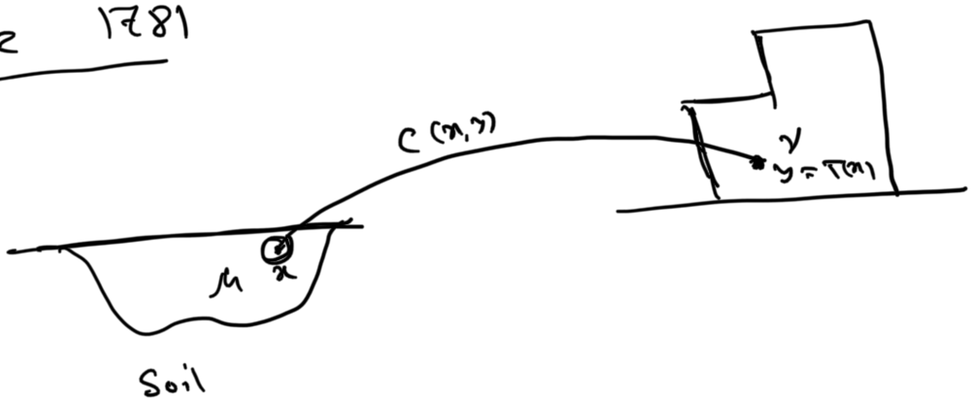
OT and Some applications to PDE

Monge 1781



μ, ν Borel measures $\mathbb{R}^n$

- $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n) = 1$
- $T: \mathbb{R}^n \to \mathbb{R}^n$ Borel
- $\nu(A) = \mu(T^{-1}(A))$ $\quad A \subseteq \mathbb{R}^n$

$T \# \mu = \nu$ push forward.

Equivalently

$$\int_{\mathbb{R}^n} \varphi(y)\, d\nu(y) = \int_{\mathbb{R}^n} \varphi(Tx)\, d\mu(x)$$

- Cost: $c: \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$

  $c(x,y)$ = cost of transporting one unit of mass from x to y.

- Total cost w.r.to T

$$I(T) = \int_{\mathbb{R}^n} \underbrace{\underbrace{c(x, Tx)}\, \underbrace{d\mu(x)}}, \quad T\#\mu = \nu$$

$$\min_{T\#\mu = \nu} I(T)$$

$$c(x,y) = |x-y| \quad - \quad \text{Monge}$$

(A)

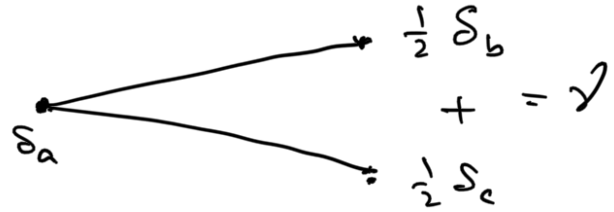
$$\boxed{\begin{aligned} & c(x,y) = \tfrac{1}{2}|x-y|^2 \\ & \int |x|^2\, d\mu + \underbrace{\int |y|^2\, d\nu}_{M_2(\nu)} < \infty \end{aligned}}$$

$$\tfrac{1}{2}|x-y|^2 \le (|x|^2 + |y|^2)$$

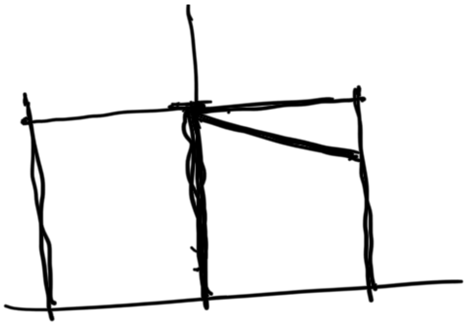
$$\int |x - Tx|^2\, d\mu(x) \le M_2(\mu) + M_2(\nu) < \infty.$$

Challenges:

1. $\mu = \delta_a \quad , \quad T\#\mu = \delta_{T(a)}$



2.

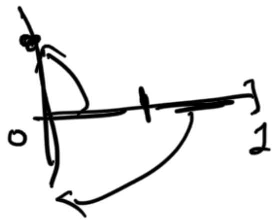


$\mathbb{R}^2$

$\mu = \mathcal{H}^1(\{0\}\times\mathbb{R})$

$\nu = \frac{1}{2}\mathcal{H}^1\{\{-1\}\times\mathbb{R}\} + \frac{1}{2}\mathcal{H}^1(\{1\}\times\mathbb{R})$

$\exists$ a $T\#\mu = \nu$ but no optimal one



3. $T\#\mu = \nu$

$$\int \varphi(y)\,d\nu(y) = \int \varphi(Tx)\,d\mu(x)$$

$$\mu = f\,dx, \quad \nu = g\,dy$$

$$\int \varphi(y)\,g(y)\,dy = \int \underline{\varphi(Tx)}\,f(x)\,dx$$

$y = Tx$

$$\parallel$$

$$\int \underline{\varphi(Tx)}\,\underbrace{g(Tx)\,|\det \nabla T(x)|}\,dx$$

$$g(Tx)\,|\det \nabla T(x)| = f(x)$$

$T\#\mu = \nu$

- $\cancel{\int |T_n|^2 f(x)\, dx}$ $\underbrace{\int |T_n(x)|\, f(x)\, dx < \infty}$

$T_n \# \mu = \nu$

$$T_n \in L^2(f\, dx)$$

- It is possible

$$T_n \# \mu = \nu$$

$$T_n \rightharpoonup T \text{ in } L^p \quad \forall\, p < \infty$$

but $\underline{T \# \mu \neq \nu}$.

Kantorovich: 1941

$$\Pi(\mu,\nu). \begin{cases} \pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n) \\ P_1(x,y) = x, \quad P_2(x,y) = y \\ P_1 \# \pi = \mu, \quad P_2 \# \pi = \nu \end{cases}$$

$$(KP) \qquad \inf_{\pi \in \Pi(\mu,\nu)} \iint c(x,y)\, d\underline{\pi}(x,y).$$

$\uparrow$ linear

π

- $\pi \mapsto \iint \underline{c}\, d\pi$ linear

$\cup\!\cup$

- $\Pi(\mu,\nu) \neq \emptyset$ $\quad \mu\times\nu \in \Pi(\mu,\nu)$

  plans

- $\Pi(\mu,\nu)$ is compact (against C_b).

- If c is l.s.c. (≥ 0) then $\exists$ a minimizer.

~~MP~~

Relation

$$\underline{T \# \mu = \nu}$$

define $\pi = (Id, T)\#\mu \in \Pi(\mu,\nu)$

$$\int c(x, Tx)\, d\mu(x) = \iint c(x,y)\, d\pi(x,y)$$

$$(MP) \geq (KP)$$

- If π is optimal, &

  π is concentrated on a m'ble G.

  $(x_1, y_1) \in G$ then if $x_1 = x_2$ then

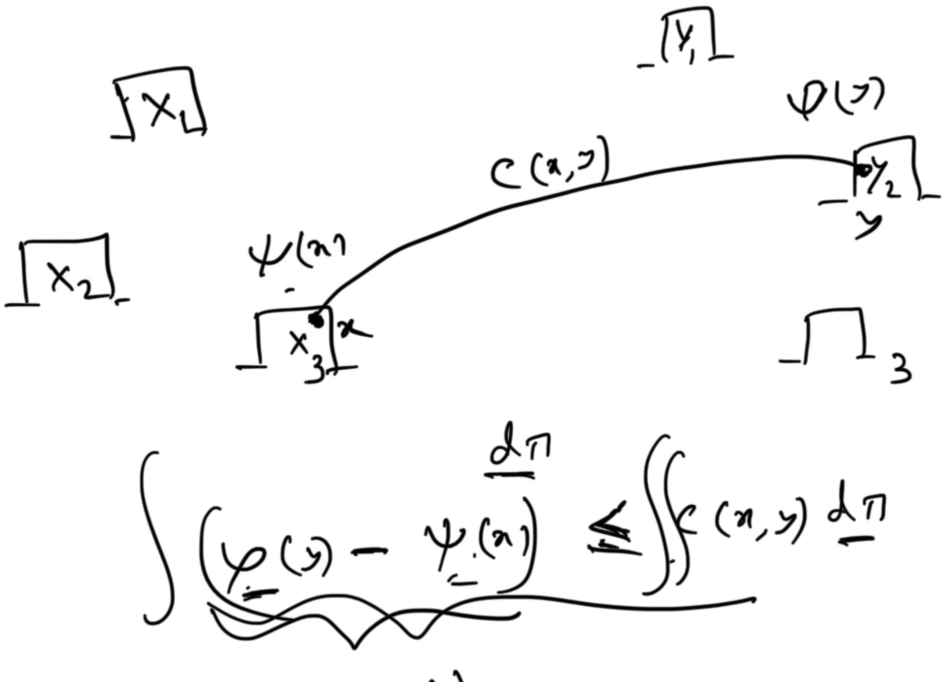
$(x_1, y_1) \in \mathcal{G}$ $\qquad y_1 = y_2$

$$T(x) = y \quad \text{s.t.} \quad (x,y) \in \mathcal{G}.$$

If an optimal π, concentrated on a graph $\mathcal{G}$ then (MP) admits a sol.

<u>K 1941</u>

(KP) is linear programing prob.



$c(x,y) = \frac{1}{2}|x-y|^2$

$$\int \underbrace{\left(\varphi(y) - \psi(x)\right) d\pi \le \int c(x,y)\, d\pi}$$

$$\|$$

$$\text{(DP)}\ \sup \left\{ \int \varphi(y)\, d\nu(y) - \int \psi(x)\, d\mu(x) : \varphi(y) - \psi(x) \le c(x,y) \right\}$$

$(DP) \le (KP)$

Kantorovich duality $(DP) = (KP)$

Let (φ, ψ) optimal

π optimal

$$\boxed{\varphi(y) - \psi(x) \le c(x,y) \quad \pi\text{. a.e. } (x,y)}$$

$\frac{1}{2}|x-y|^2$

$-\nabla\psi(x) = \frac{x-y}{|x-y|}$

$$c(x,y) = \frac{1}{2}|x-y|^2$$

$$= \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - x\cdot y$$

$$\iint c\, d\pi = \frac{1}{2} M_2(\mu) + \frac{1}{2} M_2(\nu) - \iint x\cdot y\, d\pi$$

$$\boxed{\varphi(y) - \psi(x) \le \frac{1}{2}|x|^2 + |y|^2 - x\cdot y}$$

$(DP) = (KP)$

$$\sup_{\pi\in\Pi(\mu,\nu)} \iint x\cdot y \, d\pi = \inf \left\{ \int \tilde\varphi \, d\nu + \int \tilde\psi \, d\mu \;:\; \tilde\varphi(y) + \tilde\psi(x) \ge x\cdot y \right\}$$

Let $(\tilde\varphi, \tilde\psi)$ is a sol. in DP

$$\tilde\varphi(y) \ge x\cdot y - \tilde\psi(x)$$

$$\tilde\varphi_1(y) = \sup\{x\cdot y - \tilde\psi(x)\} \le \tilde\varphi(y)$$

$$\sup\{x\cdot y - \tilde\psi(x)\} \to \tilde\psi^*(y)$$

$$\tilde\psi_1(x) = \sup\{x\cdot y - \tilde\varphi_1(y)\} = \tilde\varphi_1^*(x) = \tilde\psi^{**}$$

$(\tilde\psi_1, \tilde\varphi_1)$ is another sol of (DP).

$$\begin{gathered}(\tilde\psi_1, \tilde\varphi_1) \\ \| \\ (\tilde\psi^{**}, \tilde\psi^*)\end{gathered}$$

(φ, φ^*) is a sol. to (DP)

$\hookrightarrow \varphi$ convex $\Rightarrow$ Loc. Lips. $\Rightarrow$ diff a.e

- $\underbrace{\varphi(x) + \varphi^*(y) = x\cdot y}$ π a.e (x,y)

$$\boxed{\nabla\varphi(x) = y}$$

- $\nabla\varphi$ exists μ-a.e

If $\mu \ll \mathcal{L}$ $\Rightarrow$ $\nabla\varphi$ consists ...

<u>Brenier 91</u> $\left(c(x,y) = \frac{1}{2}|x-y|^2 \right)$.

If $\mu \ll \mathcal{L}^n$, then $\exists$ unique $\nabla\varphi$, where φ is convex and solves (MP).

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<u>Ref.</u> $\underbrace{\text{McCann, Gangbo}}_{c(x,y) = h(x-y),\ h \text{ strictly convex}}$, Caffarelli;

$c(x,y) = |x-y|$, $\begin{cases} \text{Sudakov 77, Evans - Gangbo 99, (PDE)} \\ \text{Ambrosio 01, Trudinger-Wang 01.} \\ |x-y|^p,\ p \geq 1 \end{cases}$

Champion et. all ~ 2011

<u>McCann:</u> 97.

$\mathcal{F}(\rho) = \int U(\rho) + \iint \rho(x) W(x-y) \rho(y)$

ρ^p, $p>1$

- W convex, $W(0)=0$, $W(x)=W(-x)$

$$\inf_{\int\rho=1} \tilde{F}(\rho) = \tilde{F}(\rho_0)$$

Is ρ_0 unique?

$t\rho_0+(1-t)\rho_1 = \rho_t$

$t \mapsto \tilde{F}(\rho_t)$ not convex

Displacement interpolation:

$$\int \rho_0 \, dx = \int \rho_1 \, dy = 1, \quad \rho_0, \rho_1 \geq 0$$

$$\exists \ \nabla\varphi \# \rho_0 = \rho_1$$

$$\rho_t = \big((1-t)\,Id + t\nabla\varphi\big) \# \rho_0$$

$t \mapsto \tilde{F}(\rho_t)$ convex.

Transport eqn:

$$\underline{v} : \mathbb{R}^n \times [0,T] \longrightarrow \mathbb{R}^n$$

$\underline{\rho_0}$

x, t

$$\begin{cases} \partial_t \rho(x,t) + \nabla \cdot (\rho\, \underline{v}) = 0 \\ \rho\big|_{t=0} = \rho_0 \end{cases}$$

$$K(\rho, v) = \frac{1}{2} \iint |v|^2 \rho \, dx\, dt$$

Benamou - Brenier 98

$$(MP) = \inf_{T\# \rho_0 = \rho_1} \frac{1}{2} \int |x - Tx|^2 \, d\mu(x)$$

$$= \underbrace{\inf \frac{1}{2} \iint |v|^2 \rho \; : \; \partial_t \rho + \nabla \cdot (v \rho) = 0, \quad \rho\big|_{t=0} = \rho_0, \ \rho\big|_{t=1} = \rho_1}$$

$$\rho = \delta_{x(t)} \qquad \begin{matrix} \rho_1, & \rho_2 \\ \Downarrow & \Downarrow \\ v_1 & v_2 \end{matrix} \qquad \underline{\iint \langle v_1, v_2 \rangle \rho}$$

$$\left\{ \begin{array}{l} \inf \left\{ \int |\gamma'(t)|^2 \, dt : \quad \gamma(0)=x_0, \ \gamma(1)=x_1 \right\} \\ = \quad |x_0 - x_1|^2 \end{array} \right.$$

- $\left(P_2(\mathbb{R}^n), \ d_W\right)$ is a complete M.S.

$$d_W(\mu,\nu) = (M\rho)^{1/2}$$

- BB gives a diff. structure

- $F(\rho) : P_2(\mathbb{R}^n) \to \mathbb{R}$

$$\nabla_W F(\rho) = -\nabla \cdot \left(\rho \nabla F'(\rho)\right)$$

- $F(\rho) = \int \rho \ln \rho$

$$\boxed{\partial_t \rho = -\nabla_W F(\rho) = \Delta \rho}$$

- Jordan - Kinderlehrer - Otto. 98.