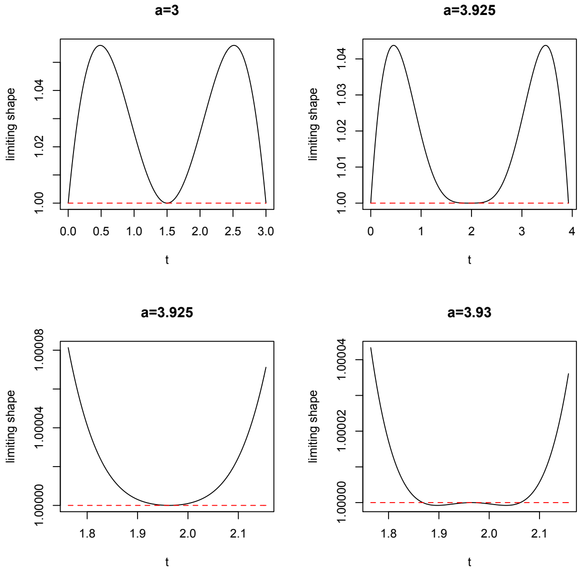
The second spectral moment is finite: for $a > 0$ sufficiently small this process satisfies the conditions of Proposition 1.

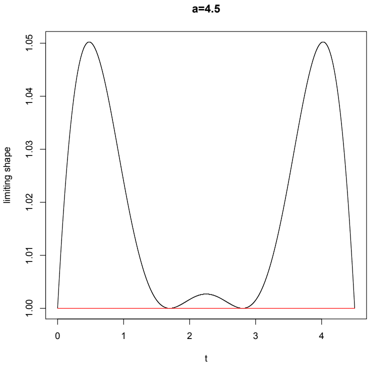
The measure $\mu = (\delta_0 + \delta_1)/2$ remains optimal for $a \leq a_1 \approx 2.2079$.



In the next regime the optimal measure acquires a point in the middle of the interval. This continues for $a_1 < a \leq a_2 \approx 3.9283$.



In the next regime the middle point of the optimal measure splits in two and starts moving away from the middle. This continues for $a_2 < a \leq a_3 \approx 5.4508$.



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Even the first spectral moment is infinite. Proposition 1 does not apply.

The optimal probability measure:

$$\mu = \frac{1}{a+2}\delta_0 + \frac{1}{a+2}\delta_1 + \frac{a}{a+2}\lambda\,,$$

λ: the Lebesgue measure on $(0,1)$.

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The limiting shape x_a: identically equal to 1 on $[0,a]$.

Long intervals

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Theorem 4 Assume that $R_{\mathbf{X}}$ is positive, $\int_0^\infty R(t)\,dt < \infty$. Then

$$\lim_{a\to\infty} \frac{1}{a} C_{\mathbf{X}}(a) = \left(\lim_{a\to\infty} a \int_0^1 \int_0^1 R_{\mathbf{X}}\big(a(u-v)\big)\,\lambda(du)\lambda(dv) \right)^{-1}$$

$$= \frac{1}{2\int_0^\infty R(t)\,dt}\,.$$

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An optimal measure μ_β exists, but it is not the uniform measure.

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In particular, $\mathcal{C}_{\mathbf{X}}(\mathbf{a})$ is regularly varying with exponent β.

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More precise one-dimensional asymptotics

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- $[a, b]$ a compact interval, $u > 0$ a high level.
- The event: the entire sample path of $\mathbf{X}$ on $[a, b]$ is above u.
- Can one obtain more precise information than what can be learned from large deviations?

The 4 questions

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Question 2. Given the event

$$B_u := \left\{\min_{a\le t\le b} X(t) > u\right\},$$

how does the conditional distribution of $(X(t), t \in [a, b])$ behave as $u \to \infty$?

Question 3. Conditionally on B_u, what is the overshoot

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Question 4. What is the asymptotic conditional distribution, given B_u, of the location of the minimum

$$\arg\min_{a \le t \le b} X(t) \quad \text{as } u \to \infty?$$

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- The canonical example: the Gaussian spectral density

$$F_X(dx) = e^{-x^2/2}\, dx,\ x \in \mathbb{R}\,.$$

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3 the optimal value $\sigma_*^2(b) > 0$.

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Then $\theta_j > 0$, $j = 1, \ldots, k$,

$$P\Big(\min_{j=1,\ldots,k} X(t_j) > u\Big) \sim (2\pi)^{-k/2}(\det \Sigma)^{-1/2}(\theta_1 \ldots \theta_k)^{-1}$$
$$u^{-k} e^{-u^2/2\sigma_*^2(b)}, \;\; u \to \infty\,.$$

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The key assumption: $m'' > 0$ on $S \cap (0, b)$.

Theorem 1

Let the cardinality of S be k. Then

$$P\big(\min_{0\le t\le b} X(t) > u\big) \sim c u^{-k} e^{-u^2/2\sigma_*^2(b)},\ u \to \infty$$

for $c \in [0, \infty)$.

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for $c \in [0, \infty)$.

Furthermore, $c > 0$ if and only if the key assumption holds.

Theorem 2

Suppose the key assumption holds. Then in $C[0, b]$,

$$P\left((X(t) - um(t),\, 0 \leq t \leq b) \in \cdot \,\middle|\, \min_{t \in [0,b]} X(t) > u\right) \Rightarrow Q_W(\cdot)\,,$$

where Q_W is the law of a tilted Gaussian process on $[0, b]$.

Theorem 3

Suppose the key assumption holds.

Then, as $u \to \infty$, the conditional distribution of

$$u\Big(\min_{t\in[0,b]} X(t) - u\Big) \quad \text{given} \ \min_{t\in[0,b]} X(t) > u$$

converges weakly to the exponential distribution with mean $\sigma_*^2(b)$.

Theorem 4

Suppose the key assumption holds. Let

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- **Example** The Gaussian covariance function $R(t) = e^{-t^2/2}$.
- If $0 < b \leq 2.2079...$, $S = \{0, b\}$.
- The key assumption holds.

- Suppose $2.2079\ldots < b \leq 3.9283\ldots$.

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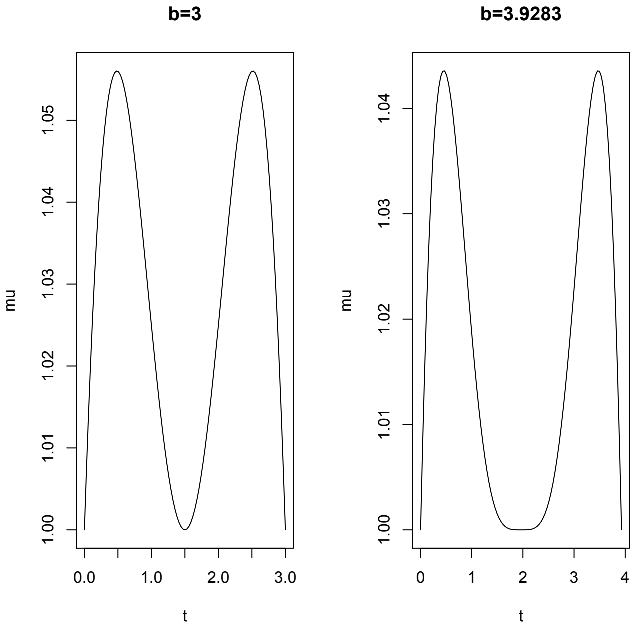
- Then $S = \{0, b/2, b\}$.



b=3
mu
1.00
1.01
1.02
1.03
1.04
1.05
0.0
1.0
2.0
3.0
t
b=3.9283
mu
1.00
1.01
1.02
1.03
1.04
0
1
2
3
4
t

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- Is there a hole in the middle of a high excursion set?
- Two probabilities:

- Fix $0 < r \leq 1$.

$$\Psi_{sp}(u;r) = P\big(\text{there exists a ball } B \text{ entirely in } T$$
$$\text{such that } X(\mathbf{t}) > u \text{ for all } \mathbf{t} \in S_B \text{ but } X(\mathbf{s}) < ru \text{ for some } \mathbf{s} \in B\big)$$

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Let

$$\Psi(u) = P\big(X(\mathbf{t}) > u \text{ for all } \mathbf{t} \in S_B\big) .$$

If

$$\lim_{u\to\infty} \frac{1}{u^2} \log \Psi_{sp;c}(u;r) = \lim_{u\to\infty} \frac{1}{u^2} \log \Psi(u)\,,$$

the hole is “likely”.

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- Large deviations approach requires solving difficult optimization problems.

Isotropic Gaussian fields

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- A centered Gaussian field is isotropic if

$$R_{\mathbf{X}}(\mathbf{t}_1, \mathbf{t}_2) = R(\|\mathbf{t}_1 - \mathbf{t}_2\|),\ \mathbf{t}_1,\ \mathbf{t}_2 \in T\,.$$

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- Start first with the hole in the center of the ball.

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- $S_\rho(\mathbf{0})$: the sphere of radius ρ centered at the origin;
- μ_h: the rotation invariant probability measure on $S_\rho(\mathbf{0})$.

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$$W_\rho(r) = \begin{cases} D(\rho) & \text{if } R(\rho) \leq rD(\rho)\,, \\ \frac{R(0)D(\rho) - \left(R(\rho)\right)^2}{R(0) - 2rR(\rho) + r^2 D(\rho)} & \text{if } R(\rho) > rD(\rho)\,. \end{cases}$$

Theorem Let $\mathbf{X}$ be isotropic. Then

$$\lim_{u\to\infty} \frac{1}{u^2} \log \Psi_{sp;c}(u;r) = -\frac{1}{2} \min_{0\le\rho\le D} \big(W_\rho(r)\big)^{-1} .$$

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For a sphere of radius ρ and $0 < r \le 1$:

- a hole of depth r is likely if $R(\rho) \le rD(\rho)$;
- a hole of depth r is unlikely if $R(\rho) > rD(\rho)$.

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- If the second spectral moment is infinite, a hole of depth $r = 1$ may or may not be unlikely.

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- In dimensions $d \geq 2$, if the memory is sufficiently long, then a deep enough hole may be unlikely even for a sphere of an infinite radius.

Suppose that $d \geq 2$, R monotone and

$$R \;\text{ is regularly varying at infinity with exponent } -(d-1)+\varepsilon.$$

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Denote

$$I(d;\varepsilon) = \int_{S_1(\mathbf{0})} \int_{S_1(\mathbf{0})} \|\mathbf{t}_1 - \mathbf{t}_2\|^{-(d-1)+\varepsilon} \, \mu_h(d\mathbf{t}_1) \, \mu_h(d\mathbf{t}_2) \,.$$

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A hole of depth $r < 1/I(d;\varepsilon)$ is unlikely even for spheres of infinite radius!

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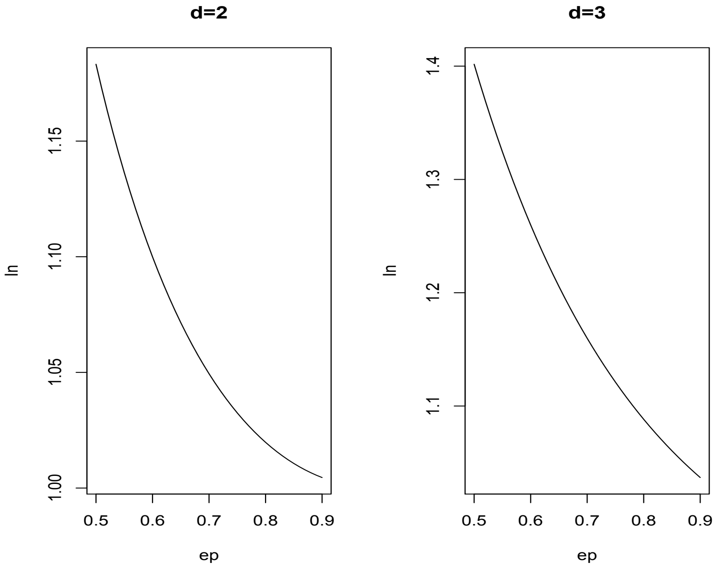
This is true even though the field is ergodic and mixing.

The value of $I(d; \varepsilon)$ in 2 and 3 dimensions.



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What is the radius of a sphere for which this event is the most likely:

1. the random field has a "peak" of height greater than u covering the entire sphere;
2. there is a "hole" in the center of the sphere where the height is smaller than ru.

Assume that R is monotone, $R(t) \to 0$, and $0 < r < 1$. Let

$$H_\rho(r) = \frac{R(0)D(\rho) - \big(R(\rho)\big)^2}{R(0) - 2rR(\rho) + r^2 D(\rho)}, \quad \rho > 0\,.$$

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Then

$$\rho_r^* = \mathrm{argmax}_{\rho \geq 0} H_\rho(r)\,.$$

is the radius of the sphere most likely to have a hole corresponding to a factor r in the center.

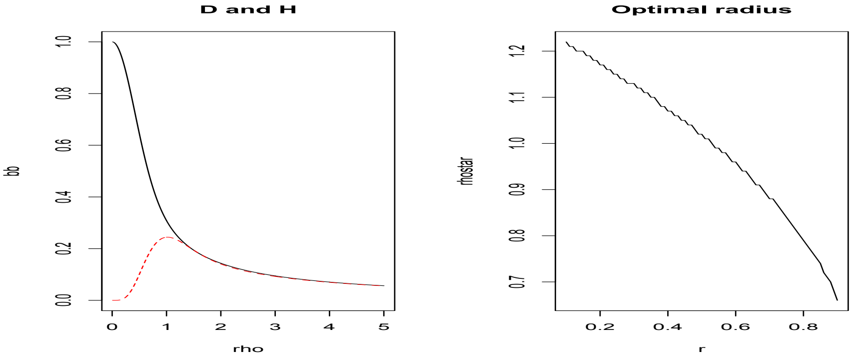

Figure: The functions $D(\rho)$ (solid line) and $H_\rho(r)$ (dashed line) for $r = 1/2$ (left plot) and the optimal radius ρ_r^* (right plot), both for $R(t) = e^{-t^2}$.

Limiting shapes

Limiting shapes

For the isotropic random field and any sphere, there is a deterministic function $(x(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d)$ such that

$$P\left(\sup_{\mathbf{t}\in T}\left|\frac{1}{u}X(\mathbf{t}) - x(\mathbf{t})\right| \geq \varepsilon \,\middle|\, X(\mathbf{t}) > u \text{ for each } \mathbf{t} \text{ on the sphere} \right.$$
$$\left. \text{and } X(\text{center}) < ru\right) \to 0$$

as $u \to \infty$.

- The shape is rotationally invariant.

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- We plot a radial crossection of the limting shape.

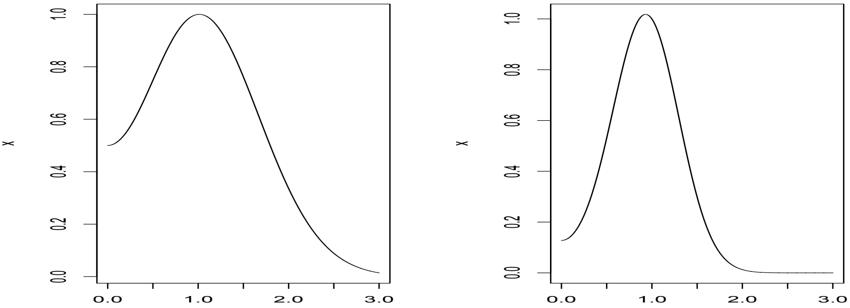

Figure: The limiting shapes for $\rho = 1$ (hole unlikely, left plot) and $\rho = 2$ (hole likely, right plot), both for $r = 1/2$ and $R(t) = e^{-t^2}$. The horizontal axes are units of t_1/ρ.

The probability $\Psi_{sp}(u; r)$

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In many cases: asymptotic behaviour of the two probabilities is the same on the logarithmic scale.

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For $\rho \geq 0$, $0 \leq b \leq 1$ and $\mu \in M_1^+(S_1)$, let

$$V(\rho, b; \mu) =$$

$$\frac{\int_{S_1} \int_{S_1} R\big(\rho \|\mathbf{t}_1 - \mathbf{t}_2\|\big)\, \mu(d\mathbf{t}_1)\mu(d\mathbf{t}_2) - \Big(\int_{S_1} R(\rho \|\mathbf{t} - b\mathbf{e}_1\|)\, \mu(d\mathbf{t})\Big)^2}{1 - 2r \int_{S_1} R(\rho \|\mathbf{t} - b\mathbf{e}_1\|)\, \mu(d\mathbf{t}) + r^2 \int_{S_1} \int_{S_1} R\big(\rho \|\mathbf{t}_1 - \mathbf{t}_2\|\big)\, \mu(d\mathbf{t}_1)\mu(d\mathbf{t}_2)}.$$

Let

$$V_*(\rho, b) = \min_{\mu \in M_1^+\left(S_1\right)} V(\rho, b; \mu)$$

subject to

$$\int_{S_1} R(\rho \|\mathbf{t} - b\mathbf{e}_1\|)\, \mu(d\mathbf{t}) \geq r \int_{S_1} \int_{S_1} R\left(\rho \|\mathbf{t}_1 - \mathbf{t}_2\|\right) \mu(d\mathbf{t}_1)\mu(d\mathbf{t}_2)\,.$$

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If, for every $0 \leq \rho \leq D$ such that $R(\rho) \geq rD(\rho)$, the function $V_*(\rho, b)$, $0 \leq b \leq 1$ achieves its maximum at $b = 0$, then

$$\lim_{u \to \infty} \frac{1}{u^2} \log \Psi_{sp}(u; r) = \lim_{u \to \infty} \frac{1}{u^2} \log \Psi_{sp;c}(u; r)\,.$$

A sufficient condition: for every $0 \le \rho \le D$ such that $R(\rho) \ge rD(\rho)$,

$$\min_{0 \le b \le 1} \int_{S_1} R(\rho \|\mathbf{t} - b\mathbf{e}_1\|)\, \mu_h(d\mathbf{t}) = \int_{S_1} R(\rho \|\mathbf{t}\|)\, \mu_h(d\mathbf{t}) = R(\rho)\,,$$

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This condition is not necessary.

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For these two covariance functions: the two probabilities are asymptotically equivalent on the logarithmic scale.