Large deviations and sum rules

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February 5th 2019

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A. Rouault (Versailles)

ISI Bangalore

Bibliography

Spectral theory

B. Simon, *Szegö's Theorem and Its Descendants* (2011)

B. Simon, *Orthogonal Polynomials on the Unit Circle I and II* (2005, 2007)

Large deviations

G. Anderson, A. Guionnet, and O. Zeitouni *An introduction to random matrices* (2010).

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Papers (LD = large deviations, SR = sum rules)

F. Gamboa, J. Nagel, A. Rouault

- Canonical moments and random spectral measures, *JoTP* (2010)
- LD for random spectral measures and SR, *AMRX* (2011)
- Operator-valued spectral measures and LD, *JSPI* (2014)
- SR via LD, *J. Funct. Anal.* (2016)
- SR and LD for spectral matrix measures, *Bernoulli* (2018)
- SR and LD for spectral measures on the unit circle, *Random Matrices Th. and Appl.* (2017)
- SR and LD for the spectral measures in the Jacobi ensemble, *arXiv* (2018)

J. Breuer, B. Simon, O. Zeitouni : LD and SR for Spectral Theory : A Pedagogical Approach, *J. Spectral Th.* (2018).

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From Barry Simon's book "Szegő's Theorem and Its Descendants", Chapter 2 :

In algebra, when one says $a = b$, it is a tautology and so uninteresting ; while in analysis, when one says $a = b$, it is two deep inequalities.
(attributed to S. Bochner)

If one only proves $a = b$ by showing $a \leqslant b$ and $b \leqslant a$, one has not understood the true reason why $a = b$.
(attributed to E. Noether)

Our contribution could be :

If a and b are two positive functionals, when one says $a = b$, a probabilist may think that a and b could be two rate functions of the same large deviation principle under two different encodings.

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Overview

1 Kullback Leibler Divergence

2 Orthogonal polynomial recursion on $\mathbb{T}$

3 Szegö Theorem

4 The three parametrizations

5 Three particular probability distributions

6 Killip Simon Theorem

7 Our sum rules

8 How to get a sum rule with a probabilistic method

Kullback Leibler Divergence

S. Kullback (1907-1994)

R. Leibler (1914-2003)

P, Q probability measures on some space E

$$K(P,Q) = \begin{cases} \int_E \log \frac{dP}{dQ} dP & \text{if } P \ll Q \text{ and } \log \frac{dP}{dQ} \in L^1(P) \\ +\infty & \text{otherwise} \end{cases}$$

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Orthogonal polynomial recursion on $\mathbb{T}$

- μ probability measure on $\mathbb{T}$
  1) (p_n) sequence of orthogonal polynomials associated to μ
  2) p_n is monic and has degree n
  3) $k \neq n$, $\int_{\mathbb{T}} p_n(z)p_k(z)\mu(dz) = 0$
- Satisfies the recursion
  - $\rightarrow$ $p_{n+1}(z) = zp_n(z) - \overline{\alpha}_n p_n^*(z)$ where $p_n^*(z) := z^n\overline{p_n(1/\overline{z})}$.
  - $\rightarrow$ $\alpha_n = -p_{n+1}(0)$ is the Verblunsky coefficient

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Szegö Theorem

G. Szegö (1895-1985)

Szegö Theorem

λ Lebesgue measure on $\mathbb{T}$.

$$K(\lambda, \mu) = -\sum_{n=0}^{\infty} \log(1 - |\alpha_n|^2)$$

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The three parametrizations

- μ probability measure on $\mathbb{R}$
  - $\rightarrow$ Assume that μ has all its **moments finite**
  - $\rightarrow$ (p_n) normalized othogonal polynomials in $L^2(\mu)$
  - $\rightarrow$ Three terms recursion

$$xp_n(x) = a_n p_{n-1}(x) + b_{n+1} p_n(x) + a_{n+1} p_{n+1}(x), \quad a_n > 0, \; b_{n+1} \in \mathbb{R}$$

- Assume moreover that μ is supported on $[0, +\infty[$
  $b_n = z_{2n-2} + z_{2n-1}$, and $a_n^2 = z_{2n-1} z_{2n}$.
- If μ is supported on $[0, 1]$, μ may be seen as the pushforward of a measure on $\mathbb{T}$ invariant by $2\pi - \theta$ by $\sin^2(\theta/2)$.
  $b_{k+1} = 1/4[2 - (1 - \alpha_{2k-1})\alpha_{2k} - (1 + \alpha_{2k-1})\alpha_{2k-2}]$ and
  $a_{k+1} = 1/4\sqrt{(1 - \alpha_{2k-1})(1 - \alpha_{2k}^2)(1 + \alpha_{2k+1})}$

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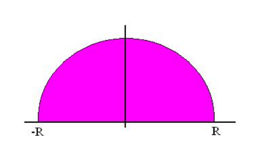
Semicircular distribution



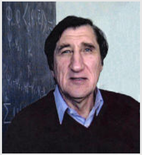
$$R = 2$$

$$SC(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}\,\mathbb{1}_{[-2,2]}(x)\,dx$$

Limit of the eigenvalues distribution of the symmetric Gaussian ensemble

$$a_k = 1,\ b_k = 0 \ \text{ for all } k \geqslant 1\,.$$

Pastur-Marchenko Distribution



L. Pastur

V. Marchenko

$$\mathrm{MP}_\tau(dx) = \frac{\sqrt{(\tau^+ - x)(x-\tau^-)}}{2\pi\tau x}\, \mathbb{1}_{(\tau^-,\tau^+)}(x)\,dx \ , \tau^{\pm} = (1 \pm \sqrt{\tau})^2$$

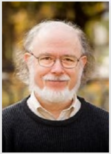
Limit of the squared singular values distribution of rectangular Gaussian matrices. $\tau \in]0, 1]$ the asymptotic ratio nb col/ nb line

$$a_k = \sqrt{\tau}\ (k \geqslant 1)\ ,\ \ b_1 = 1\ ,\ b_k = 1 + \tau\ \ (k \geqslant 2)$$

and correspond to $z_{2n-1} = 1$ and $z_{2n} = \tau$ for all $n \geqslant 1$.

Kesten Mac Kay Distribution

H. Kesten



B. Mc Kay

$$\mathrm{KMK}_{\kappa_1,\kappa_2}(dx) = \frac{(2+\kappa_1+\kappa_2)}{2\pi} \frac{\sqrt{(u^+ - x)(x-u^-)}}{x(1-x)} \mathbb{1}_{(u^-,u^+)}(x)dx$$

$$u^{\pm} := \frac{1}{2} + \frac{\kappa_1^2 - \kappa_2^2 \pm 4\sqrt{(1+\kappa_1)(1+\kappa_2)(1+\kappa_1+\kappa_2)}}{2(2+\kappa_1+\kappa_2)^2}, \kappa_1, \kappa_2 \geqslant 0.$$

Asymptotic distribution of the eigenvalues in the Jacobi-ensemble

The associated Verblunsky coefficients for $k \geqslant 0$,

$$\alpha_{2k} = \frac{\kappa_1 - \kappa_2}{2 + \kappa_1 + \kappa_2}, \quad \alpha_{2k+1} = -\frac{\kappa_1 + \kappa_2}{2 + \kappa_1 + \kappa_2}.$$

Then

$$a_1 = \frac{\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(2+\kappa_1+\kappa_2)^{3/2}}, \quad b_1 = \frac{1+\kappa_2}{2+\kappa_1+\kappa_2},$$

and for $k \geqslant 2$

$$a_k = \frac{\sqrt{(1+\kappa_1+\kappa_2)(1+\kappa_1)(1+\kappa_2)}}{(2+\kappa_1+\kappa_2)^2}, \quad b_k = \frac{1}{2}\left[1 - \frac{\kappa_1^2 - \kappa_2^2}{(2+\kappa_1+\kappa_2)^2}\right].$$

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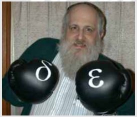
Killip Simon Theorem

R. Killip



B. Simon

$$K(SC|\mu) + \sum_{n=1}^{N^+} \mathcal{F}_H^+(\lambda_n^+) + \sum_{n=1}^{N^-} \mathcal{F}_H^-(\lambda_n^-) = \sum_{k \geqslant 1} \left(\frac{1}{2} b_k^2 + a_k^2 - 1 - \log(a_k^2)\right)$$

$$\mathcal{F}_H^+(x) := \begin{cases} \int_2^x \sqrt{t^2 - 4}\, dt = \frac{x}{2}\sqrt{x^2 - 4} - 2\log\left(\frac{x + \sqrt{x^2-4}}{2}\right) & \text{if } x \geqslant 2 \\ \infty \quad \text{otherwise,} & \end{cases}$$

$$\mathcal{F}_H^-(x) := \mathcal{F}_H^+(-x)$$

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Our sum rules I

$$K(MP_\tau|\mu) + \sum_{n=1}^{N^+} \mathcal{F}_L^+(\lambda_n^+) + \sum_{n=1}^{N^-} \mathcal{F}_L^-(\lambda_n^-) = \sum_{k=1}^{\infty} \tau^{-1} G(z_{2k-1}) + G(\tau^{-1} z_{2k})$$

$$\mathcal{F}_L^+(x) = \begin{cases} \displaystyle\int_{\tau^+}^{x} \frac{\sqrt{(t-\tau^-)(t-\tau^+)}}{t\tau}\, dt & \text{if } x \geqslant \tau^+, \\ \infty & \text{otherwise,} \end{cases}$$

$$\mathcal{F}_L^-(x) = \begin{cases} \displaystyle\int_{x}^{\tau^-} \frac{\sqrt{(\tau^- - t)(\tau^+ - t)}}{t\tau}\, dt & \text{if } x \leqslant \tau^-, \\ \infty & \text{otherwise.} \end{cases}$$

$$G(x) = x - 1 - \log x, \ (x > 0).$$

Our sum rules II

$$K(KMK_{\kappa_1,\kappa_2}|\mu) + \sum_{n=1}^{N^+} \mathcal{F}_J^+(\lambda_n^+) + \sum_{n=1}^{N^-} \mathcal{F}_J^-(\lambda_n^-) = \sum_{k=0}^{\infty} H_1(\alpha_{2k+1}) + H_2(\alpha_{2k})$$

$$\mathcal{F}_J^+(x) = \begin{cases} \displaystyle\int_{u^+}^{x} \frac{\sqrt{(t-u^+)(t-u^-)}}{t(1-t)}\,dt & \text{if } u^+ \leqslant x \leqslant 1 \\ \infty & \text{otherwise.} \end{cases}$$

$$\mathcal{F}_J^-(x) = \begin{cases} \displaystyle\int_{x}^{u^-} \frac{\sqrt{(u^- - t)(u^+ - t)}}{t(1-t)}\,dt & \text{if } 0 \leqslant x \leqslant u^- \\ \infty & \text{otherwise.} \end{cases}$$

For $-1 \leqslant x \leqslant 1$

$$H_1(x) = -(1+\kappa_1+\kappa_2)\log\left[\frac{2+\kappa_1+\kappa_2}{2(1+\kappa_1+\kappa_2)}(1-x)\right] - \log\left[\frac{2+\kappa_1+\kappa_2}{2}(1+x)\right]$$

$$H_2(x) = -(1+\kappa_1)\log\left[\frac{(2+\kappa_1+\kappa_2)}{2(1+\kappa_1)}(1+x)\right] - (1+\kappa_2)\log\left[\frac{(2+\kappa_1+\kappa_2)}{2(1+\kappa_1)}(1-x)\right].$$

Particular case $\kappa_1 = \kappa_2 = 0 \Rightarrow u^- = 0,\ u^+ = 1$, $\mathrm{KMK}_{\kappa_1,\kappa_2}$ arsine law.

We recover the Szegö Theorem pushed on $[0,1]$ by $\sin^2(\theta/2)$.

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How to get a sum rule with a probabilistic method

- Consider a compactly supported measure μ as the spectral measure of some operator M in some class, at a vector e :

$$< e, M^k e >= \int_E x^k d\mu(x)\ ,\ (k \geqslant 0)$$

  $E = \mathbb{T}$ or $\mathbb{R}$, M unitary or self-adjoint.
- Randomize in this class, a family of finite-dimensional operators $(M_n)_{n\geqslant 1}$ and their spectral measures $(\mu_n)_{n\geqslant 1}$ at $(e^{(n)})_{n\geqslant 1}$
- Consider the two encodings of the spectral measures μ_n
  1) the pair "locations, weights"

$$\mu_n = \sum_{k=1}^{n} w_k^{(n)} \delta_{\lambda_k^{(n)}}$$

  2) the recursion coefficients (Jacobi or Verblunsky) : $(a_k^{(n)}, b_k^{(n)})$ or $(\alpha_k^{(n)})$.

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- Prove two Large Deviation Principles :

$$\frac{1}{n}\log P(\mu_n \cong \mu) \cong -\mathcal{I}_{sp}(\mu)$$
$$\frac{1}{n}\log P(\mu_n \cong \mu) \cong -\mathcal{I}_{Jac}(a_1, b_1, a_2, \dots)$$

- Write equality of both rate functions :

$$\mathcal{I}_{sp}(\mu) = \mathcal{I}_{Jac}(a_1, b_1, a_2, \dots)$$

Notice the difference between the two measures

$$\mu_n = \mu_n^{SP} = \sum_{k=1}^{n} w_k \delta_{\lambda_k} \quad \text{(spectral measure)}$$

$$\mu_n^{ESD} = \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k} \quad \text{(empirical spectral distribution)} .$$

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$$\mu_n^{ESD} = \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k} \quad \text{(empirical spectral distribution)} .$$

- Prove two Large Deviation Principles :

$$\frac{1}{n}\log P(\mu_n \cong \mu) \cong -\mathcal{I}_{sp}(\mu)$$
$$\frac{1}{n}\log P(\mu_n \cong \mu) \cong -\mathcal{I}_{Jac}(a_1, b_1, a_2, \dots)$$

- Write equality of both rate functions :

$$\mathcal{I}_{sp}(\mu) = \mathcal{I}_{Jac}(a_1, b_1, a_2, \dots)$$

Notice the difference between the two measures

$$\mu_n = \mu_n^{SP} = \sum_{k=1}^{n} w_k \delta_{\lambda_k} \ \text{(spectral measure)}$$
$$\mu_n^{ESD} = \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k} \ \text{(empirical spectral distribution)}.$$

Randomization for the KS/SR

- Suppose the distribution of M_n has the GUE-density
$$\mathcal{Z}_n^{-1} \exp -\frac{n}{2} \mathrm{tr} M^2$$
- Dumitriu-Edelman ('02) proved that the Jacobi parameters are independent and
$$b_k^{(n)} \sim \mathcal{N}(0; n^{-1}) \quad (1 \leqslant k \leqslant n),$$
$$(a_k^{(n)})^2 \sim \text{Gamma}\,(n-k; n^{-1}) \;\; (1 \leqslant k \leqslant n-1)\,.$$

Note that $b_k^{(n)} \to 0$, $a_k^{(n)} \to 1$, the Jacobi coefficients of SC.

Theorem (GR '11)

μ_n^{SP} *satisfies the LDP with speed* n *and rate function*
$$\mathcal{I}_{Jac} = \sum_1^\infty \frac{1}{2} b_k^2 + \sum_1^\infty G(a_k^2)\,,\;\; G(x) = x - 1 - \log x\,.$$

LDP for the *measure side*, general potential

- M_n random complex Hermitian $n \times n$ matrix with density

$$(\mathcal{Z}_n^V)^{-1} \exp(-n \mathrm{tr} V(M))$$

- Potential $V : \mathbb{R} \to (-\infty, \infty]$ smooth, e.g. $V(x) = x^2/2$, (GUE).
- $$\mu_n^{SP} = \sum_1^n w_i \delta_{\lambda_i}$$

  with $w_i = |U_{1,i}|^2$ for U unitary matrix of eigenvectors.
- The joint density of eigenvalues is

$$(Z_n^V)^{-1} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_i^n \exp(-nV(\lambda_i))$$

  and $(w_1, \dots, w_n)$ is uniformly distributed on the simplex, and independent of the eigenvalues.

Theorem (GNR '16)

Under assumptions on V, the sequence of random spectral measures $\mu^{(n)}$ satisfies the LDP with speed n with good rate function

$$\mathcal{I}_{sp}(\mu) = \mathcal{K}(\mu_V \,|\mu) + \sum_k \mathcal{F}_V(E_k^+) + \sum_k \mathcal{F}_V(E_k^-)$$

for probability measures μ on $\mathbb{R}$ satisfying

$$\mathit{Supp}(\Sigma) = [a_V, b_V] \cup \{E_j^-\}_{j=1}^{N^-} \cup \{E_j^+\}_{j=1}^{N^+}$$

where N^+ (resp. N^-) is 0, finite or infinite, $E_j^- \uparrow a_V$ and $E_j^+ \downarrow b_V$ are isolated points of the support .

New sum rules need LDPs for the coefficient side.

We need matrix models whose spectral measure admits coefficients with *nice probabilistic properties* !

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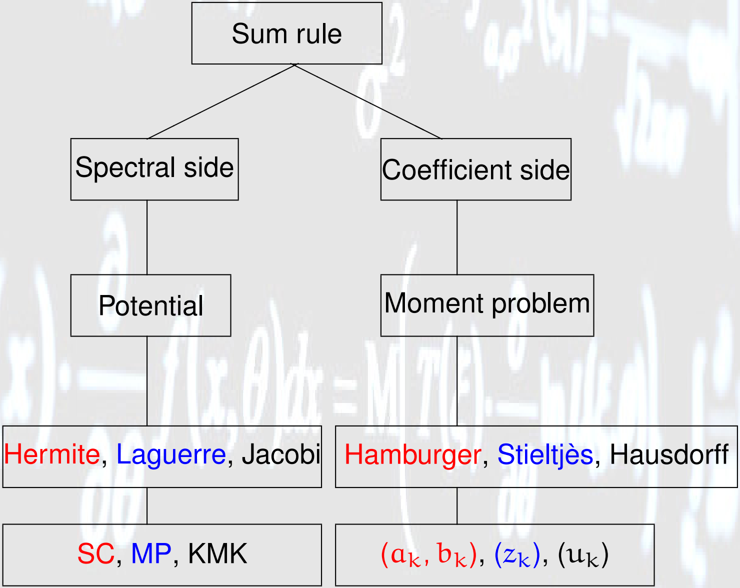
Sum rule
Spectral side
Coefficient side
Potential
Moment problem
Hermite, Laguerre, Jacobi
Hamburger, Stieltjès, Hausdorff
SC, MP, KMK
(a_k, b_k), (z_k), (u_k)

THANK YOU FOR YOUR ATTENTION!