# The free energy of pure spherical models: computation from the TAP approach

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מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

# Spherical spin glass models

A sequence of random functions on the sphere in $\mathbb{R}^N$, (configuration space)

$$\mathbb{S}^N := \left\{\mathbf{x} = (x_1, ..., x_N) \in \mathbb{R}^N : \|\mathbf{x}\| = 1\right\}.$$

The **spherical pure $p$-spin Hamiltonian:**

$$\boxed{H_{N,p}(\mathbf{x}) = \sqrt{N} \sum_{i_1,\dots,i_p=1}^{N} J_{i_1,\dots,i_p} x_{i_1} x_{i_2} \cdots x_{i_p},}$$

where $\mathbf{x} = (x_1, \dots, x_N)$ and $J_{i_1,\dots,i_p} \sim \mathrm{Normal}(0,1)$ i.i.d.

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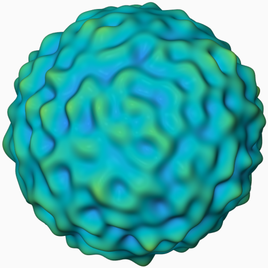
$^\star$ Models with **Ising spins**: (not today...)

$$\mathbb{S}^N \text{ replaced by } \Sigma_N := \{+1, -1\}^N.$$

# Scaling of basic quantities

Typical values: $H_N(\mathbf{x}) = O(\sqrt{N})$.



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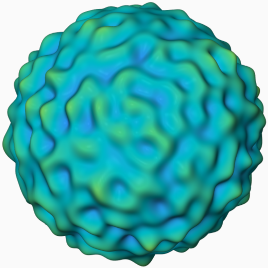

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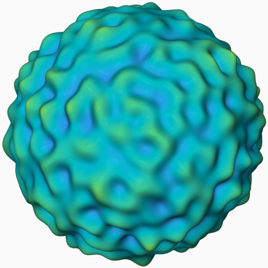

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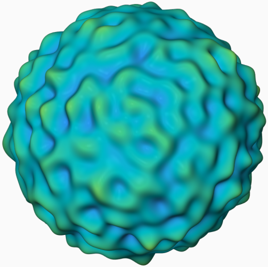

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Critical points: $\forall E \in [0, E_\star)$ :

$$\#\Big\{\mathbf{x} \in \mathbb{S}^N : \nabla H_N(\mathbf{x}) = 0,\ H_N(\mathbf{x}) \approx NE\Big\} = e^{C_E N + o(N)}$$

[Auffinger-Ben Arous-Cerny ‘13], [Auffinger-Ben Arous ‘13], [S. ‘17], [Ben Arous-S.-Zeitouni ‘18], [S.-Zeitouni ‘21]

Consider the super-level sets of the random function $H_N(\mathbf{x})$,

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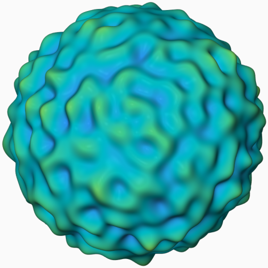

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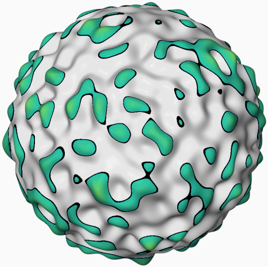

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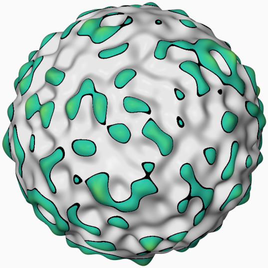
**As $N \to \infty$, what is the asymptotic behavior of $\mathbf{Vol}\big(A_N(E)\big)$?**



# Concentration of measure on the sphere

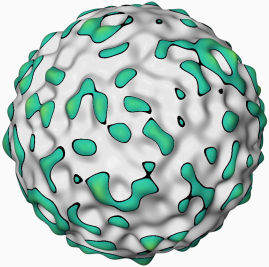
Let $f : \mathbb{S}^N \to \mathbb{R}$ be a deterministic function on the unit sphere in $\mathbb{R}^N$.

Consider $$A(t) := \left\{ \mathbf{x} \in \mathbb{S}^N : f(\mathbf{x}) \geq Nt \right\}.$$

Suppose $f$ is Lipschitz with constant $NL$,

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq NL\|\mathbf{x} - \mathbf{y}\|,$$

and for simplicity that $\int_{\mathbb{S}^N} f(\mathbf{x}) d\mathbf{x} = 0$.



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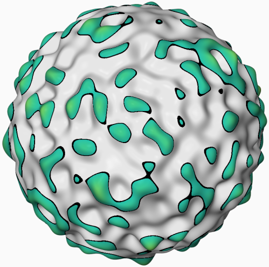

**Lévy's inequality (1919)**

$$\mathrm{Vol}\big(A(t)\big) \leq K e^{-\frac{CNt^2}{L^2}}.$$

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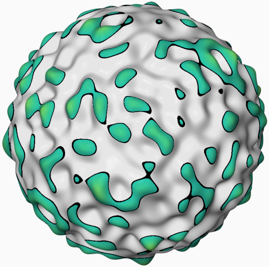

If $t$ not too close to $\max_{\mathbf{x}} f(\mathbf{x})$, then for large $N$,

$$\boxed{e^{-cN} \leq \mathrm{Vol}\big(A(t)\big) \leq e^{-CN}.}$$

Consider the super-level sets of the random function $H_N(\mathbf{x})$,

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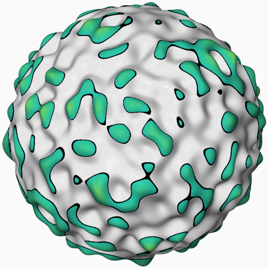

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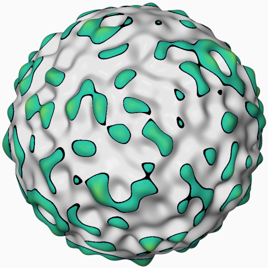
Since $$e^{-cN} \leq \mathrm{Vol}\big(A_N(E)\big) \leq e^{-CN},$$

the right question is to compute (it exists?)

$$V(E) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log \Big( \mathrm{Vol}\big(A_N(E)\big)\Big).$$



# The free energy

The **free energy** is defined by:

$$F_N(\beta) = \frac{1}{N}\log Z_N(\beta) = \frac{1}{N}\log \int_{\mathbb{S}^N} e^{\beta H_N(\mathbf{x})} d\mathbf{x}.$$

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One of the most important problems is to compute the limit

$$F(\beta) := \lim_{N\to\infty} \mathbb{E}F_N(\beta).$$

# The free energy

For any energy level $E$,

$$\begin{aligned} F(\beta) &:= \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log\int_{\mathbb{S}^N} e^{\beta H_N(\mathbf{x})}d\mathbf{x} \\ &\geq \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log\int_{A_N(E)} e^{\beta NE}d\mathbf{x} \\ &= \beta E + \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log\Big(\mathrm{Vol}\big(A_N(E)\big)\Big) =: \beta E + V(E). \end{aligned}$$

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In fact, $F(\beta)$ and $-V(E)$ are **convex conjugates:**

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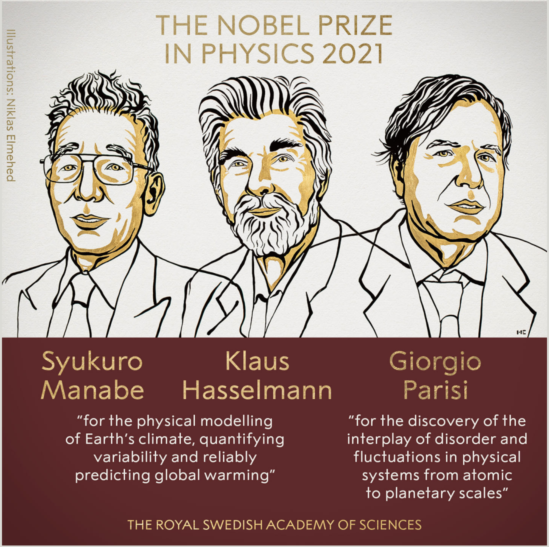
$\big(F(\beta)\big)_{\beta>0}$ and $\big(V(E)\big)_{E\in(0,E_\star)}$ contain the same information!

# Parisi's formula for the free energy



# The Parisi formula

The **'replica method'** (Edwards-Anderson '75) suggests that to compute $F(\beta)$ one can compute $\mathbb{E}Z_{N,\beta}^t$ for **integer** $t \geq 1$, extend to real $t$, and use

$$F(\beta) := \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z_{N,\beta} = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\lim_{t\to 0} \frac{Z_{N,\beta}^t - 1}{t} \approx \lim_{t\to 0} \lim_{N\to\infty} \frac{1}{N} \frac{\mathbb{E}Z_{N,\beta}^t - 1}{t}.$$

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Upper bound proved by **Guerra '03**, lower bound by **Talagrand '06** for even models ($\gamma_p = 0$ for odd $p$). Extended to general mixtures by **Panchenko '14** (cube) and **Chen '13** (sphere).

Notable related breakthroughs: **Ghirlanda-Guerra identities '98**, **Aizenman-Sims-Starr scheme '03**, **ultrametricity** by **Panchenko '13**.

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For spherical models, the **Crisanti-Sommers '92** representation is

$$\mathcal{P}_{\xi,\beta}(y) = \frac{1}{2}\Big(\beta^2 \int_0^1 y(q)\xi'(q)dq + \int_0^{\hat{q}} \frac{dq}{\int_q^1 y(s)ds} + \log(1-\hat{q})\Big).$$

For Ising models, more complicated.

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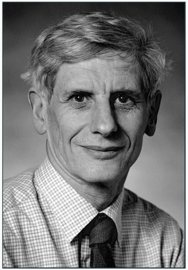
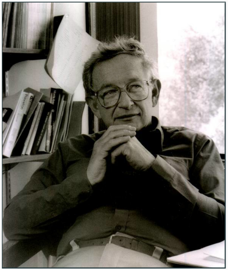
The minimizing distribution in Parisi's formula is equal to the (averaged) distribution of the 'overlap' of two independent samples $\mathbf{x}_1, \mathbf{x}_2$ from the Gibbs measure,

$$y(t) = \mathbb{P}\big(\mathbf{x}_1 \cdot \mathbf{x}_2 \leq t\big).$$

# Thouless-Anderson-Palmer Approach (1977)



D. J. Thouless



P. W. Anderson

R. Palmer

Thouless and Anderson are Nobel Prize laureates (2016 & 1977).

# Overview

- **Generalized TAP representation for the free energy**
- **Computation of the free energy for <u>pure models</u>**
- **Multi-species models**

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**Def.:** $q \in [0,1)$ is a good overlap if $\forall k \geq 1, \epsilon > 0$,

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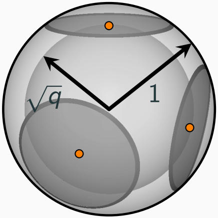


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**Lemma.** Any overlap in the support of the Parisi distribution is good!

Denote $\quad E_\star(q) := \lim_{N\to\infty} \frac{1}{N} \max_{m \in \sqrt{q}\cdot\mathbb{S}^N} H_N(m).$

Define

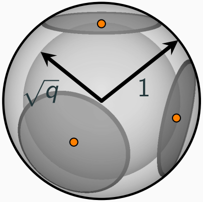
$$\mathrm{Band}(m) = \big\{\mathbf{x} \in \mathbb{S}^N : |(\mathbf{x} - m)\cdot m| < \delta_N\big\}.$$

# TAP representation for the free energy

$F(\beta, q)$ — free energy of the mixture

$$\xi_q(t) = \xi\big((1-q)t + q\big) - \xi(q) - \xi'(q)(1-q)t,$$

and $$\frac{1}{2}\log(1-q) = \lim_{N\to\infty} \frac{1}{N} \log \mathrm{Vol}(\mathrm{Band}(m)).$$




**Theorem (TAP representation) [S. '18]**

Consider a Hamiltonian with general mixture $\xi(t)$. $q$ is good if and only if

$$F(\beta) = \beta E_\star(q) + \frac{1}{2}\log(1-q) + F(\beta, q).$$

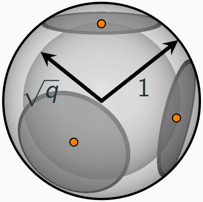
If $q$ is not good, same holds with inequality $>$.

# TAP representation for the free energy

$F(\beta, q)$ — free energy of the mixture

$$\xi_q(t) = \xi\big((1-q)t + q\big) - \xi(q) - \xi'(q)(1-q)t,$$

and $$\frac{1}{2}\log(1-q) = \lim_{N\to\infty} \frac{1}{N} \log \mathrm{Vol}(\mathrm{Band}(m)).$$




**Theorem (TAP representation) [S. '18]**

Consider a Hamiltonian with general mixture $\xi(t)$. $q$ is good if and only if

$$F(\beta) = \beta E_\star(q) + \frac{1}{2}\log(1-q) + F(\beta, q).$$

If $q$ is not good, same holds with inequality $>$.

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For the maximal good $q$, (so-called **Onsager correction**)

$$F(\beta, q) = \frac{1}{2}\beta^2\Big(\xi(1) - \xi(q) - (1-q)\xi'(q)\Big).$$

# Previous works

The 'classical' formula with $\boxed{\text{Onsager correction}}$ was proved for:

- **S. '17:** spherical pure $p$-spin with $p \geq 3$ and $\beta \gg 1$.
- **Ben Arous-S.-Zeitouni '18:** same as above, for mixed models 'close' to pure.
- **Belius-Kistler '18:** spherical pure 2-spin.
- **Chen-Panchenko '17:** general mixed models, Ising spins, a similar, but more complicated formula.

The general formula with $\boxed{\text{with good } q}$ proved for:

- **Chen-Panchenko-S. '18:** general mixed models with Ising spins.

# Computing the free energy from the TAP representation for pure models

# TAP representation for pure models

Note that by Jensen's inequality, always

$$F(\beta) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z_{N,\beta} \le \lim_{N\to\infty} \frac{1}{N}\log \mathbb{E} Z_{N,\beta} = \frac{1}{2}\beta^2\xi(1).$$

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For any spherical model, there exists a critical $\beta_c > 0$ such that

$$F(\beta) = \frac{1}{2}\beta^2\xi(1) \iff \beta \le \beta_c,$$
$$F(\beta) < \frac{1}{2}\beta^2\xi(1) \iff \beta > \beta_c.$$

# TAP representation for pure models

For the pure models, the Hamiltonian is homogeneous

$$H_N(\mathbf{x}) := \sqrt{N} \sum_{i_1,\dots,i_p=1}^{N} J_{i_1,\dots,i_p} x_{i_1} x_{i_2} \cdots x_{i_p}.$$

Therefore, $$E_\star(q) := \lim_{N\to\infty} \frac{1}{N} \max_{m \in \sqrt{q}\cdot\mathbb{S}^N} H_N(m) = q^{\frac{p}{2}} E_\star.$$

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The TAP representation becomes

$$\begin{aligned} q \text{ good}: \quad & F(\beta) = \beta E_\star q^{\frac{p}{2}} + \frac{1}{2}\log(1-q) + F(\beta, q), \\ q \text{ not good}: \quad & F(\beta) > \beta E_\star q^{\frac{p}{2}} + \frac{1}{2}\log(1-q) + F(\beta, q). \end{aligned}$$

# TAP representation for pure models

Denote by $q_c$ the maximal good overlap at $\beta_c$.

**Theorem (S. '21)**

For the spherical pure $p$-spin model with $p \geq 3$:

1. $q_c$ is the unique solution in (0,1) of
$$p(1-q)\log(1-q) + pq - (p-1)q^2 = 0.$$
2. The critical inverse-temperature is
$$\beta_c = \frac{q_c^{-\frac{p}{2}+1}}{\sqrt{p(1-q_c)}}.$$
3. The ground-state energy is
$$E_\star = \sqrt{\frac{p-1}{p}}\left(\sqrt{(p-1)(1-q_c)} + \frac{1}{\sqrt{(p-1)(1-q_c)}}\right).$$

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**Theorem (S. '21)**

For the spherical pure $p$-spin model with $p \ge 3$ and any $\beta \ge \beta_c$, the maximal good overlap $q$ is the larger of the two solutions in $(0,1)$ of

$$q^{\frac{p}{2}-1}(1-q) = \frac{1}{\beta\sqrt{p(p-1)}}\left(\frac{E_\star}{E_\infty} - \sqrt{\frac{E_\star^2}{E_\infty^2} - 1}\right).$$

With the same $q$, the free enegy is

$$F(\beta) = \beta E_\star q^{\frac{p}{2}} + \frac{1}{2}\log(1-q) + \frac{1}{2}\beta^2\Big(1 - q^p - p(1-q)q^{p-1}\Big).$$

# Multi-species models

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---

$p = (p(s))_{s \in \mathscr{S}}, \qquad p(s) \in \mathbb{Z}_+, \qquad |p| := \sum_{s \in \mathscr{S}} p(s).$

**Hamiltonian:** $\quad H_N(\mathbf{x}) = H_{N,p}(\mathbf{x}) = C_{N,p} \sum J_{i_1, \ldots, i_{|p|}} x_1 \cdots x_{|p|}$

$J_{i_1, \ldots, i_{|p|}}$ - iid normal variables.

Sum over indices s.t. for any $s \in \mathscr{S}$, $\quad \#\{j : i_j \in I_s\} = p(s)$.

# Multi-species mixed $p$-spin models

Same configuration space, and

$$\boxed{H_N(\boldsymbol{\sigma}) = \sum_p \Delta_p H_{N,p}(\mathbf{x}),}$$

for some numbers $\Delta_p \geq 0$.

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**Covariance function:** define

$$R_s(\mathbf{x}, \mathbf{y}) = \frac{N}{|I_s|} \sum_{i \in I_s} x_i y_i \in [-1, 1],$$

$$\xi(x) = \sum_p \Delta_p^2 \prod_{s \in \mathscr{S}} x(s)^{p(s)}.$$

Then,

$$\mathbb{E} H_N(\mathbf{x}) H_N(\mathbf{y}) = N\xi\big((R_s(\mathbf{x}, \mathbf{y}))_s\big).$$

# The free energy

**Free energy:**

$$F(\beta) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log \int_{S_N} e^{\beta H_N(\mathbf{x})} d\mu(\mathbf{x}),$$

where $\mu$ is the product of uniform measures on each of the $|\mathscr{S}|$ spheres.

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**Barra, Contucci, Mingione and Tantari '15** and **Panchenko '15** proved a Parisi formula for the free energy for multi-species Ising models.

More recently, **Bates and Sohn '21** proved a Parisi formula for the multi-species spherical models.

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The upper bound in both cases assumes that $\xi(x)$ is a convex function.

For the pure models, $\xi(x)$ is concave everywhere in $x$!

However, the computation from the TAP representation still works in the multi-species case.

# The free energy

Let $p = (p(s))_{s \in \mathscr{S}} \in \mathbb{Z}_+$ and $q = (q(s))_{s \in \mathscr{S}} \in [0,1]^{\mathscr{S}}$.

Define

$$V(q) := -\sum_{s \in \mathscr{S}} \lambda(s) \log(1 - q(s)),$$
$$U(q) := 1 + \sum_{s \in \mathscr{S}} p(s) \frac{1 - q(s)}{q(s)},$$

and

$$\Phi(q) := \frac{V(q)}{U(q)},$$
$$\Omega(q) := V(q)U(q).$$

# The free energy

**Theorem (S. '21)**

For the multi-species spherical pure $p$-spin model with $|p| \geq 3$:$^\star$

(1) At $\beta_c$ there is a unique maximal good $q_c \in (0,1)^{\mathscr{S}}$ and it is equal to the unique solution of

$$\forall s \in \mathscr{S} : \quad \frac{\lambda(s)}{p(s)} \frac{q(s)^2}{1 - q(s)} = \Phi(q).$$

(2) The critical inverse-temperature is given by

$$\beta_c = \sqrt{\frac{\Phi(q_c)}{\xi(q_c)}}.$$

(3) The ground-state is given by

$$E_\star = \sqrt{\Omega(q_c)}.$$

$\star$ Assuming the convergence of the free energy.

# The free energy

**Theorem (S. '21)**

For the multi-species spherical pure $p$-spin model with $|p| \geq 3$ and $\beta > \beta_c$: there is a unique maximal good $q$ and it is defined by

$$\forall s \in \mathscr{S} : \quad \frac{1-q(s)}{q(s)} = \frac{-\sqrt{\Phi(q_c)} + \sqrt{\Phi(q_c) + 4\frac{\lambda(s)}{p(s)}}}{2y_\star(\beta)},$$

where $y_\star(\beta)$ is the larger of the two solutions in $(0, \infty)$ of

$$y^2 \prod_{s \in \mathscr{S}} \left( \frac{-\sqrt{\Phi(q_c)} + \sqrt{\Phi(q_c) + 4\frac{\lambda(s)}{p(s)}}}{2y} + 1 \right)^{p(s)} = \beta^2.$$

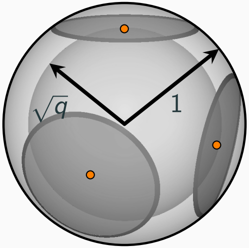
With the same $q$,

$$F(\beta) = \beta\sqrt{\xi(q)}E_\star + \frac{1}{2}\sum_{s \in \mathscr{S}} \lambda(s) \log(1 - q(s)) + \frac{1}{2}\beta^2 \left( \xi(1) - \xi(q) - \xi(q) \sum_{s \in \mathscr{S}} p(s) \frac{1-q(s)}{q(s)} \right).$$

# Free energy landscapes

## Free energy landscape I

Similarly to the approach of TAP, we wish to associate to each point $m$ inside the sphere a free energy.



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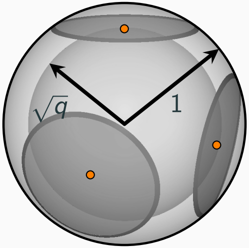


We first associate to $m$ the subset

$$\mathrm{Band}(m) = \left\{\mathbf{x} \in \mathbb{S}^N : |(\mathbf{x} - m)\cdot m| < \delta_N\right\}.$$

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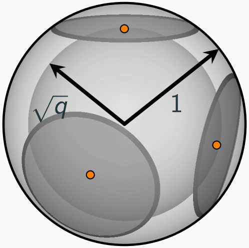


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And then define

$$\boxed{F_{N,\beta}(m) = \frac{1}{N} \log \int_{\mathrm{Band}(m)} e^{\beta H_N(\mathbf{x})} d\mathbf{x}.}$$

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Can we find a meaningful characterization for points such that

$$F_{N,\beta}(m) \approx F_{N,\beta}?$$

That is, points such that computing the free energy over the band roughly gives the same result as computing it over the whole sphere.

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Unfortunately, the set of such points is 'too large' to work with...

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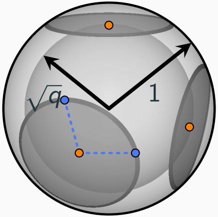


$$\mathrm{Band}(m,k,\epsilon) := \big\{(\mathbf{x}_1,\ldots,\mathbf{x}_k) \in \mathrm{Band}(m)^k : \forall i \neq j,\ |(\mathbf{x}_i - m)\cdot(\mathbf{x}_j - m)| < \epsilon\big\},$$

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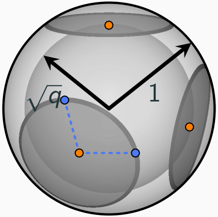


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Note that

$$F_{N,\beta}(m,k,\epsilon) \le F_{N,\beta}(m) \le F_{N,\beta}.$$

# Why work with orthogonal replicas?

Define the centered versions by replacing $H_N(\mathbf{x})$ by $H_N(\mathbf{x}) - H_N(m)$:

$$F^{\mathrm{c}}_{N,\beta}(m) = \frac{1}{N}\log\int_{\mathrm{Band}(m)} e^{\beta(H_N(\mathbf{x}) - H_N(m))}d\mathbf{x},$$

and similarly define $F^{\mathrm{c}}_{N,\beta}(m, k_N, \epsilon_N)$, so that

$$F_{N,\beta}(m) = \frac{\beta}{N}H_N(m) + F^{\mathrm{c}}_{N,\beta}(m),$$

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Let $k_N \to \infty$ and $\epsilon_N \to 0$, going the their limit slowly.

**Theorem (S. '18)**

**Uniform** concentration of the **centered** free energy:

$$\max_{m:\, \|m\|<1} \left| F^{\mathrm{c}}_{N,\beta}(m, k_N, \epsilon_N) - \mathbb{E} F^{\mathrm{c}}_{N,\beta}(m, k_N, \epsilon_N) \right| \to 0 \ \text{ a.s.}$$

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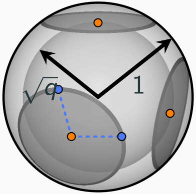
Recall that for general $m$,

$$F_{N,\beta}(m, k_N, \epsilon_N) \leq F_{N,\beta}.$$



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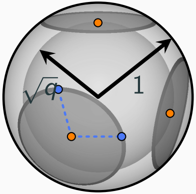


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$$\exists m_\star \in \sqrt{q}\cdot\mathbb{S}^N : F_{N,\beta}(m_\star, k_N, \epsilon_N) = F_{N,\beta} + o(1).$$

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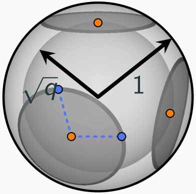


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We can substitute $F_{N,\beta}(m, k_N, \epsilon_N) = \frac{\beta}{N} H_N(m) + F^{\mathrm{c}}_{N,\beta}(m, k_N, \epsilon_N)$,

and use **concentration** to add expectations:

$$\forall m \in \sqrt{q} \cdot \mathbb{S}^N : \quad \frac{\beta}{N} H_N(m) + \mathbb{E} F^{\mathrm{c}}_{N,\beta}(m, k_N, \epsilon_N) \le \mathbb{E} F_{N,\beta},$$

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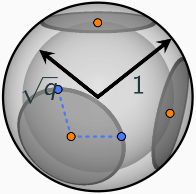


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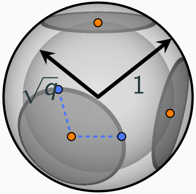


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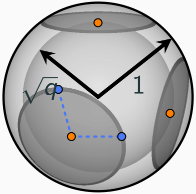


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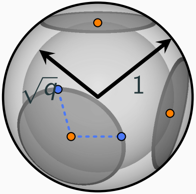
# Why work with orthogonal replicas?

Recall that for general $m$,

$$F_{N,\beta}(m, k_N, \epsilon_N) \leq F_{N,\beta}.$$

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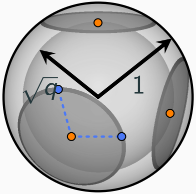
# TAP representation

We proved a representation for the free energy:

if $q$ is a good overlap, for arbitray $m \in \sqrt{q} \cdot \mathbb{S}^N$,

$$\boxed{F(\beta) = \beta E_\star(q) + \lim_{N\to\infty} \mathbb{E} F^{\mathrm{c}}_{N,\beta}(m, k_N, \epsilon_N).}$$



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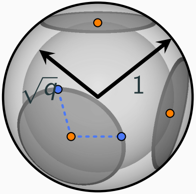


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Recall that we work with a thin band, approximately a sphere.

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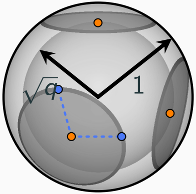


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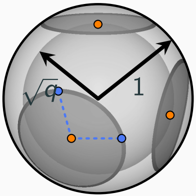


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We can map the band to the sphere of radius 1.

This mapping gives rise to the entropy term

$$\frac{1}{2}\log(1-q) = \lim_{N\to\infty} \frac{1}{N} \log \mathrm{Vol}(\mathrm{Band}(m)),$$

and (after several additional steps) to the last term in the representation

$$F(\beta, q),$$

corresponding to the mixture $\xi_q(t) = \xi\big((1-q)t+q\big) - \xi(q) - \xi'(q)(1-q)t.$

# Proof sketch: free energy of pure $p$-spin

# TAP representation for pure models

For the pure models, the Hamiltonian is homogeneous

$$H_N(\mathbf{x}) := \sqrt{N} \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} x_{i_1} x_{i_2} \cdots x_{i_p}.$$

Therefore, $E_\star(q) := \lim_{N\to\infty} \frac{1}{N} \max_{m\in\sqrt{q}\cdot\mathbb{S}^N} H_N(m) = q^{\frac{p}{2}} E_\star.$

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The TAP representation becomes

$$q \text{ good} : \quad F(\beta) = \beta E_\star q^{\frac{p}{2}} + \frac{1}{2}\log(1-q) + F(\beta, q),$$

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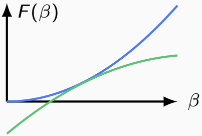


At any good $q > 0$, derivatives in $\beta$ or $q$ of both sides are equal.

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Denote by $q_c$ the maximal good overlap at $\beta_c$.

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From the TAP representation (with the Onsager correction),

$$\frac{1}{2}\beta_c^2 = F(\beta_c) = \beta_c E_\star q_c^{\frac{p}{2}} + \frac{1}{2}\log(1-q_c) + \frac{1}{2}\beta_c^2\Big(1 - q_c^p - p(1-q_c)q_c^{p-1}\Big). \tag{Eq. I}$$

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From equality of derivatives in $q$,

$$0 = \frac{d}{dq}F(\beta_c) = \beta_c E_\star \frac{p}{2} q_c^{\frac{p}{2}-1} - \frac{1}{2}\frac{1}{1-q_c} - \frac{1}{2}\beta_c^2 p(p-1)(1-q_c)q_c^{p-2}. \tag{Eq. III}$$

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We got <u>three equations</u> in <u>three variables</u> $\beta_c$, $E_\star$ and $q_c$.
Solving them yields the first theorem we saw.

# TAP representation for pure models

Denote by $q_c$ the maximal good overlap at $\beta_c$.

**Theorem (S. '21)**

For the spherical pure $p$-spin model with $p \geq 3$:

1. $q_c$ is the unique solution in $(0,1)$ of
$$p(1-q)\log(1-q) + pq - (p-1)q^2 = 0.$$
2. The critical inverse-temperature is
$$\beta_c = \frac{q_c^{-\frac{p}{2}+1}}{\sqrt{p(1-q_c)}}.$$
3. The ground-state energy is
$$E_\star = \sqrt{\frac{p-1}{p}}\left(\sqrt{(p-1)(1-q_c)} + \frac{1}{\sqrt{(p-1)(1-q_c)}}\right).$$

# TAP representation for pure models

For $\beta > \beta_c$, we only have the last equation: if $q$ is the maximal good overlap then

$$0 = \frac{d}{dq}F(\beta) = \beta E_\star \frac{p}{2} q^{\frac{p}{2}-1} - \frac{1}{2}\frac{1}{1-q} - \frac{1}{2}\beta^2 p(p-1)(1-q)q^{p-2}. \tag{Eq. III}$$

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But $\beta$ is a given, and we already know the value of $E_\star$.

Solving for $q$ we get 4 possible solutions.
We prove that the maximal good overlap can only be equal to one of them.

This gives the value of the maximal good overlap. Substituting back to the TAP representation gives the free energy, and the second theorem we saw.

# TAP representation for pure models

For $\beta \leq \beta_c$, $F(\beta) = \frac{1}{2}\beta^2$. For $\beta > \beta_c$, the free energy is given by the following.

# TAP representation for pure models

For $\beta \leq \beta_c$, $F(\beta) = \frac{1}{2}\beta^2$. For $\beta > \beta_c$, the free energy is given by the following. Define $E_\infty = 2\sqrt{\frac{p-1}{p}}$.

**Theorem (S. '21)**

For the spherical pure $p$-spin model with $p \geq 3$ and any $\beta \geq \beta_c$, the maximal good overlap $q$ is the larger of the two solutions in $(0, 1)$ of

$$q^{\frac{p}{2}-1}(1-q) = \frac{1}{\beta\sqrt{p(p-1)}}\left(\frac{E_\star}{E_\infty} - \sqrt{\frac{E_\star^2}{E_\infty^2} - 1}\right).$$

With the same $q$, the free enegy is

$$F(\beta) = \beta E_\star q^{\frac{p}{2}} + \frac{1}{2}\log(1-q) + \frac{1}{2}\beta^2\Big(1 - q^p - p(1-q)q^{p-1}\Big).$$

# Thank You!