Percolation of worms

Balázs Ráth (BME)
Joint work with
Sándor Rokob (BME)

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- the trace of (x, H) is the translated set $x + H$

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- alternatively: let $N_x, x \in \mathbb{Z}^d$ i.i.d. $\mathrm{POI}(v)$,
  put N_x i.i.d. copies of animals with law ν translated by x

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- law of $\mathcal{S}^v$ is ergodic under spatial translations

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Expectation of total cardinality of animals that contain o is $v \cdot m_2$.

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Thus the m_1 condition can be sharp. How about the m_2 lemma?

Can the m_2 lemma be strengthened?

Question

Given $d \geq 2$, is there a function $f : \mathbb{N} \to \mathbb{R}_+$ satisfying

$$\lim_{n\to\infty} f(n)/n^2 = 0$$

such that for any choice of ν the condition

$$\sum_{H\in\mathcal{H}} f(|H|) \cdot \nu(H) < +\infty$$

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We do not know, but we will show that the m_2 condition is quite close to being sharp for a specific choice of ν.

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- $\mathcal{S}^v$ is the called the random length worms set at level v
- Alternatively: start $\mathrm{POI}(v)$ worms from each site of $\mathbb{Z}^d$
  $\mathcal{S}^v$ is the set of sites visited by these worms

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Theorem

Let $d \geq 5$. Let $\varepsilon > 0$ and $\ell_0 \geq e^e$. If

$$m(\ell) := \mathbb{P}(\mathcal{L} = \ell) = c\frac{\ln(\ln(\ell))^\varepsilon}{\ell^3 \ln(\ell)} \mathbb{1}[\ell \geq \ell_0], \quad \ell \in \mathbb{N}$$

then $v_c = 0$.

Discussion

$$m(\ell) = c\frac{\ln(\ln(\ell))^{\varepsilon}}{\ell^3 \ln(\ell)} \mathbb{1}[\ell \geq \ell_0], \quad \ell \in \mathbb{N}$$

- if $\varepsilon < -1$ then $\mathbb{E}(\mathcal{L}^2) < +\infty$, thus $v_c > 0$
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If the answer to this question is positive then
the answer to our previous question is negative for $d \geq 5$.

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- if we have already used worms from a spatial region then that region develops a shortage of worms

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- Wiener sausage percolation [Erhard, Poisat, 2016]
  like worms where $\mathcal{L} = T$

$$v_c(T) \asymp \begin{cases} T^{-d/2} & d = 2, 3 \\ \ln(T)/T^2 & d = 4 \\ 1/T^2 & d \geq 5 \end{cases}$$

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If $\alpha = 2$ then $m_2 = +\infty$, but $v_c > 0$

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Theorem (Sapozhnikov, Chang, 2016)

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Note: $m_2 < +\infty$ if $d \geq 5$, but $m_2 = +\infty$ if $d = 3, 4$

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Thank you for your attention!