# Smooth Gaussian Fields

Dmitry Beliaev (Oxford)

joint work with
Michael McAuley (Helsinki)
Stephen Muirhead (Melbourne)

Bangalore Probability Seminar

# Gaussian fields are everywhere: Oceanography

[ 321 ]

## THE STATISTICAL ANALYSIS OF A RANDOM, MOVING SURFACE

By M. S. LONGUET-HIGGINS

*National Institute of Oceanography, Wormley*

(*Communicated by G. E. R. Deacon, F.R.S.—Received* 29 *March* 1956—
*Revised* 31 *July* 1956)

The following statistical properties are derived for a random, moving, Gaussian surface: (1) the probability distribution of the surface elevation and of the magnitude and orientation of the gradient; (2) the average number of zero-crossings per unit distance along a line in an arbitrary direction; (3) the average length of the contours per unit area, and the distribution of their direction; (4) the average density of maxima and minima per unit area of the surface, and the average density of specular points (i.e. points where the two components of gradient take given values); (5) the probability distribution of the velocities of zero-crossings along a given line; (6) the probability distribution of the velocities of contours and of specular points; (7) the probability distribution of the envelope and phase angle, and hence (8) when the spectrum is narrow, the probability distribution of the heights of maxima and minima and the distribution of the intervals between successive zero-crossings along an arbitrary line. All the results are expressed in terms of the two-dimensional energy spectrum of the surface, and are found to involve the moments of the spectrum up to a finite order only. (1), (3), (4), (5) and (6) are discussed in detail for the special case of a narrow spectrum.

The converse problem is also studied and solved: given certain statistical properties of the surface, to find a convergent sequence of approximations to the energy spectrum.

The problems arise in connexion with the statistical analysis of the sea surface.

(More detailed summaries are given at the beginning of each part of the paper.)

# Gaussian fields are everywhere: Oceanography

JOURNAL OF THE OPTICAL SOCIETY OF AMERICA VOLUME 44, NUMBER 11 NOVEMBER, 1954

## Measurement of the Roughness of the Sea Surface from Photographs of the Sun's Glitter

CHARLES COX AND WALTER MUNK
*Scripps Institution of Oceanography,* La Jolla, California*

(Received April 28, 1954)

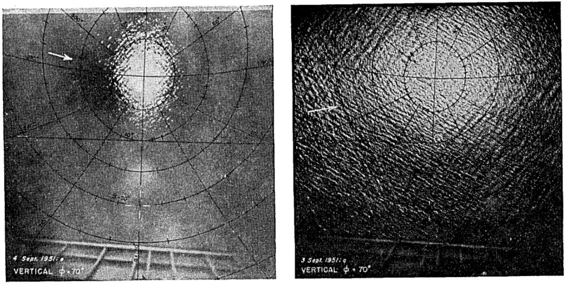
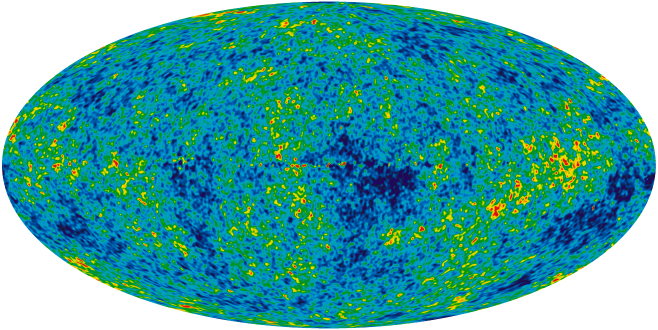



FIG. 1. Glitter patterns photographed by aerial camera pointing vertically downward at solar elevation of $\phi=70°$. The superimposed grids consist of lines of constant slope azimuth $\alpha$ (radial) drawn for every 30°, and of constant tilt $\beta$ (closed) for every 5°. Grids have been translated and rotated to allow for roll, pitch, and yaw of plane. Shadow of plane can barely be seen along $\alpha=180°$ within white cross. White arrow shows wind direction. *Left:* water surface covered by natural slick, wind 1.8 m sec$^{-1}$, rms tilt $\sigma=0.0022$. *Right:* clean surface, wind 8.6 m sec$^{-1}$, $\sigma=0.045$. The vessel *Reverie* is within white circle.

# Gaussian fields are everywhere: Cosmology



The cosmic microwave background (CMB, CMBR)

## Gaussian fields are everywhere: Laplace eigenfunctions

In 1977 M. Berry conjectured that high energy eigenfunctions in the chaotic case have statistically the same behaviour as random plane waves. (Figures from Bogomolny-Schmit paper)

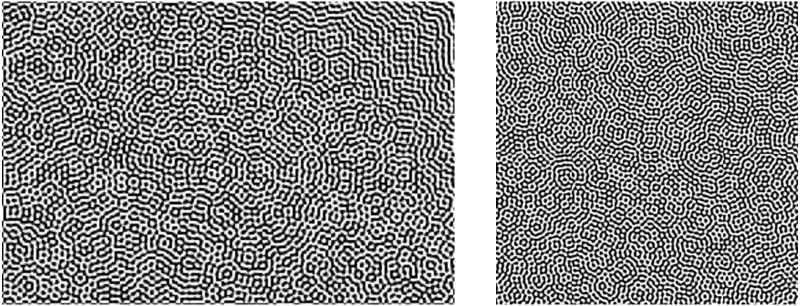


Figure: Nodal domains of an eigenfunction (left) of a stadium and of a random plane wave (right)

# Gaussian functions and fields

Two different perspectives

- Analytic: Random series A field is the white noise in some Hilbert space. Take any orthonormal basis $\{\phi_i\}$ in some functional Hilbert space $H$ and define

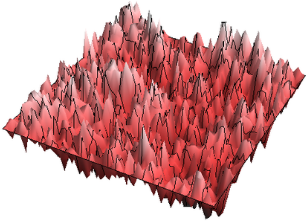
$$f = \sum a_i \phi_i, \qquad a_i \text{ i.i.d. } N(0,1)$$

- Probabilistic: Collection of random variables A field $f(x)$ is a collection of jointly Gaussian random variables indexed by $x$. Could be defined by its covariance function $K(x,y) = \mathbb{E}\left[\Psi(x)\Psi(y)\right]$.

Covariance function

$$K(x,y) = \sum \phi_i(x)\phi_i(y)$$

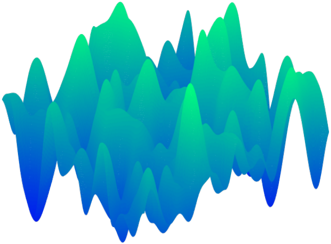
# Examples: Gaussian Free Field



Gaussian Free Field (GFF). Hilbert space is the Sobolev space $H_0^1(\Omega)$. The covariance kernel is the Green's function.

Note: GFF is not a function!

# Examples: Bargmann-Fock



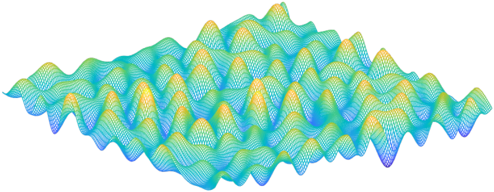
The space is the Bargmann-Fock space. Orthonormal basis

$$\psi_{n,m}(x) = \frac{1}{\sqrt{n!m!}} x_1^n x_2^m e^{-\frac{|x|^2}{2}}.$$

Covariance kernel

$$K(x,y) = e^{-\frac{|x-y|^2}{2}}.$$

# Examples: Random Plane Wave



The space is $\mathcal{F}^{-1}L^2(\mathbb{T}, d\theta)$. Orthonormal basis

$$\psi_n(re^{i\theta}) = \cos(n\theta)J_n(r), \qquad \phi_n(re^{i\theta}) = \sin(n\theta)J_n(r)$$

Covariance kernel

$$K(x, y) = J_0(|x - y|).$$

## Stationary Gaussian fields

If the field is stationary i.e. $K(x, y) = K(x - y)$ then by Bochner's Theorem

$$K(x) = \int e^{2\pi i x \cdot t} d\rho(t)$$

where $\rho$ is a symmetric probability measure. It is called the spectral measure of the field.

Properties of $f$, $H$, $K$, and $\rho$ are closely related. In particular, smoothness of $K$ at zero or finite moments of $\rho$ imply smoothness of $f$.

## Main questions

What can we say about large scale geometry and topology of level sets $\{f(x) = \ell\}$ and excursion sets $\{f(x) \geq \ell\}$?

Observables:

- Length
- Area
- Number
- Homology

## Main questions

What can we say about large scale geometry and topology of level sets $\{f(x) = \ell\}$ and excursion sets $\{f(x) \geq \ell\}$?

Observables:

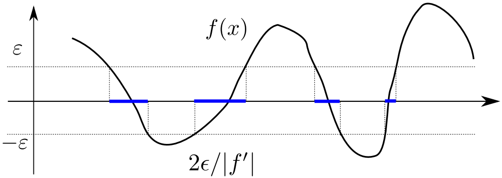
- Length
- Area
- Number
- Homology

Questions:

- Expectation
- Variance
- Central Limit Theorem

# Local quantities

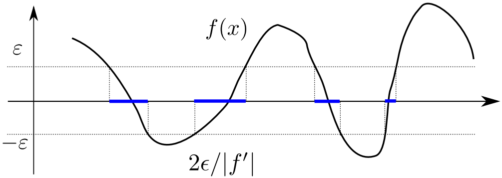
Many quantities like length, area and Euler characteristic are local.
They are easy.
Kac-Rice formula:




$$\#\{x \in [a,b] : f(x) = 0\} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_a^b |f'(x)| \mathbb{1}_{|f(x)| \leq \epsilon} dx$$

# Local quantities

Many quantities like length, area and Euler characteristic are local.
They are easy.
Kac-Rice formula:




$$\mathbb{E}\#\{x \in [a,b] : f(x) = 0\} = \int_a^b \phi_f(0)\mathbb{E}[|f'(x)| \mid f(x) = 0]dx$$

where $\phi_f(0)$ is the probability density of $f$.
This expectation is explicitly computable for Gaussian fields!

# Non-local quantities are hard!

The main example of a non-local quantity is the number of components. There is no integral formula!

**Theorem (Nazarov-Sodin)**

*Under mild regularity conditions (smoothness, non-degeneracy, ergodicity) for a stationary field $f$ in $\mathbb{R}^n$*

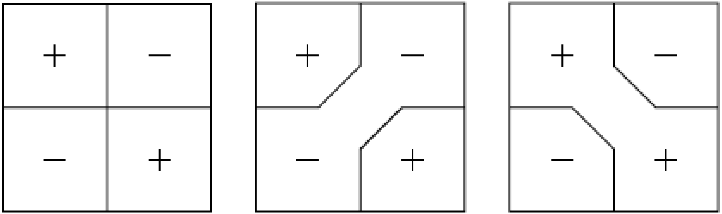
$$\frac{N(f, R\cdot\Omega)}{R^n \mathrm{Vol}(\Omega)} \to c, \quad R \to \infty$$

*where $N(f,\Omega)$ is the number of level/excursion sets of $f$ inside $\Omega$.*

Similar statements are true for the number of nodal domains of given area, perimeter, topological type etc (B., Sarnak, Wigman ...)

# Bogomolny-Schmit Percolation Model

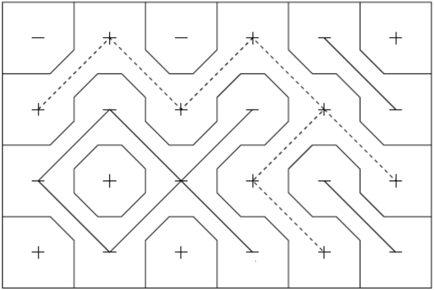
Bogomolny-Schmit proposed that the nodal lines of the random plane wave form a perturbed square lattice




Picture from Bogomolny-Schmit paper.

## Bogomolny-Schmit Percolation Model

Using this analogy we can think of the nodal domains as percolation clusters on the square lattice.



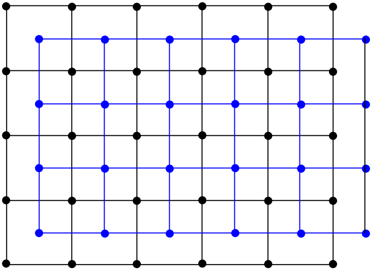
This leads to the conjecture that

$$\mathbb{E}(N(f, R\Omega)) = R^2 \mathrm{Area}(\Omega) \frac{3\sqrt{3}-5}{4\pi^2}$$

$$\mathrm{Var}(N(f, \Omega)) \approx \mathrm{const}\, R^2.$$

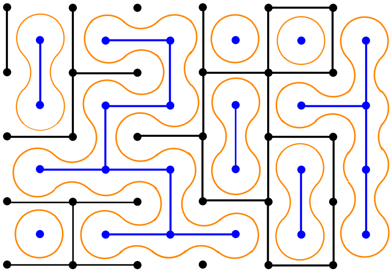
# Critical Square Lattice Bond Percolation

Each edge of the lattice is preserved with probability $p_c = 1/2$. If an edge is preserved, then the dual edge is removed and vice versa. Primal and dual clusters create an loop model of interfaces.



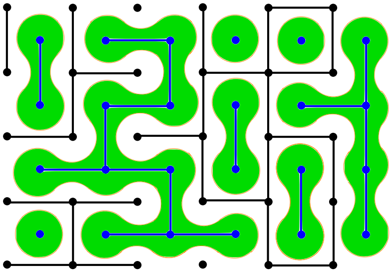
# Critical Square Lattice Bond Percolation

Each edge of the lattice is preserved with probability $p_c = 1/2$. If an edge is preserved, then the dual edge is removed and vice versa. Primal and dual clusters create an loop model of interfaces.



# Critical Square Lattice Bond Percolation

Each edge of the lattice is preserved with probability $p_c = 1/2$. If an edge is preserved, then the dual edge is removed and vice versa. Primal and dual clusters create an loop model of interfaces.



# Off-critical Percolation

Off-critical percolation is a model for excursion and level sets

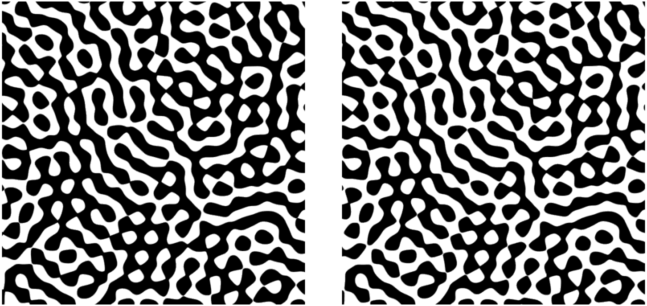


Figure: Excursion sets for levels 0 (nodal domains) and level 0.1

# Off-critical Percolation

Off-critical percolation is a model for excursion and level sets

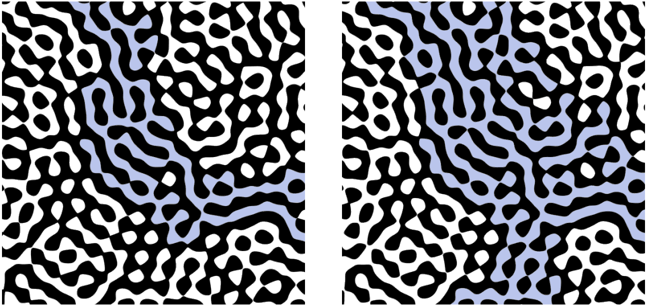


Figure: Excursion sets for levels 0 (nodal domains) and level 0.1

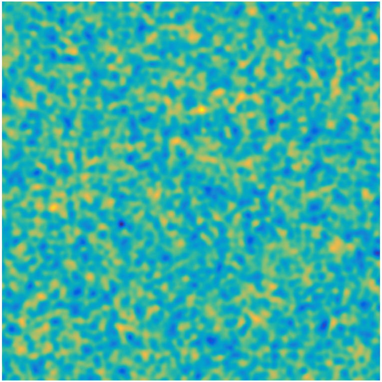
## Generalized Bogomolny-Schmit conjecture

For a wide class of Gaussian fields the nodal domains behave like the critical percolation. Other excursion sets behave like off-critical percolation.

Assumptions:

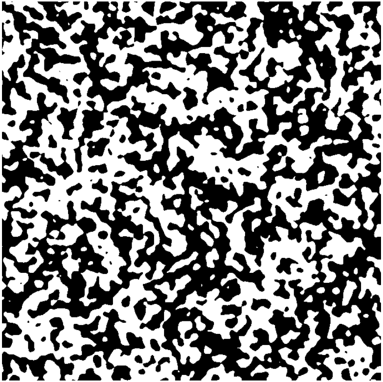
- Smooth (nodal lines are nice curves)
- Stationary (percolation is almost stationary)
- Isotropic or symmetric enough (uniform conformal structure)
- Weakly correlated (percolation is local)

# A Good Example: Bargmann-Fock field



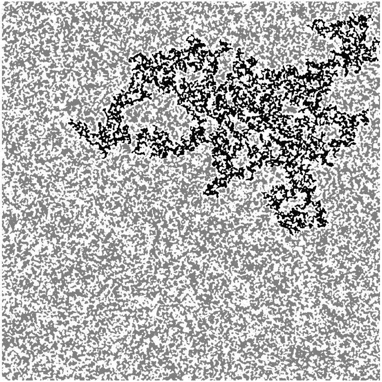
Bargmann-Fock field heat-map

# A Good Example: Bargmann-Fock field



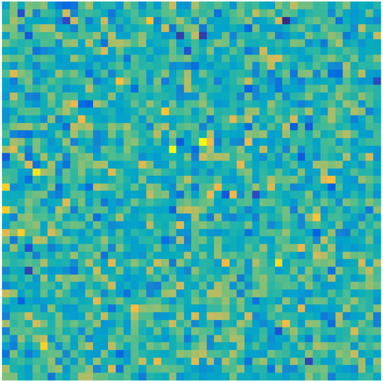
Nodal domains

# A Good Example: Bargmann-Fock field



Nodal domains with highlighted largest domain

# A Bad Example: discrete white noise



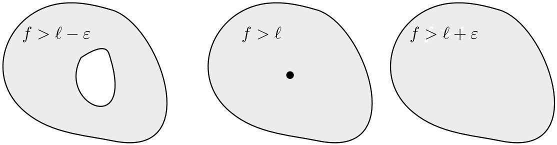
Nodal domains are exactly Bernoulli site percolation clusters with $p = 1/2$ which is not critical.

# A Bad Example: discrete white noise

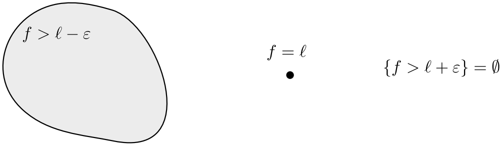
Nodal domains are exactly Bernoulli site percolation clusters with $p = 1/2$ which is not critical.

# Back to the number of level/excursion sets

Morse theory caricature



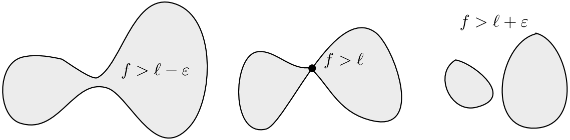

Local minimum



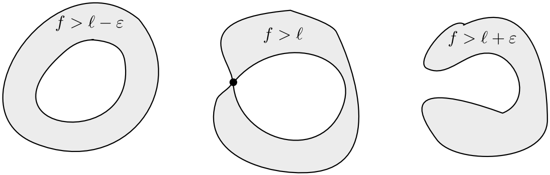

Local maximum

# Back to the number of level/excursion sets

Morse theory caricature




Lower connected saddle




Upper connected saddle

# Number of level/excursion sets

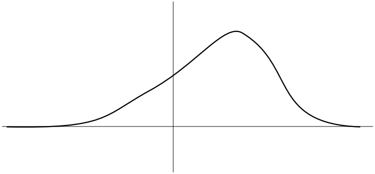
**Theorem (B.-McAuley-Muirhead)**

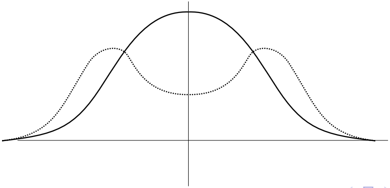
*Under minimal assumptions*

$$c_{ES}(\ell) = \int_{\ell}^{\infty} p_{m^+}(x) - p_{s^-}(x) dx,$$
$$c_{LS}(\ell) = \int_{\ell}^{\infty} p_{m^+}(x) - p_{s^-}(x) + p_{s^+}(x) - p_{m^-}(x) dx$$

*where $c_{ES}(\ell)$ and $c_{LS}(\ell)$ are the densities (per volume) of the number of excursion/level sets, $p_{m^+}(\ell)$, $p_{m^-}(x)$, $p_{s^+}(\ell)$ and $p_{s^-}(\ell)$ are densities of the number of local maxima, minima, upper and lower connected saddles at level $\ell$.*

# Leveel/excursion set functionals



# Back to the number of level/excursion sets

**Theorem (B.-McAuley-Muirhead)**

*If we additionally assume positive 'one-arm' exponent and non-vanishing spectral density in a neighbourhood of the origin then* $p_{m^+}(\ell)$ *and* $p_{s^-}(\ell)$ *are continuous and*

$$c'_{ES}(\ell) = p_{m^+}(\ell) - p_{s^-}(\ell).$$

Note: assumptions hold for BF but not for RPW

Under other assumptions (which include RPW) we show that continuous differentiability of $c_{ES}$ is equivalent to 'no infinite four arm saddle'.

Proof is based on analysis of the field conditioned to have a critical point at a given point at given level and Morse theory type perturbation analysis.

## Back to the number of level/excursion sets

**Theorem (B.-McAuley-Muirhead)**

*If we assume mild regularity of $f$, decay of correlations and non-vanishing spectral density in a neighbourhood of the origin then* $\mathrm{Var}[N_{ES}(B(R),\ell)] \gtrsim R^2$ *provided* $c'_{ES} \neq 0$. *Similar result holds for* $N_{LS}(\ell)$.

Note: assumptions hold for BF but not for RPW.

Note: $c'_{LS}(0) = 0$. Result is not applicable for nodal domains.

**Theorem (B.-McAuley-Muirhead)**

*For the random plane wave field* $\mathrm{Var}[N_{ES}(B(R),\ell)] \gtrsim R^3$ *provided* $c'_{ES} \neq 0$ *and* $\ell \neq 0$. *Similar result holds for* $N_{LS}(\ell)$.

## Complementary result

**Theorem (Nazarov-Sodin 2020)**

*Let $f$ be a sufficiently smooth stationary Gaussian field with polynomial decay of correlations, then there is $\sigma > 0$ such that*

$$\mathrm{Var}(N(B(R), 0) \gtrsim R^{\sigma}.$$

**Thank you!**