

Erdős-Rényi
random graph model

+

forest fires

=

Self Organized Criticality

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In familiar models of stat. phys. critical phenomena at particularly tuned values of some parameter phase transitions

e.g. percolation p_c

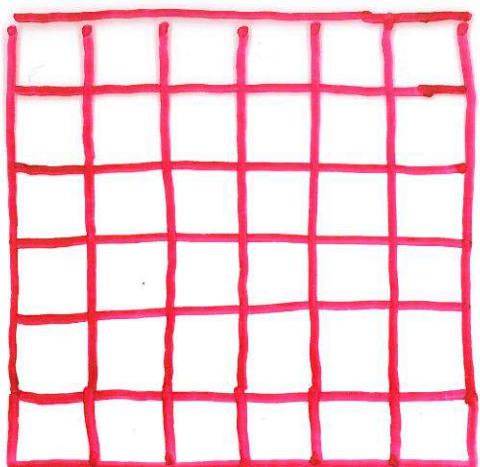
Ising, Heisenberg β_c

S.O.C. = dynamical phenomena driven by dynamics to a permanent critical state

e.g. sandpile models

* forest fire models *

Drossel-Schwabl model ③ (1992)



$$\lambda_n \in \mathbb{Z}^d$$

- (A) trees appear on empty sites with rate 1
- (B1) connected clusters of trees (forests) burn down instantly when touching $\partial \lambda_n$
- or
(B2) lightnings hit trees with rate λ_n and connected clusters hit by lightning burn down instantly

A + B1

or

A + B2

decent finite state Markov processes

What happens when $n \rightarrow \infty$? ④

A+B1 "dynamical percolation with the incipient infinite cluster permanently burnt down"
sticks to permanent criticality if the process exists

A+B2

$$\lambda_n < \frac{1}{T\lambda_n}$$

no lightnings, just pure percolation

$$\lambda_n = \frac{c}{T\lambda_n}$$

percolation with the ∞ cluster burnt down with rate $c \cdot \theta$

J. van den Berg - R. Brouwer:
self destructive percolation
in $d=2$ (2006)

(5)

$$\frac{1}{|\lambda_n|} \ll \lambda_n \ll 1$$

something very similar
to $A + B_1$

$$\lambda_n = \lambda \times 1$$

subcritical

- existence of the infinite
dynamics:

M. Dürre (2006)

- $\lambda \rightarrow 0$ asymptotics in $d=1$

J. van den Berg - A. Järai
(2005)

Erdős-Rényi + lightnings:

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$A + B_2$ on K_N

N sites, $\binom{N}{2}$ edges.

edges in state 0 or 1

(A) edges switch $0 \rightarrow 1$ with rate $\frac{1}{N}$, independently.

(B2) sites are hit by lightnings with rate λ_N , all edges of the connected cluster switch $1 \rightarrow 0$ (i.e. the cluster hit by lightning falls apart to singletons)

$$\mathcal{V} := \left\{ \underline{v} = (v_k)_{k=1}^{\infty} : v_k \geq 0, \sum_{k=1}^{\infty} v_k \leq 1 \right\}$$

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$$\Theta(\underline{v}) := 1 - \sum_{k=1}^{\infty} v_k \quad \text{"the gel"}$$

topology: component-wise *cvg.*

$$\mathcal{E} := \left\{ t \mapsto \underline{v}(t) : v_k(t) \text{ BV loc.} \right\}$$

topology: component-wise weak

$$N_{n,k}(t) := n^{-1} \cdot \# \left\{ \begin{array}{l} \text{vertices in class-} \\ \text{clusters of size } k \end{array} \right\}$$

$$\underline{v}_n(t) = (v_{n,k}(t))_{k=1}^{\infty}$$

Q: The $n \rightarrow \infty$ asymptotics
of the process $t \mapsto \underline{v}_n(t)$?

The Erdős-Rényi model ⑧

$\lambda_n = 0$ — well known:

$$V_n(\cdot) \xrightarrow{P} V(\cdot)$$

where $V(\cdot)$ is the solution of the Cauchy problem:

$$\dot{N}_k(t) = \frac{k}{2} \sum_{\ell=1}^{k-1} N_\ell(t) N_{k-\ell}(t) - k N_k(t)$$

$k=1, 2, \dots$

$$N_k(0) = S_{k,1}$$

↑
Smoluchowski
Coagulation eq

other initial conditions with

$$\sum_{k=1}^{\infty} N_k(0) = 1, \quad \sum_{k=1}^{\infty} k^3 N_k(0) < \infty$$

are OK.

Qualitative behaviour:

$$t_c = \left(\sum_{k=1}^{\infty} k N_k(0) \right)^{-1}$$

$t < t_c$ subcritical

$$\theta = 0$$

$k \mapsto N_k(t)$ decays expo.

$t > t_c$ supercritical

$$\theta > 0$$

$k \mapsto N_k(t)$ decays expo.

$t = t_c$ critical

$$\theta = 0$$

$k \mapsto N_k(t_c) \sim k^{-3/2}$

$C_k^N(t) := \# \text{ of clusters of size } k \text{ at time } t$

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$$\sum_{k \geq 1} k C_k^N(t) \equiv N$$

$$\underline{C}^N(t) := (C_1^N(t), C_2^N(t), \dots)$$

is a Markov process !!!

Coagulations:

$$(C_k^N, C_\ell^N, C_{k+\ell}^N) \rightarrow (C_{k-1}^N, C_{\ell-1}^N, C_{k+\ell+1}^N)$$

with rate $\frac{1}{N} \cdot k \cdot \ell \cdot C_k^N \cdot C_\ell^N$ k ≠ ℓ

$$(C_k^N, C_{2k}^N) \rightarrow (C_{k-2}^N, C_{2k}^N + 1)$$

with rate $\frac{1}{N} \cdot k^2 \cdot \frac{1}{2} \cdot C_k^N (C_k^N - 1)$

Fragmentations:

$$(C_1^N, C_k^N) \rightarrow (C_1^N + k, C_{k-1}^N)$$

with rate $\lambda(N) \cdot k \cdot C_k^N$ k > 1

Regimes:

① $\lambda_n \ll n^{-1}$

no lightnings, simply E-R.

② $\lambda_n = c \cdot n^{-1}, c \in (0, \infty)$

③ $n^{-1} < \lambda_n < 1$

the interesting case

④ $\lambda_n = \lambda \in (0, \infty)$

subcritical.

Asymptotic behaviour:

① just like E-R

② $\underline{V}_n(\cdot) \Rightarrow \underline{V}(\cdot)$

$t \mapsto \underline{V}(t)$ driven by

Smoluchowski coag. eq. +
random jumps :

$$(\underline{N}_1, \underline{N}_2, \dots) \rightarrow (\underline{N}_1 + \theta, \underline{N}_2, \dots)$$

with rate $c \cdot \theta$

④ $\underline{V}_n(\cdot) \xrightarrow{\text{P}} \underline{V}(\cdot)$

$$\dot{\underline{N}}_k = \frac{k}{2} \sum_{l=1}^{k-1} \underline{N}_l \underline{N}_{k-l} - (1+\lambda) k \underline{N}_k + \lambda \delta_{k,1} \sum_{l=1}^{\infty} l \underline{N}_l$$

unique

subcritical solution

$k=1, 2, \dots$

③ The interesting range:

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$$\bar{n}^{-1} < \lambda_n < 1$$

Thm:

$$\underline{v}_n(\cdot) \xrightarrow{P} v(t)$$

where $t \mapsto v(t)$ is the unique solution of the constrained Smoluchowski system

$$\dot{N}_k = \frac{k}{2} \sum_{l=1}^{k-1} \gamma_l N_{k-l} - k N_k, \quad k \geq 2$$

$$\sum_{k=1}^{\infty} N_k = 1 \quad + \text{I.C.}$$

$$\text{For } t \geq t_c := \left(\sum_{k=1}^{\infty} k N_k(0) \right)^{-1},$$

$$\sum_{l=k}^{\infty} N_l(t) \sim \sqrt{\frac{\varphi(t)}{2}} k^{-1/2}$$

where $\varphi(t)$ is strictly positive, bounded, Lipschitz continuous.

Rewrite as PDE:

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$$V(t,x) := \sum_{k=1}^{\infty} e^{-kx} v_k(t) - 1$$

the constrained Smoluchowski system becomes:

$$\begin{cases} \partial_t V(t,x) + \lambda_x \frac{1}{2} V(t,x)^2 = e^{-x} \varphi(t) \\ V(t,0) \equiv 0 \\ V(0,x) = \sum_{k=1}^{\infty} e^{-kx} v_k(0) - 1 \end{cases}$$

$$\varphi(t) = \lim_{x \rightarrow 0} V(t,x) \partial_x V(t,x)$$

$$= \bar{N}_1(t) + \dot{v}_1(t) \geq 0$$

seems to be ill posed

actually it is a Burgers control problem.