Topics in spherical stochastic geometry - part II

# Spherical splitting tessellations

Christoph Thäle

(based on joint work with Daniel Hug)

- $\mathbb{S}_{d-1}$ space of great hyperspheres in $\mathbb{S}^d \subset \mathbb{R}^{d+1}$
- $[c]$ great hyperspheres intersecting a set $c \subset \mathbb{S}^d$
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- Fix $t > 0$

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For each cell $c \in Y_{\tau_{n-1}}$ generate an exponential random variable $E_c \sim \mathrm{Exp}(\kappa([c]))$

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Pick a cell $c \in Y_{\tau_{n-1}}$ with probability $\dfrac{\kappa([c])}{\sum_{c \in Y_{\tau_{n-1}}} \kappa([c])}$ ← larger cells are picked with higher probability

Pick a great hypersphere $S$ with distribution $\dfrac{\kappa(\cdot \cap [c])}{\kappa([c])}$

Update $Y_{\tau_n} := \oslash\,(c_n, S_n, Y_{\tau_{n-1}})$ and increase $n$

(split $c_n \in Y_{\tau_{n-1}}$ by $S_n$)

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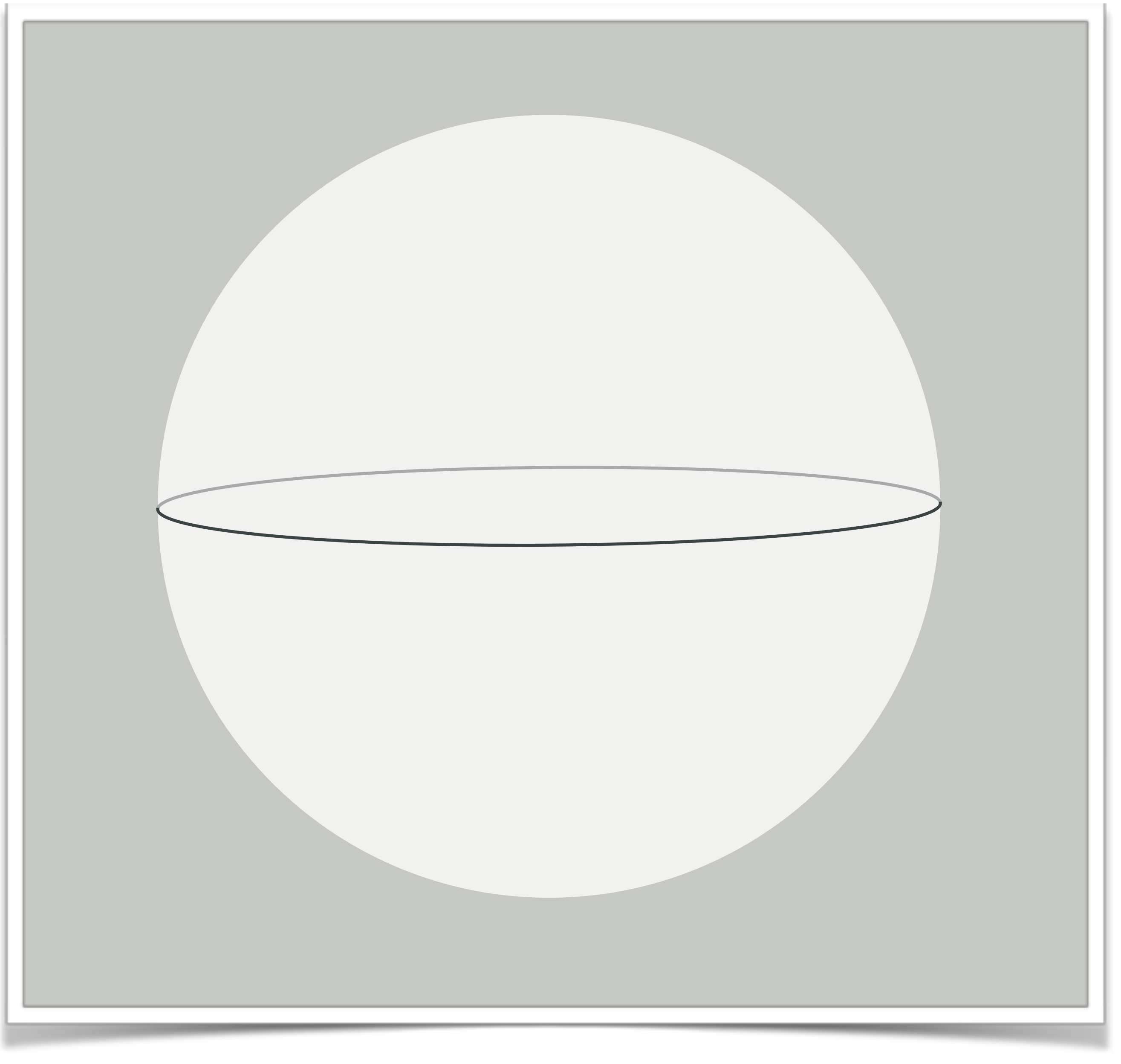
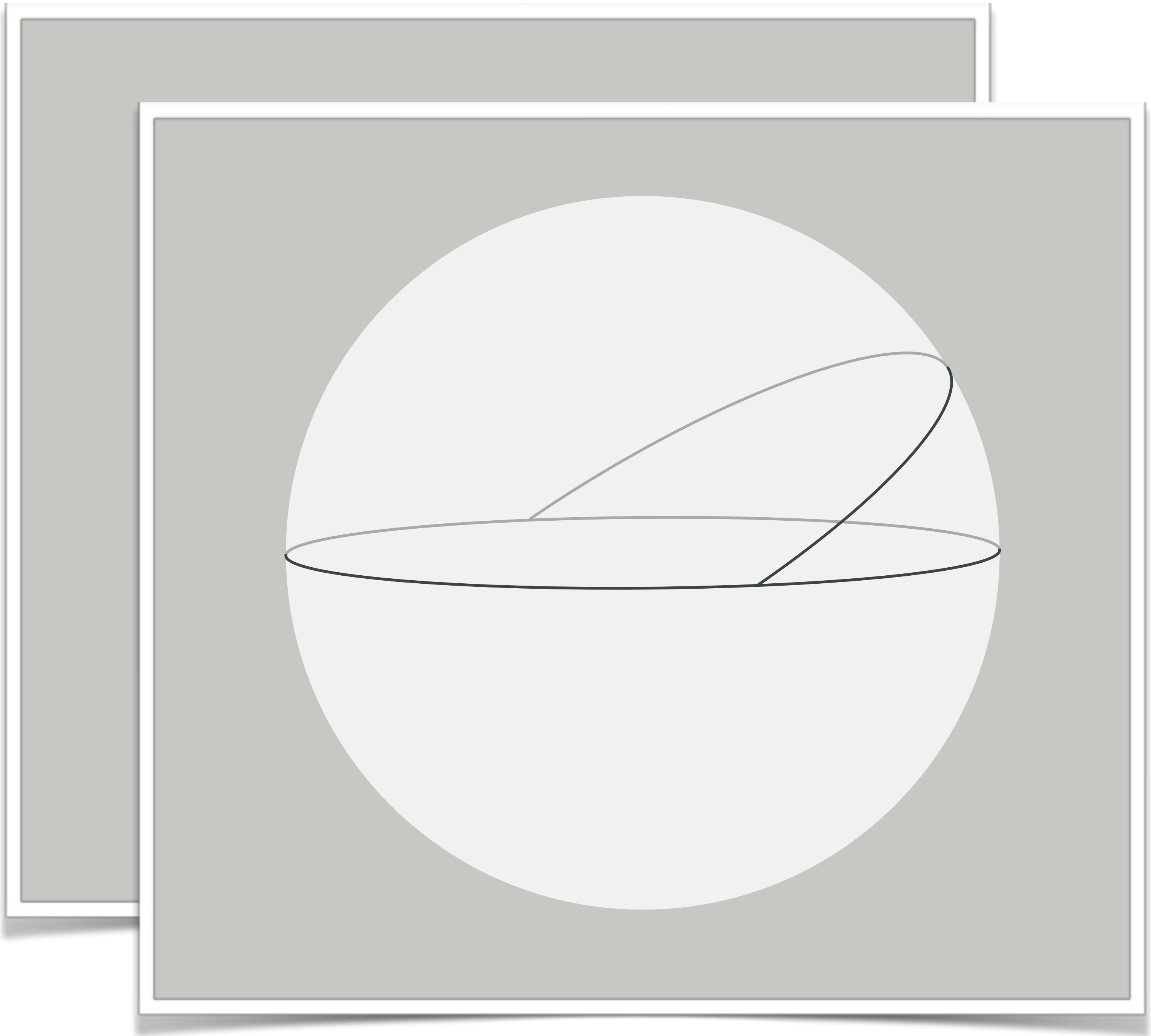
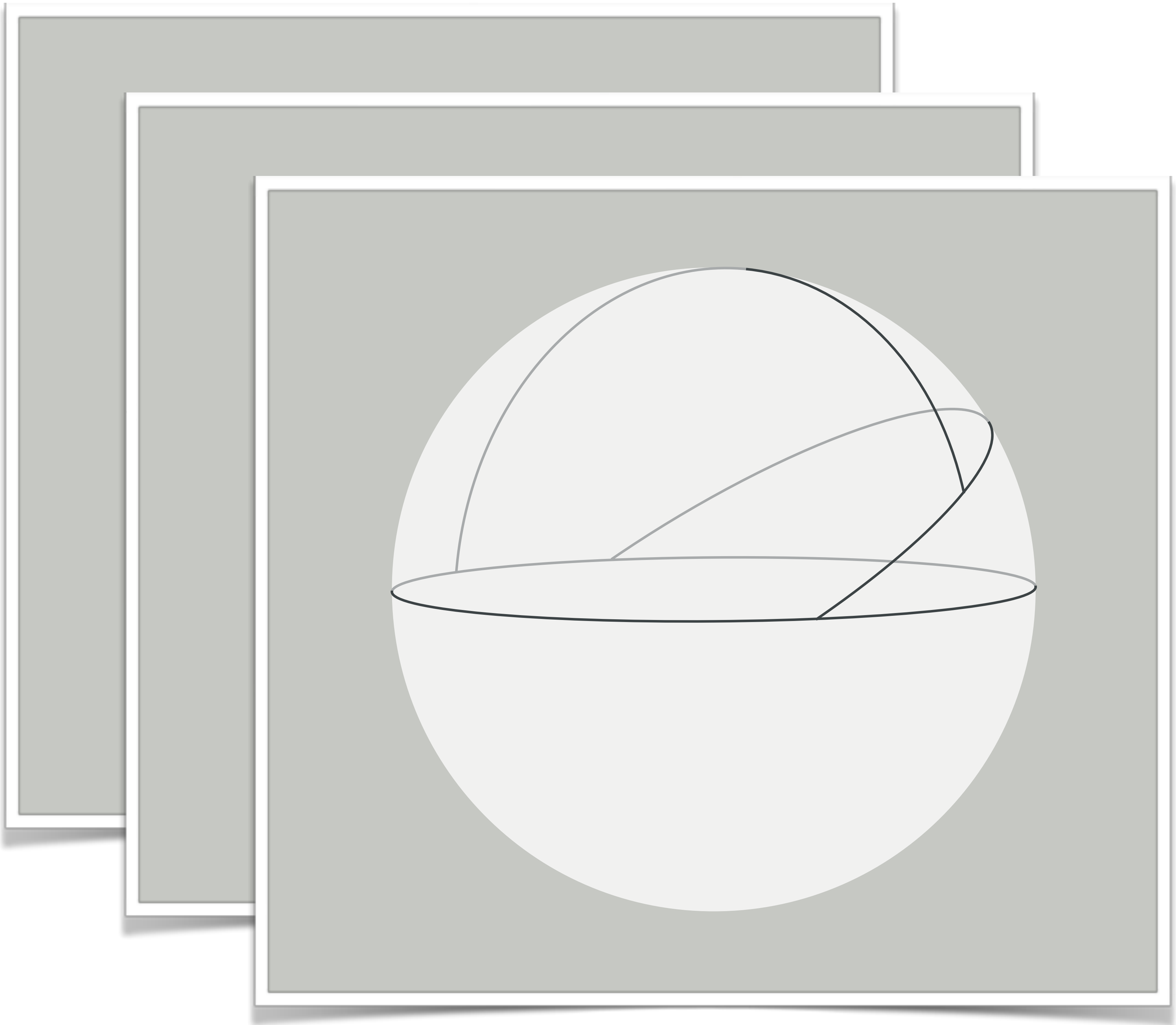
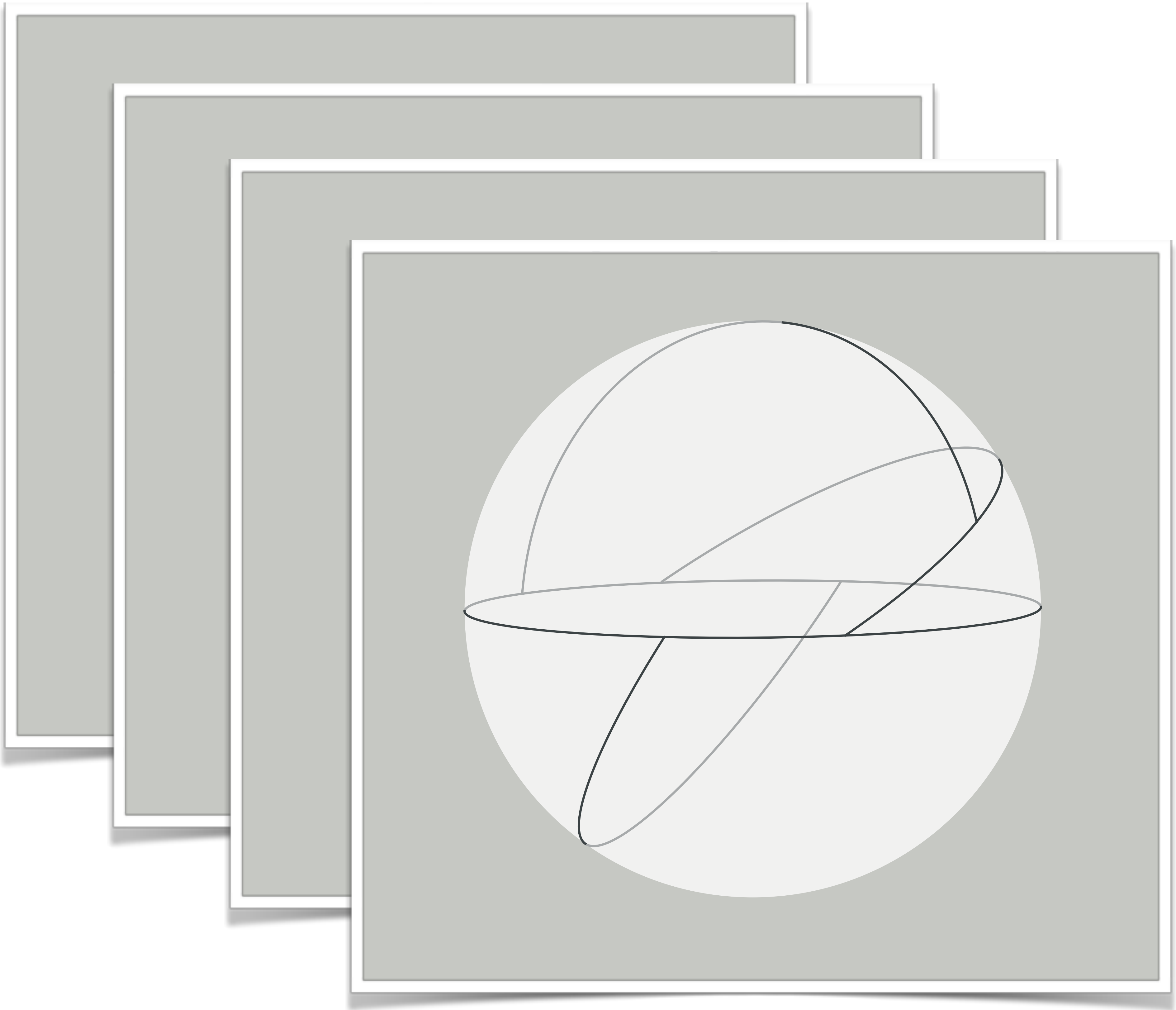
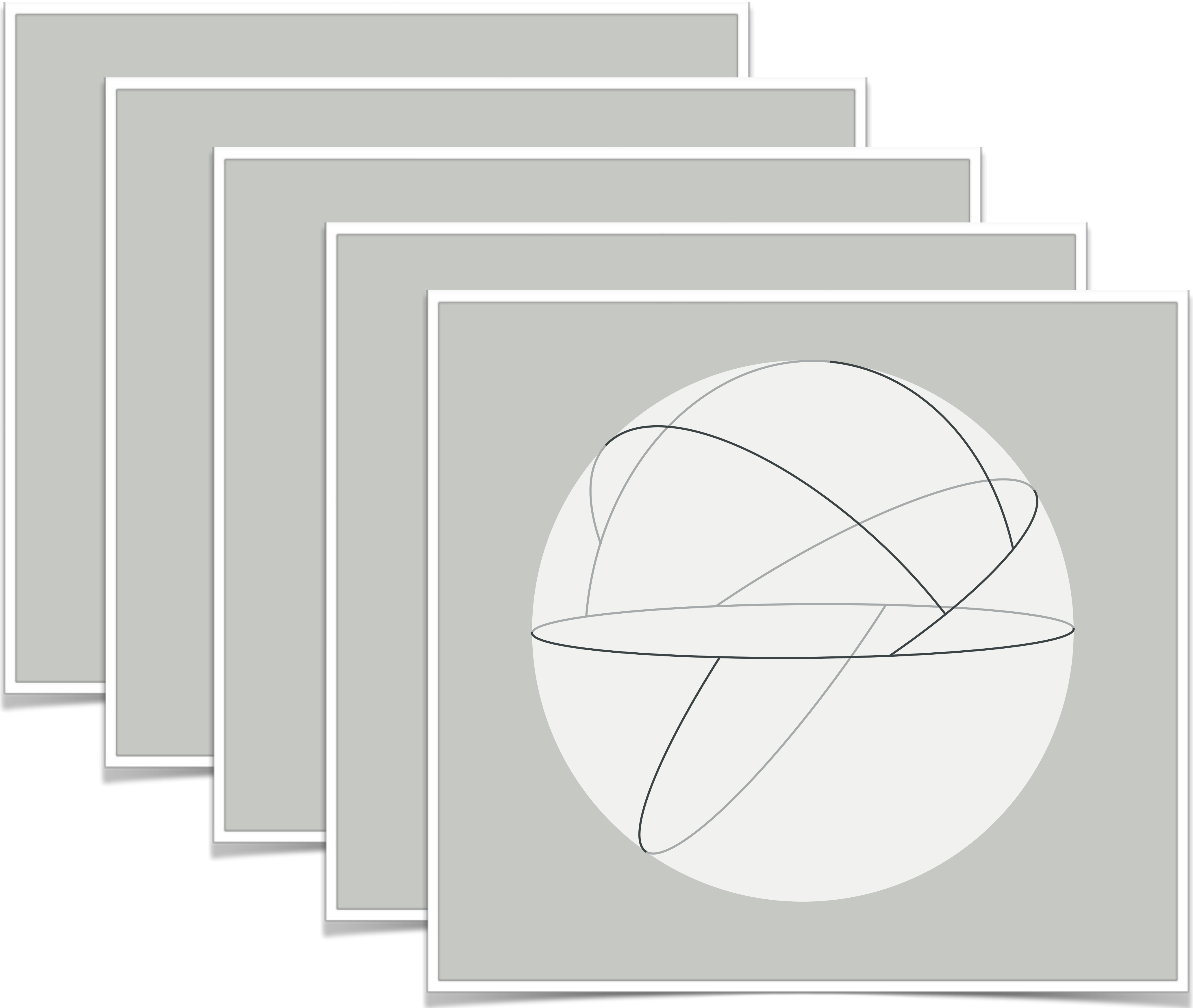
Output $Y_{\tau_{n-1}}$

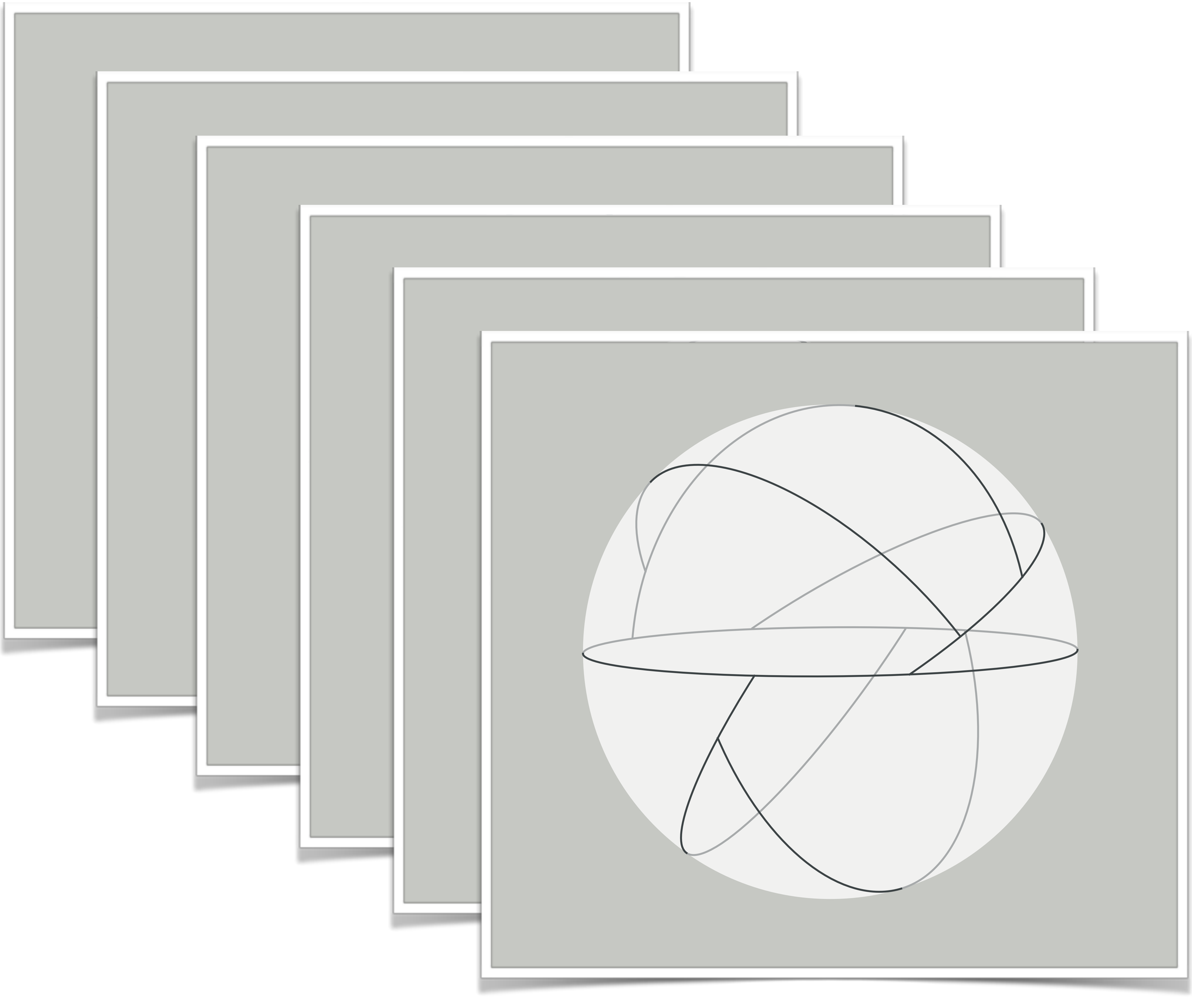


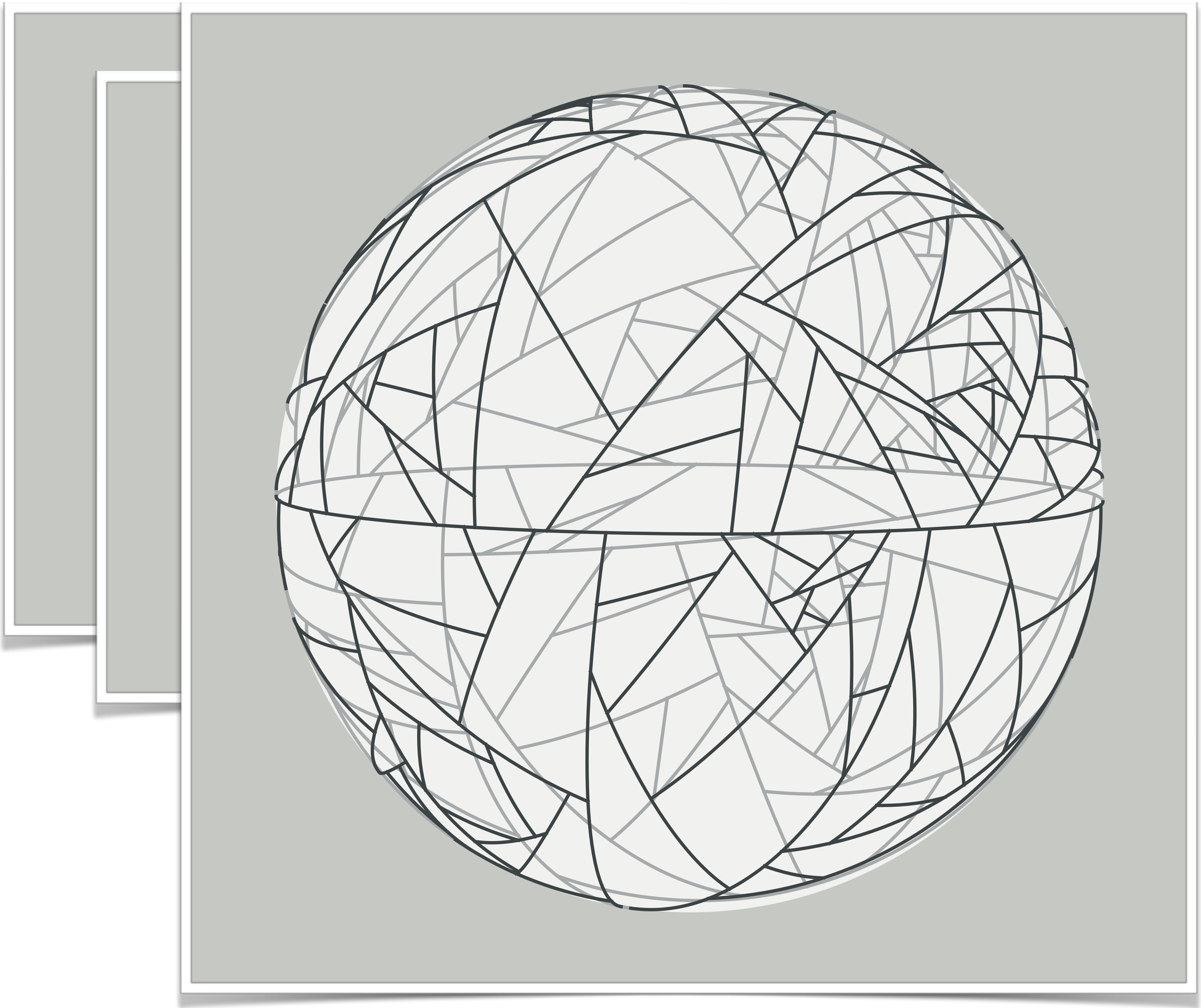












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$\oslash(c, S, T) := (T \setminus \{c\}) \cup \{c \cap S^+, c \cap S^-\} \in \mathbb{T}^d$ if $c \in T, S \in [c]$

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The **splitting tessellation process** $(Y_t)_{t \geq 0}$ with initial tessellation $Y_0 = \{\mathbb{S}^d\}$ is the continuous time, pure-jump Markov process on $\mathbb{T}^d$ with generator

$$(\mathscr{A}f)(T) = \sum_{c \in T} \int_{[c]} \left[ f(\oslash(c, S, T)) - f(T) \right] \kappa(\mathrm{d}S), \qquad T \in \mathbb{T}^d,$$

where $f$ is a bounded measurable function on $\mathbb{T}^d$.

Let $(Y_t)_{t\geq 0}$ be a Markov process taking values in a Borel space and with generator $\mathscr{A}$ and domain $D(\mathscr{A})$. Then, for $f \in D(\mathscr{A})$ the random process

$$f(Y_t) - f(Y_0) - \int_0^t (\mathscr{A}f)(X_s)\,\mathrm{d}s\,, \qquad t \geq 0\,,$$

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For $\phi : \mathbb{P}^d \to \mathbb{R}$ consider $\Sigma_\phi(T) := \sum_{c\in T} \phi(c) \quad (T \in \mathbb{T}^d)$

The stochastic process

$$M_t(\phi) := \Sigma_\phi(Y_t) - \Sigma_\phi(Y_0) - \int_0^t (\mathscr{A}\Sigma_\phi)(Y_s)\,\mathrm{d}s$$

is a martingale.

- Consider a spherical convex body $K \subset \mathbb{S}^d$ and its spherical parallel set

$$K_r = \{x \in \mathbb{S}^d : \ell(x, K) \leq r\}$$

- Spherical Steiner formula

$$\mathscr{H}^d(K_r \backslash K) = \sum_{j=0}^{d-1} \beta_j \beta_{d-j-1} V_j(K) \int_0^r (\cos t)^j (\sin t)^{d-j-1} \, \mathrm{d}t$$

surface area of $\mathbb{S}^j$

spherical intrinsic volumes

additive (valuations)
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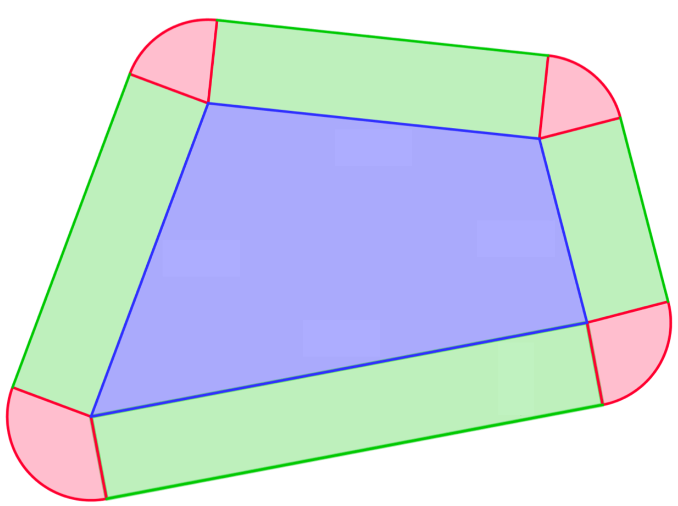
- For spherical polytopes $c \in \mathbb{P}^d$

$$V_j(c) = \frac{1}{\beta_j} \sum_{F \in \mathscr{F}_j(c)} \gamma(F, c) \, \mathscr{H}^j(F)$$

external angle



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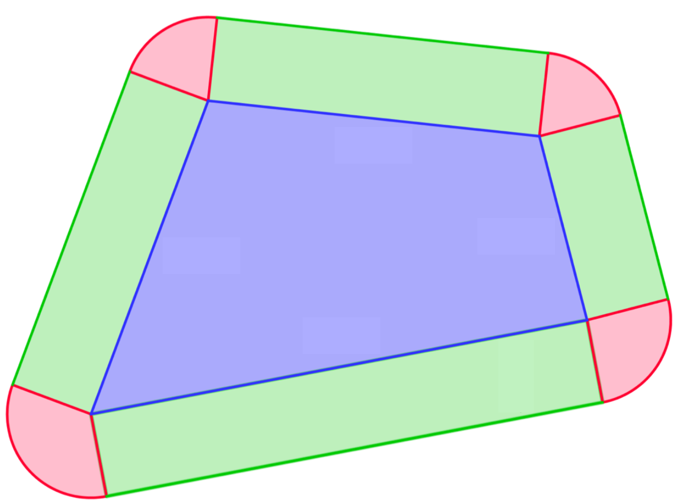

- local extension $\phi_j(K, \cdot)$ spherical curvature measures

- For $t \geq 0$, $j \in \{0,1,\ldots,d\}$ and bounded $h : \mathbb{S}^d \to \mathbb{R}$ define

$$\Sigma_j(t;h) = \sum_{c \in Y_t} \phi_j(c,h) = \sum_{c \in Y_t} \int_c h(x)\, \phi_j(c, \mathrm{d}x)$$

Total surface area: $j = d-1, h \equiv 1$ (times $\beta_{d-1}$)

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$$\Sigma_d(s_{d-j};h) = \sum_{c \in Y_s} \frac{\mathscr{H}^d(h\mathbf{1}_c)}{\beta_d} = \frac{\mathscr{H}^d(h)}{\beta_d}$$

Let $t \geq 0, j \in \{0,1,\ldots,d\}$. Then $\mathbf{E}\Sigma_j(t;h) = \frac{t^{d-j}}{(d-j)!}\frac{\mathscr{H}^d(h)}{\beta_d}$.

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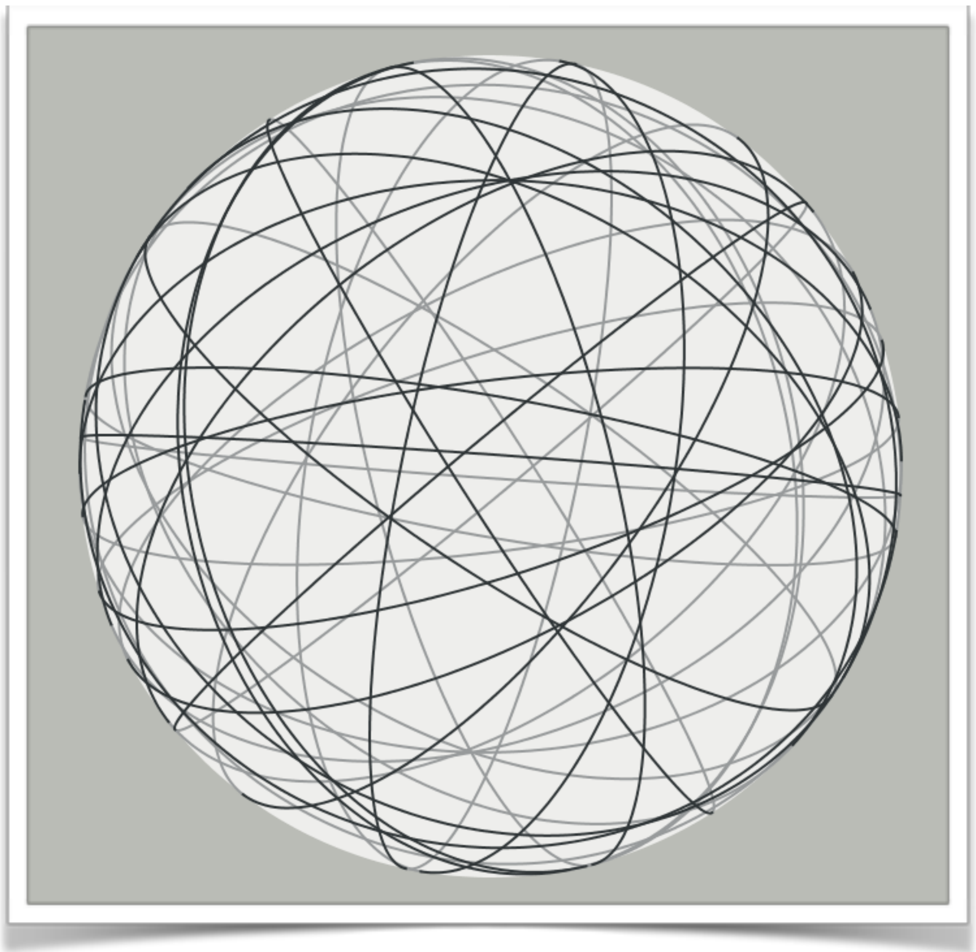

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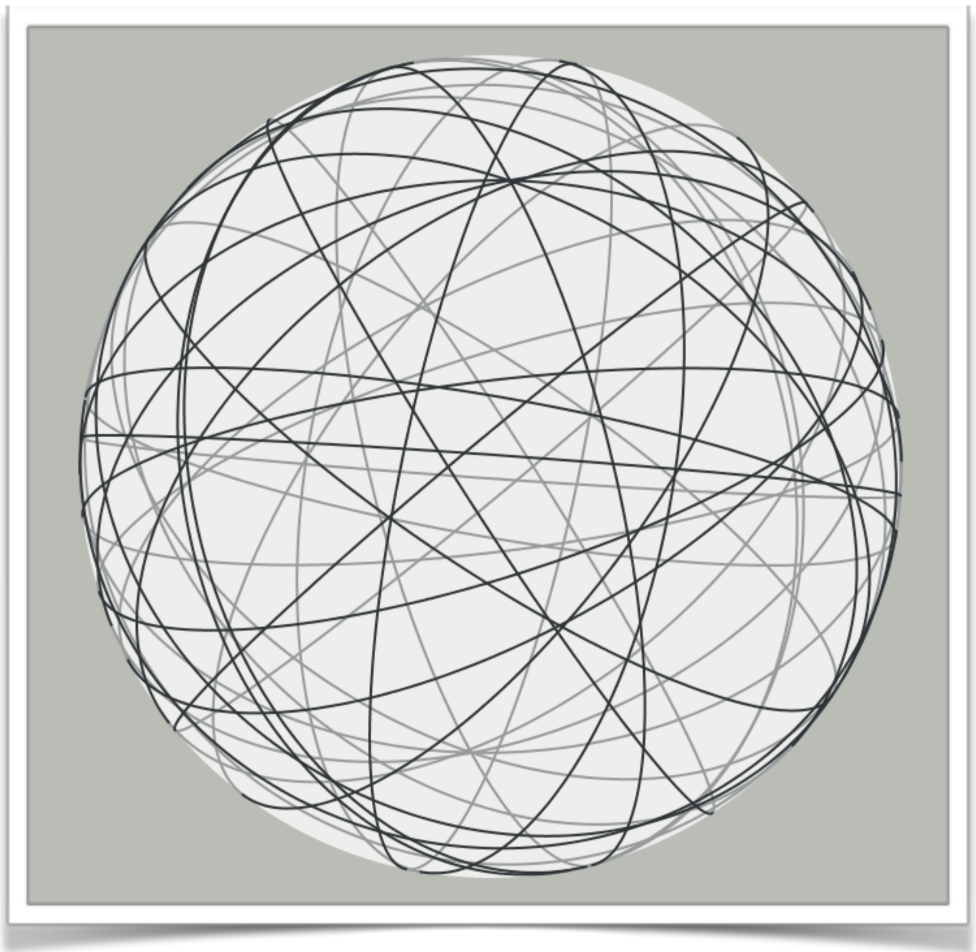

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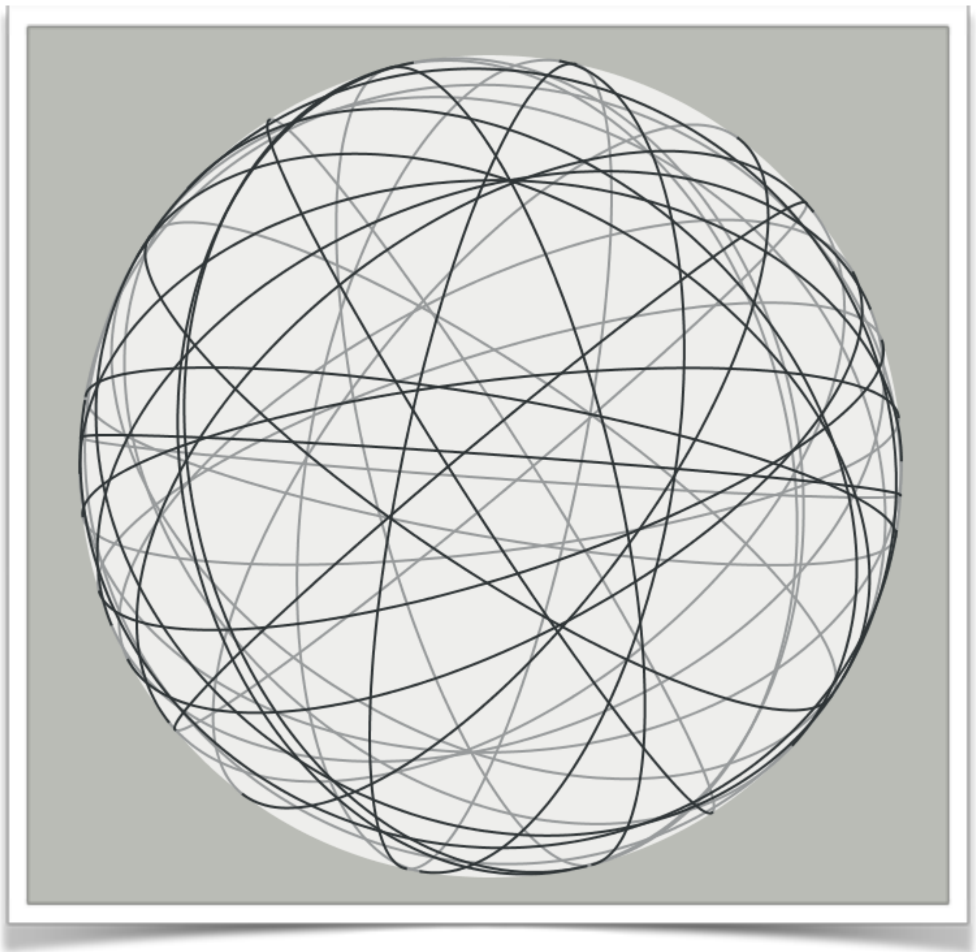

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How can we distinguish the two types of tessellations by a simple characteristic?

Let $t \geq 0$ and $h : \mathbb{S}^d \to \mathbb{R}$ be bounded. Then

$$\mathrm{Var}\Sigma_{d-1}(t;h) = \frac{\pi\beta_{d-2}}{\beta_d\beta_{d-1}^2}\int_{\mathbb{S}^d}\int_{\mathbb{S}^d}\frac{1-\exp\left(-\frac{1}{\pi}\ell(x,y)t\right)}{\ell(x,y)\sin(\ell(x,y))}h(x)h(y)\,\mathscr{H}^d(\mathrm{d}x)\mathscr{H}^d(\mathrm{d}y)$$

- The proof uses further auxiliary martingales
- Spherical integral-geometric transformation formulas of Blaschke-Petkantschin-type
- Covariances (and variances) for different functions and lower-order curvature measures can also be determined
- The variance of the total surface area is a special case

Let $t \geq 0$ and $h : \mathbb{S}^d \to \mathbb{R}$ be bounded. Then

$$\operatorname{Var}\Sigma_{d-1}(t;h) = \frac{\pi\beta_{d-2}}{\beta_d \beta_{d-1}^2} \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \frac{1-\exp\left(-\frac{1}{\pi}\ell(x,y)t\right)}{\ell(x,y)\sin(\ell(x,y))} h(x)h(y)\,\mathscr{H}^d(\mathrm{d}x)\mathscr{H}^d(\mathrm{d}y)$$

- The proof uses further auxiliary martingales

$\phi_1, \phi_2 : \mathbb{P}^d \to \mathbb{R}$ bounded, $b_1, b_2, v_1, v_2 \geq 0$

$$\Psi_{\phi_1,\phi_2}(T,t) := (\Sigma_{\phi_1}(T) - b_1 t^{v_1})(\Sigma_{\phi_2}(T) - b_2 t^{v_2})$$

$$N_t(g) := g(Y_t, t) - g(Y_0, 0) - \int_0^t (\mathscr{A}g(\cdot\ ,s))(Y_s) + \frac{\partial g}{\partial s}(\cdot\,, s)(Y_s)\,\mathrm{d}s$$

$g \in D(\mathscr{A}) \otimes C_0^1([0,\infty))$

Then $N_t(\Psi_{\phi_1,\phi_2})$ is a marginale.

- Spherical integ... ...f Blaschke-Petkantschin-type
- Covari... meas...
- The varian...

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- Spherical integral-geometr

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$$\mathrm{Var}\mathscr{H}^{d-1}(Z_t) = \pi\beta_{d-2} \int_{\mathbb{S}^d} \frac{1 - \exp\big(-\frac{1}{\pi}\ell(x,e)t\big)}{\ell(x,e)\sin(\ell(x,e))} \mathscr{H}^d(\mathrm{d}x)$$

$$= \frac{(2\pi)^d}{(d-2)!} \int_0^1 \sin(\pi z)^{d-2} \frac{1 - \exp(-zt)}{z} \mathrm{d}z$$

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$d = 2$

$$\mathrm{Var}\,\mathscr{H}^1(Z_t) = 4\pi^2\int_0^1 \frac{1-e^{-tz}}{z}\,\mathrm{d}z = 4\pi^2\big(\gamma + \ln t + E_1(t)\big) \sim 4\pi^2 \ln t \to \infty$$

$$\gamma \approx 0.5772, \qquad E_1(t) := \int_t^\infty s^{-1}e^{-s}\,\mathrm{d}s$$

$d \geq 3$

$$\mathrm{Var}\,\mathscr{H}^{d-1}(Z_t) \leq \frac{(2\pi)^d}{(d-2)!}\int_0^1 \pi\sin(\pi z)^{d-3}\,\mathrm{d}z < \infty$$

Consider an isotropic random measure $\mathbf{M}$ on $\mathbb{S}^d$

- $\mu := \mathbf{E}[\mathbf{M}(\mathbb{S}^d)]$ intensity of $\mathbf{M}$

- $K_{\mathbf{M}}(r) := \frac{1}{\mu^2}\mathbf{E}\int_{(\mathbb{S}^d)^2} \mathbf{1}(\ell(x,y) \leq r)\,\mathbf{M}^2(\mathrm{d}(x,y))$ spherical K-function of $\mathbf{M}$

Spherical analogues to Ripley's K-function
J. Royal Stat. Soc. 1977

- $g_{\mathbf{M}}(r) := \frac{\beta_d}{\beta_{d-1}(\sin r)^{d-1}} K'_{\mathbf{M}}(r)$ spherical pair-correlation function of $\mathbf{M}$

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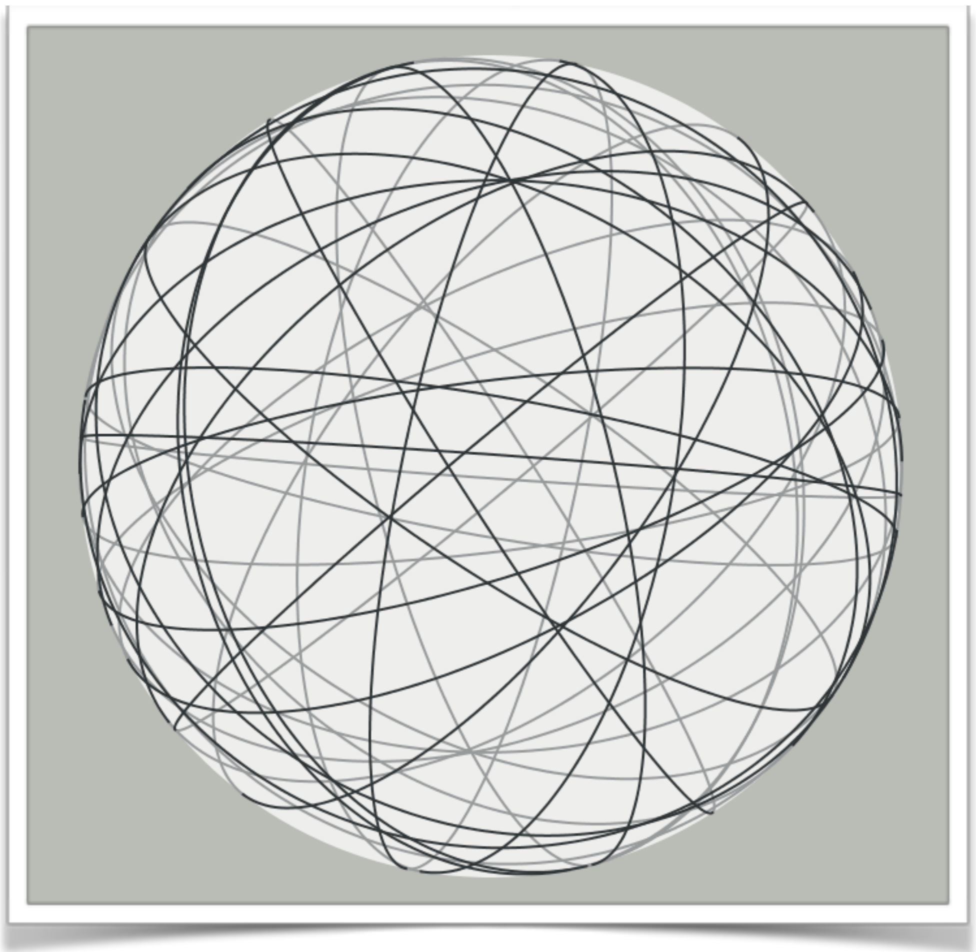

$$\overline{Z}_t = \bigcup_{u\in\eta_t} (u^\perp \cap \mathbb{S}^d)$$ Poisson great hypersphere tessellation

$$\overline{K}_{d,t}(r) = \frac{\beta_{d-1}}{\beta_d}\int_0^r (\sin\varphi)^{d-1}\,\mathrm{d}\varphi + \frac{1}{t}\frac{\beta_{d-2}}{\beta_{d-1}}\int_0^r (\sin\varphi)^{d-2}\,\mathrm{d}\varphi$$

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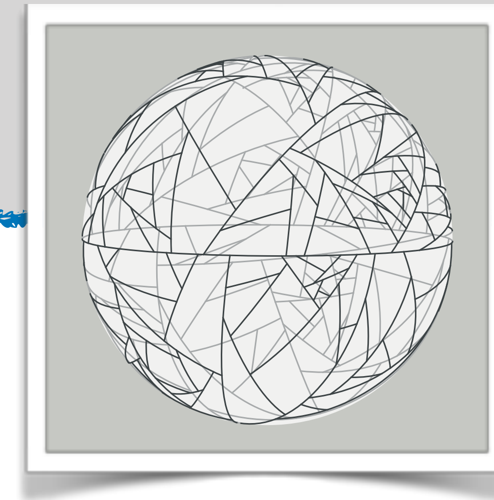

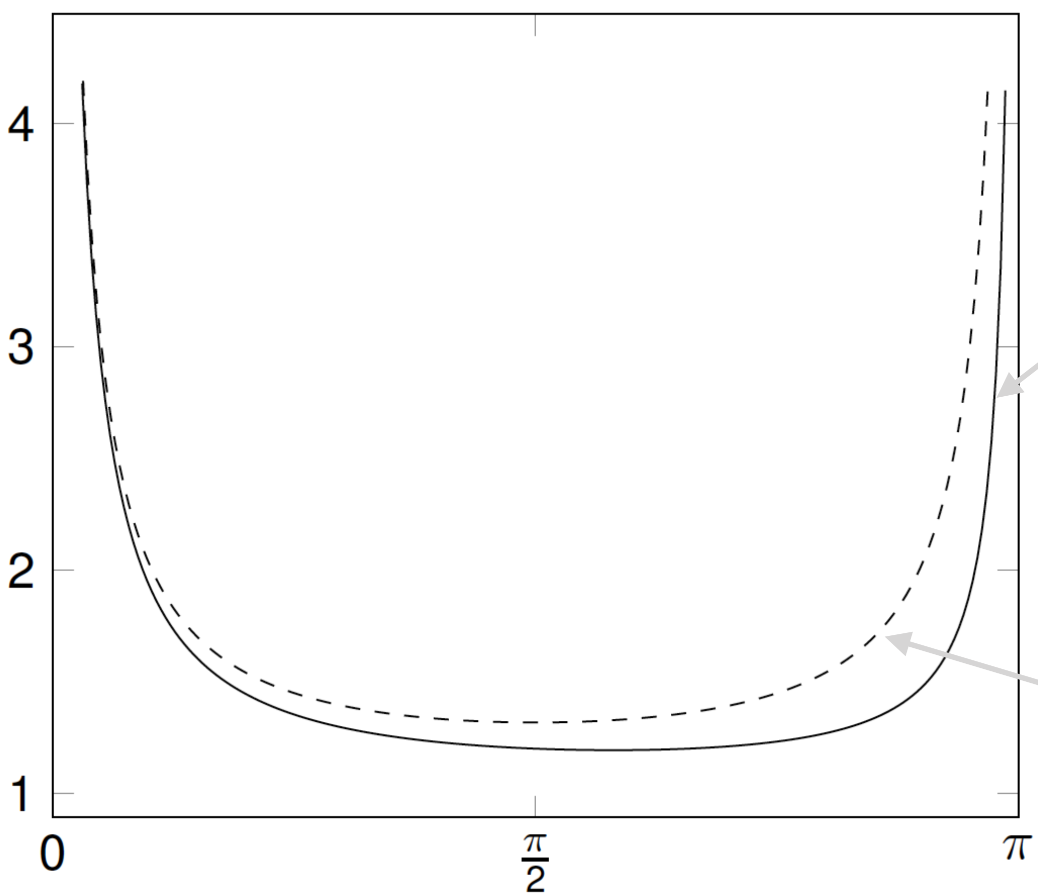


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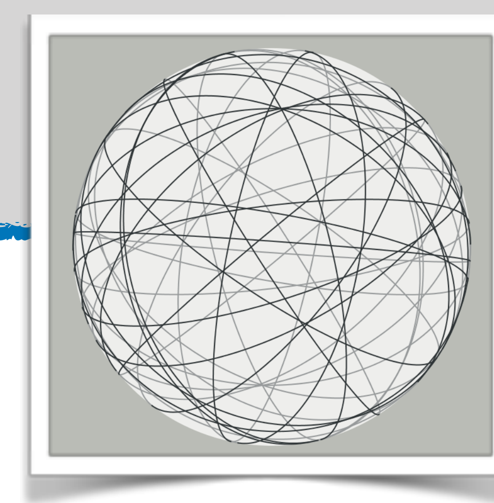

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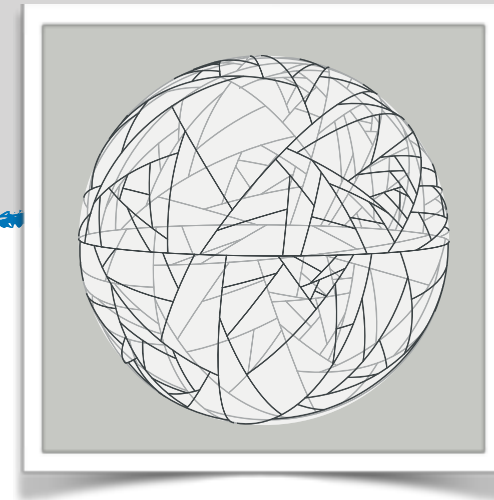

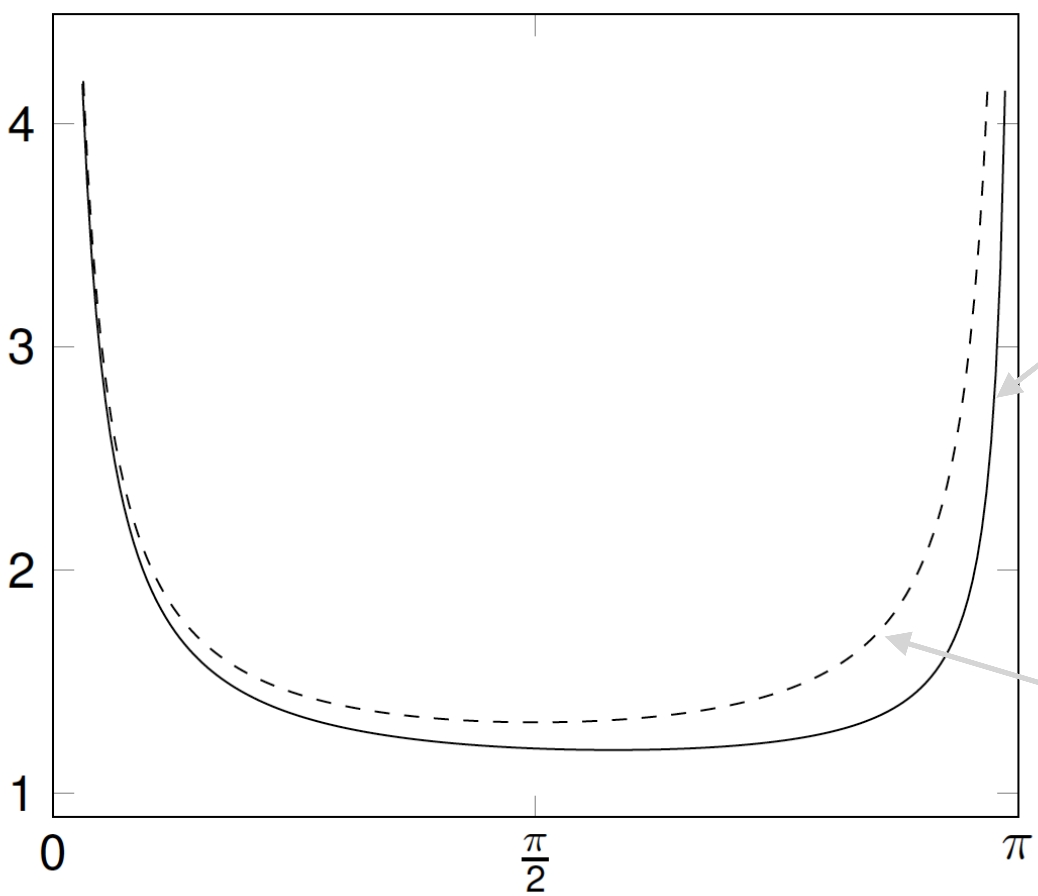


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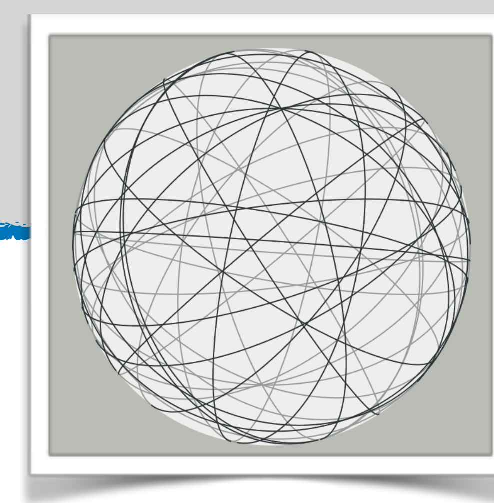

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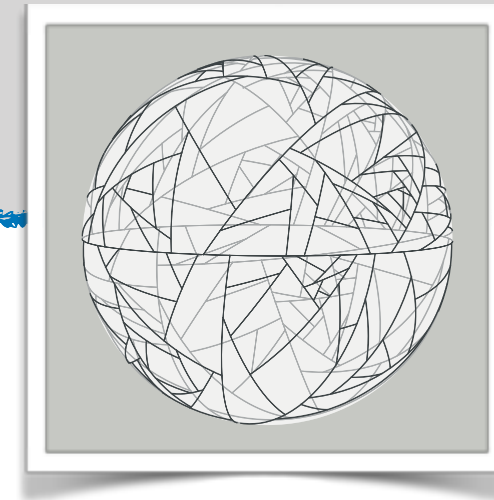
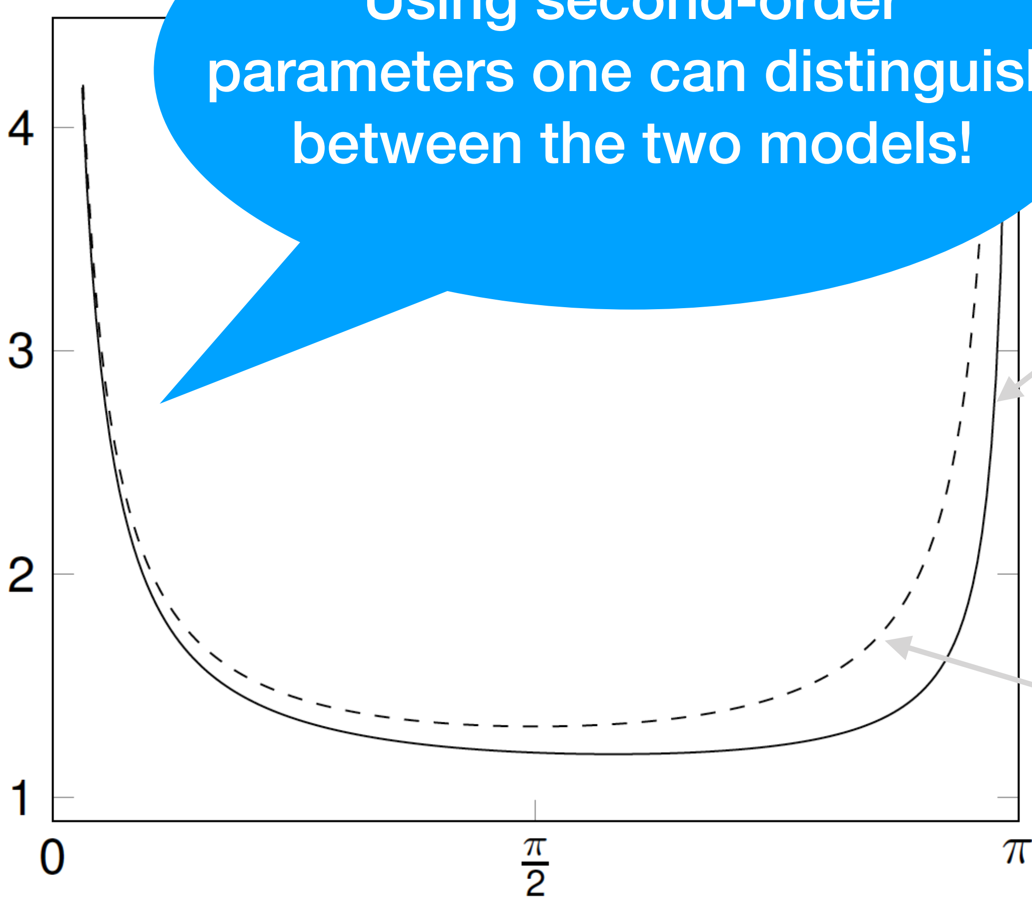


Using second-order parameters one can distinguish between the two models!

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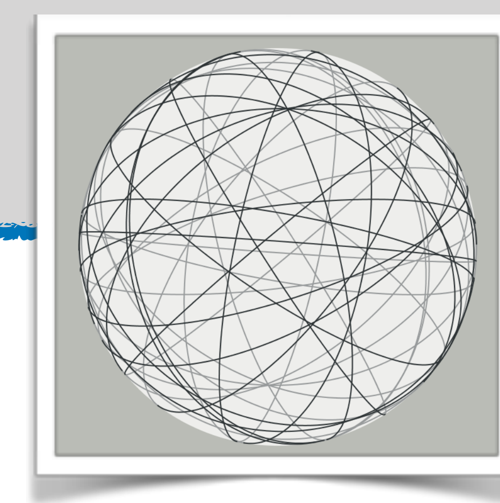

# Thank you!

*D. Hug & C.T.*

*Splitting tessellations in spherical spaces*
*EJP 24, article 24 (2019)*