Geometric statistics of clustering points.

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Joint work with:

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What if not independent ?

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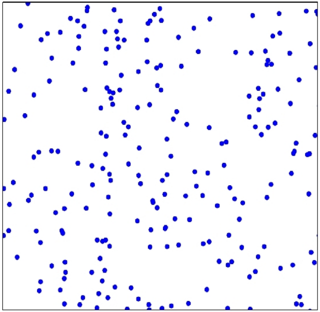

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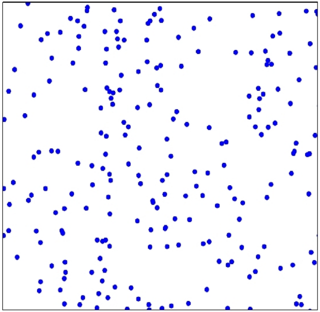

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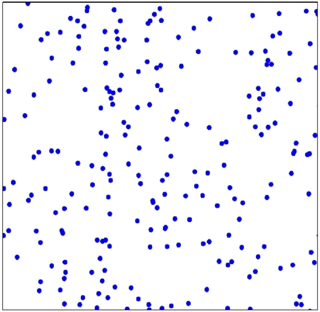

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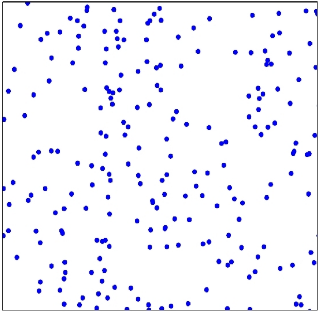

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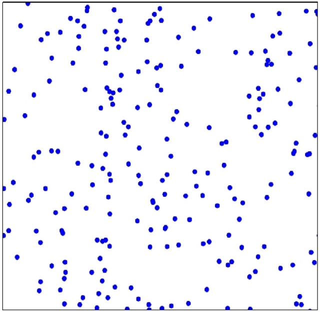

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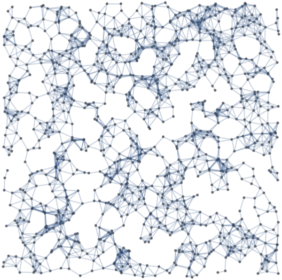

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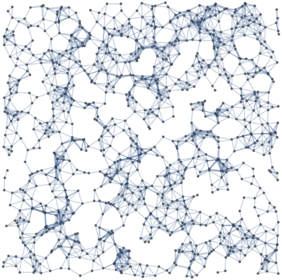

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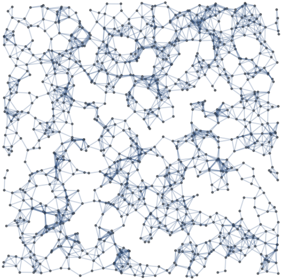

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$$= \sum^{\neq}_{(x_2,\ldots,x_k)\in\mathcal{P}^{k-1}} h(x_1,\ldots,x_k) = \sum^{\neq}_{(x_2,\ldots,x_k)\in\mathcal{P}^{k-1}} \frac{1[x_i \sim x_j \ \ \forall i, j]}{k!}.$$

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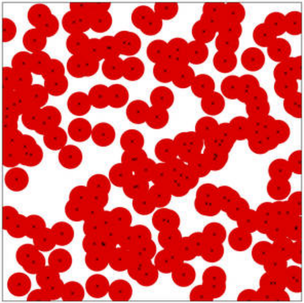

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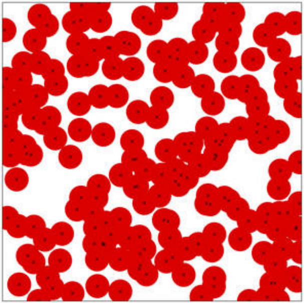

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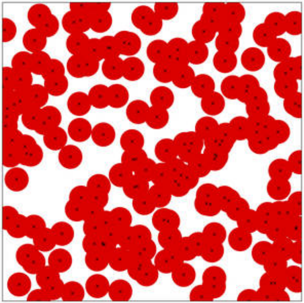

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- $H_n :=$ Intrinsic volume of $C_B(\mathcal{P}_n, r)$, $\mathcal{P}_n = \mathcal{P} \cap W_n$.

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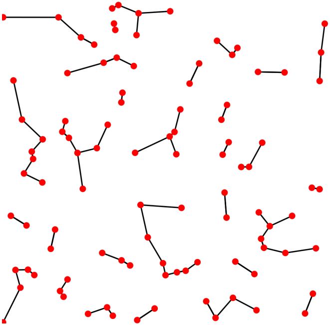

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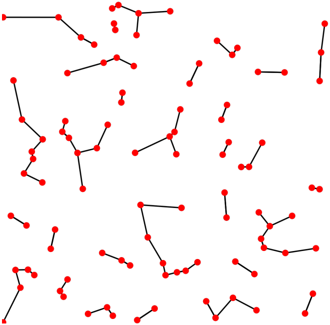

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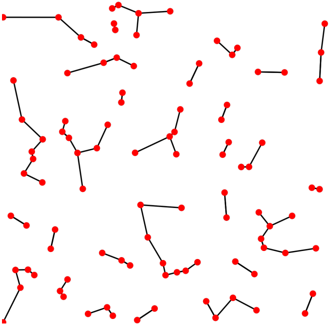

- $\xi(x, \mathcal{P})$- Sum of length of edges incident on x.
- $H_n = \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P})$ - Total edge-length of NNG on $\mathcal{P}_n$.

Point Processes

- **Point process** - locally finite random collection of points in $\mathbb{R}^d$.
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  - J. Yukich Limit theorems in discrete stochastic geometry,
  - G. Peccati & M. Reitzner Stochastic analysis for Poisson point processes
  - P. Calka Tessellations
  - Etc....

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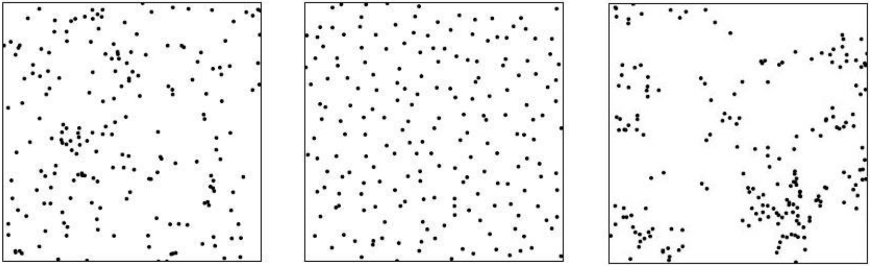
- Geometric statistics of general point processes ?

‘Not Poisson in Disguise’



Do not listen to the prophets of doom who preach that every point process will eventually be found out to be a Poisson process in disguise!” - G. C. Rota

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- $R(x, \mathcal{P}) = R(O, \mathcal{P} - x)$.
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Stabilizing functionals

- $H_n = \sum_{X \in \mathcal{P}_n} \xi(X, \mathcal{P})$
- Stabilizing: $\exists R(O, \mathcal{P}) = \inf\{r : ...\}$ a.s. finite, such that $\forall$ locally finite $A \subset B_r(O)^c$,

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- Examples 1 and 2 : $R(x, \mathcal{P}_{x_1,\ldots,x_p}) \leq 3r$ a.s. for any $\mathcal{P}$.

Clustering point processes

- 'Clustering' - Borrowed from Statistical Physics.

Clustering point processes

- *k*-correlation functions : $\rho^{(k)}(x_1, \ldots, x_k)$ -

$$\mathsf{E}\left(\prod_{i=1}^{k} \mathcal{P}(B_i)\right) = \int_{\prod_{i=1}^{k} B_i} \rho^{(k)}(x_1, \ldots, x_k) \mathrm{d}x_1 \ldots \mathrm{d}x_k .$$

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- *U*-statistics :

$$\xi(x,\mathcal{P}) := \sum_{X_1,\dots,X_{k-1}\in\mathcal{P}\cap B_r(x)}^{\neq} h(x,X_1,\dots,X_{k-1}).$$

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- Examples of Scores : Intrinsic Volumes (Example 2), Edge-length in NNG (Example 3).

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Proof Sketch : CLT

- Mixed moments:

$$m_k(x_1, \ldots, x_k) := \mathsf{E}\Big(\prod_{i=1}^{k} \xi(x_i, \mathcal{P}_{x_1,\ldots,x_k})\Big)\rho^{(k)}(x_1, \ldots, x_k).$$

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- Two proofs - Generalizing both Baryshnikov-Yukich and Nazarov -Sodin.

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- Growth rates of $\tilde{C}_k, \tilde{c}_k$ (**?**) $\Rightarrow$ Moderate deviations, Law of iterated logarithms, Berry-Esseen bounds.

References

- B. Błaszczyszyn, DY and J. E. Yukich (2016), Limit theory for geometric statistics of clustering point processes, *arXiv :1606.03988*
- Ph. A. Martin and T. Yalcin (1980), The charge fluctuations in classical Coulomb systems, *J. Statist. Phys.*.
- Yu. Baryshnikov and J. E. Yukich (2005), Gaussian limits for random measures in geometric probability. *Ann. Appl. Prob.*.
- F. Nazarov and M. Sodin (2012), Correlation functions for random complex zeroes: Strong clustering and local universality, *Comm. Math. Phys.*.
- B. Błaszczyszyn, E. Merzbach and V. Schmidt (1997), A note on expansion for functionals of spatial marked point processes, *Statistics & Probability Letters*.