Spectral phases of Erdős-Rényi graphs

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Part I: overview

Universality conjecture for disordered quantum systems

$$H = \text{random Hermitian operator}$$
$$= \text{Hamiltonian of quantum system with disorder}$$

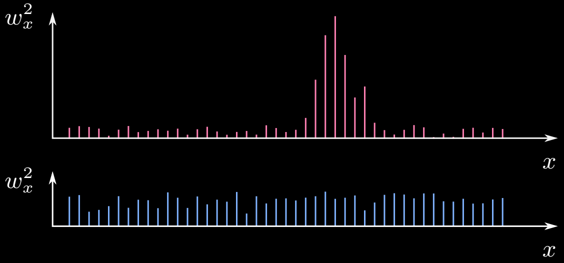
Universality conjecture: spectrum of H splits:

(1) Localized (insulator): Eigenvectors are localized.
Local spectral statistics are Poisson.

(2) Delocalized (metal): Eigenvectors are delocalized.
Local spectral statistics follow random matrix theory (e.g. GOE).

Let Λ be a finite set (physical space) and suppose that $H : \mathbb{R}^\Lambda \to \mathbb{R}^\Lambda$.

ℓ^2-normalized eigenvector $\mathbf{w} = (w_x)$:



How to quantify localization vs. delocalization?

Let $N := |\Lambda|$.

- Localization exponent: $\mathcal{D}_q \in [0,1]$:

$$\|\mathbf{w}\|_{2q} = N^{-\frac{q-1}{2q}\mathcal{D}_q}\,, \qquad 1 < q \leqslant \infty\,.$$

  Remarks:

  - $\mathbf{w}$ localized at single site $\iff$ $\mathcal{D}_q = 0$
  - $\mathbf{w}$ perfectly delocalized $\iff$ $\mathcal{D}_q = 1$
  - $\mathbf{w}$ uniform over N^γ sites $\impliedby$ $\mathcal{D}_q = \gamma$

- Scarring: there exists a small ε and a small $\mathcal{B} \subset \Lambda$ such that

$$\sum_{x \in \mathcal{B}} w_x^2 \geqslant 1 - \varepsilon\,.$$

Example: Anderson model

$$-\Delta + \lambda V \quad \text{on } \Lambda \subset \mathbb{Z}^d\,, \qquad V = (V_x)_{x\in\Lambda} \text{ i.i.d. } \mathcal{N}(0,1)\,.$$

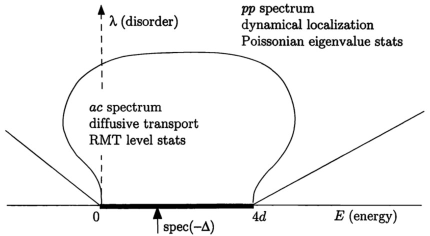


Figure 1.2. The predicted shape of the phase diagram of the Anderson model (1.2) in dimensions $d > 2$ for site potentials given by bounded iid random variables with a distribution similar to (1.5).

(From M. Aizenman, S. Warzel, *Random Operators*, AMS.)

Localized phase very well understood ([Fröhlich, Spencer; 1983], [Aizenman, Molchanov; 1993], [Molchanov; 1981], [Minami; 1996], ...)

Delocalized phase wide open (extended states conjecture).

Other models of quantum disorder

- Wigner matrices with light tails are in the delocalized phase.
  [Erdős, Schlein, Yau, Yin; 2009–. . .], [Tao, Vu; 2009–. . .]

- Heavy-tailed Wigner matrices proposed as a simple model that exhibits a phase transition.
  [Cizeau, Bouchaud; 1994], [Auffinger, Ben Arous, Péché; 2009], [Tarquini, Biroli, Tarzia; 2016], [Bordenave, Guionnet; 2013–2017], [Aggarwal, Lopatto, Yau; 2020]

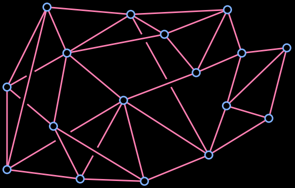
- Random band matrices proposed as a simpler alternative to the Anderson model on $\mathbb{Z}^d$.
  [Disertori, Spencer, Zirnbauer; 2009], [Sodin; 2009], [Erdős, K; 2010], [Schenker; 2010], [Erdős, K, Yau, Yin; 2013], [Bourgade, Erdős, Yau, Yin; 2017], [Bourgade, Fan, Yau, Yin; 2019], [Fan, Yau, Yin; 2021].

- This talk: Random graphs $\approx$ sparse random matrices.

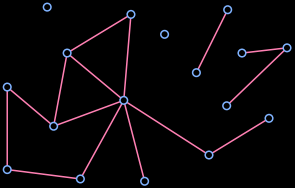
Erdős-Rényi graph and critical regime

Erdős-Rényi graph $\mathbb{G}(N, d/N)$

Critical regime: $d \approx \log N$, below which degrees do not concentrate.



$d \gg \log N$



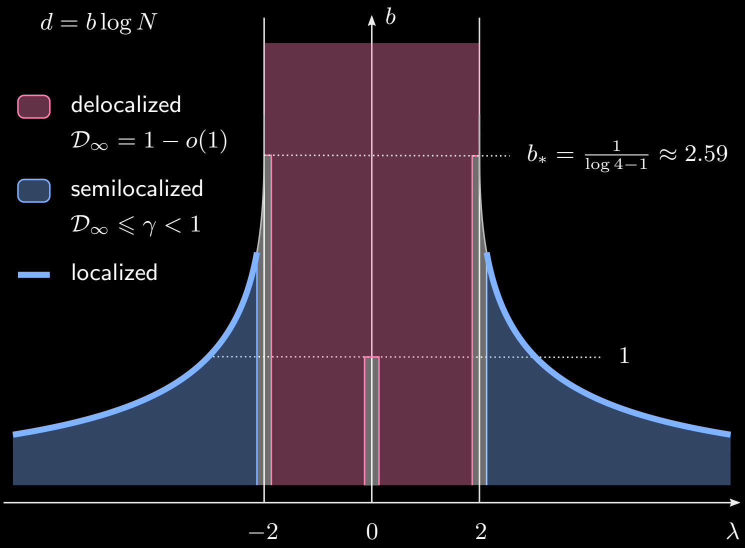
$d \ll \log N$

Supercritical $d \gg \log N$: homogeneous.

Subcritical $d \ll \log N$: inhomogeneous (hubs, leaves, isolated vertices, ...).

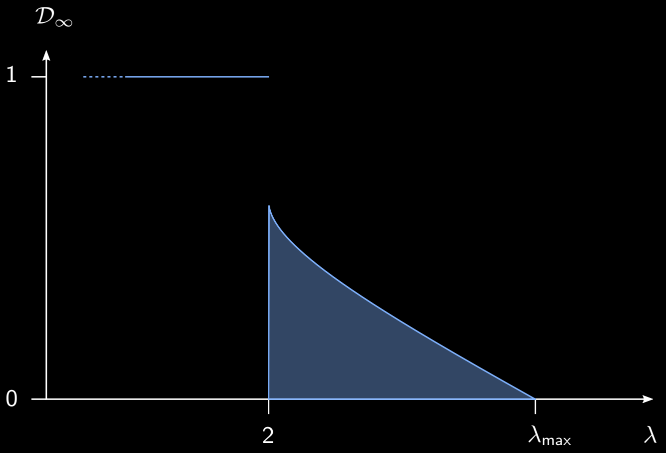
Consider the adjecency matrix $A = (A_{xy}) \in \{0,1\}^{N \times N}$.

Phase diagram for $H := d^{-1/2} A$



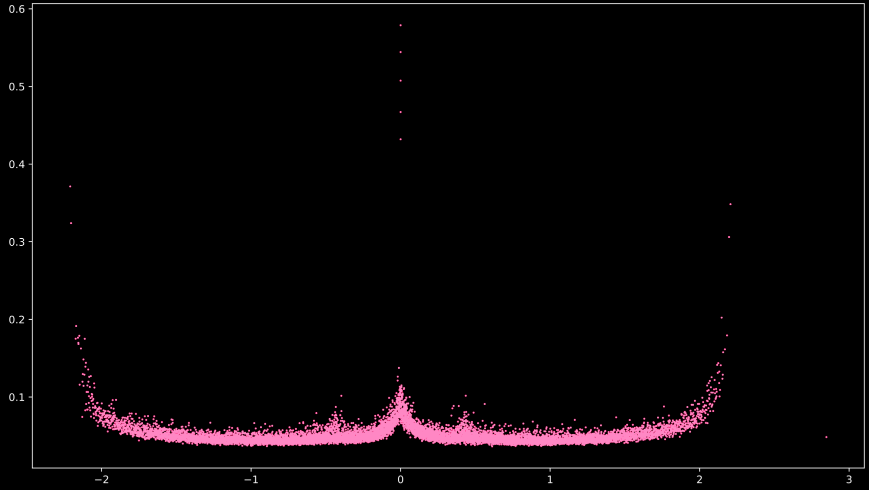
Behaviour of localization exponent

Asymptotically allowed region for $\mathcal{D}_\infty$ (plotted for $b = 1$):



Simulation of eigenvectors

Scatter plot of $(\text{eigenvalue}, \|\text{eigenvector}\|_\infty)$. ($N = 10'000$, $b = 0.6$)



Part II: results (Alt, Ducatez, K; 2019–2022)

Convention: $\kappa > 0$ tends to 0 slowly as $N \to \infty$.

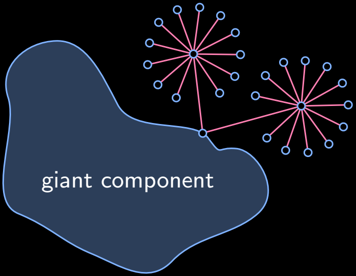
Delocalization

Theorem. Delocalization with high probability under any of the conditions

- $d \geqslant (b_* + \kappa) \log N$
- $d \geqslant (1 + \kappa) \log N$ and $|\lambda| \leqslant 2 - \kappa$
- $d \geqslant C\sqrt{\log N}$ and $\kappa \leqslant |\lambda| \leqslant 2 - \kappa$.

Remark. The assumptions are optimal (up to constant C).

Consider two identical stars of central degrees D attached to a common vertex.



This gives rise to a localized eigenvector with eigenvalue $\sqrt{D/d}$.

Such pairs occur up to $D = \mathrm{O}(1)$ if $d \leqslant C \log N$ and up to $D = \mathrm{O}(d)$ for $d \leqslant C\sqrt{\log N}$.

Semilocalization

Define the normalized degree $\alpha_x := \frac{1}{d} \sum_y A_{xy}$ and the map $\Lambda(\alpha) := \frac{\alpha}{\sqrt{\alpha - 1}}$.

Theorem. Let $\lambda \geqslant 2 + \kappa$ be an eigenvalue with eigenvector $\mathbf{w} \in \mathbb{S}^{N-1}$. Define the set of vertices in resonance with λ,

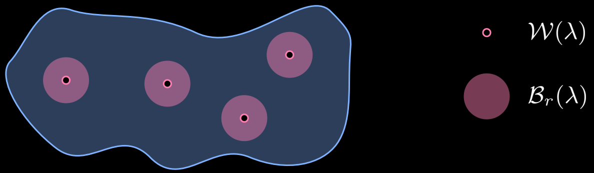
$$\mathcal{W}_\lambda := \left\{ x : \alpha_x \geqslant 2, |\Lambda(\alpha_x) - \lambda| \leqslant \kappa \right\}.$$

There is a radius $r \gg 1$ such that for each $x \in \mathcal{W}_\lambda$ there exists a normalized vector $\mathbf{v}(x)$, supported in $B_r(x)$, such that the supports of $\mathbf{v}(x)$ and $\mathbf{v}(y)$ are disjoint for $x \neq y$, and

$$\sum_{x \in \mathcal{W}_\lambda} \langle \mathbf{v}(x), \mathbf{w} \rangle^2 = 1 - o(1)$$

with high probability. Moreover, $\mathbf{v}(x)$ decays exponentially around x:

$$\sum_{y \notin B_r(x)} (\mathbf{v}(x))_y^2 \leqslant \frac{1}{(\alpha_x - 1)^{r+1}}.$$


$\mathcal{W}(\lambda)$
$\mathcal{B}_r(\lambda)$

Spectral edge: Poisson eigenvalue statistics

Theorem. Suppose that

$$(\log \log N)^4 \leqslant d \leqslant (b_* - \kappa) \log N \,.$$

There exist deterministic u, σ, τ, θ (which are explicit functions of d and N) such that the rescaled eigenvalue process

$$\Phi := \sum_i \delta_{d\tau(\lambda_i(H) - \sigma)}$$

is asymptotically close to a Poisson point process Ψ on $\mathbb{R}$ on intervals $[-\kappa, \infty)$ containing at most $\mathcal{K} \gg 1$ points.

Corollary. Asymptotic equality in law of $k = O(\mathcal{K})$ largest points.

Intensity of Ψ

The intensity of Ψ is

$$\rho(\mathrm{d}s) := \sum_{\ell\in\mathbb{Z}} u^{\langle du\rangle+\ell}\, g\big(s+\theta(\langle du\rangle+\ell)\big)\,\mathrm{d}s\,,$$

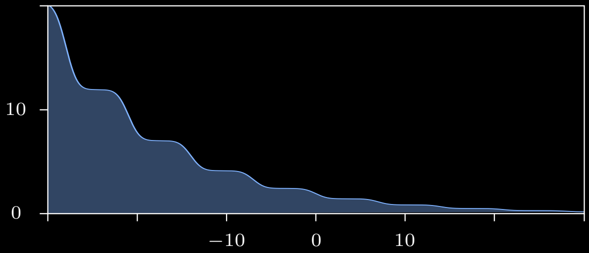
where $\langle\cdot\rangle$ is the periodic representative in $[-1/2,1/2)$, and $g(s) := \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}s^2}$.

Scaling laws

$$u \asymp \tau \asymp \sigma^2 \asymp \theta^2 \asymp \frac{t}{\log(t\vee 2)}\,, \qquad t := \frac{\log N}{d}\,.$$

Distribution of ρ in critical regime $t \asymp 1$:

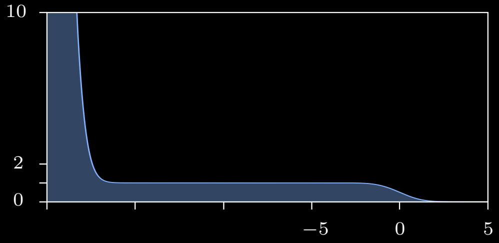
Top eigenvalue not governed by Gumbel law.



Distribution of ρ in subcritical regime $t \gg 1$:

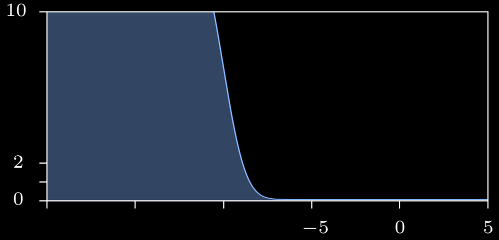
Resonance $\langle du \rangle = 0$:

Top eigenvalue not governed by Gumbel law.



Off-resonance $|\langle du \rangle| \geqslant c$:

Top eigenvalue governed by Gumbel law.



Spectral edge: eigenvector localization

Theorem. Suppose that

$$(\log\log N)^4 \leqslant d \leqslant (b_* - \kappa)\log N\,.$$

Let $\mathbf{w} = (w_x)$ be an eigenvector associated with one of the top (or bottom) $\mathcal{K}$ eigenvalues. Then with high probability there exists a vertex x with $\alpha_x > 2$ such that $\|\mathbf{w} - \mathbf{v}(x)\| = o(1)$.

Remark. The vector $\mathbf{v}(x)$ is explicit, radial, and exponentially decaying:

$$\mathbf{v}(x) := \sum_{i=0}^{r} u_i(x) \frac{\mathbf{1}_{S_i(x)}}{\|\mathbf{1}_{S_i(x)}\|}\,,$$

where

$$u_1(x) = \frac{\sqrt{\alpha_x}}{\sqrt{\alpha_x - 1}} u_0(x)\,, \qquad u_{i+1}(x) = \frac{1}{\sqrt{\alpha_x - 1}} u_i(x) \quad (i \geqslant 1)\,.$$

Part III: overview of the proof in the localized phase

Basic intuition: one-to-one correspondence between eigenvalues and vertices of large degree.

Main steps of proof:

Step 1. Characterize the fluctuations of an eigenvalue associated with a vertex of large degree.

Step 2. Establish a one-to-one relation between such eigenvalues and the eigenvalues of H near the edge.

Step 1

Consider neighbourhood of vertex in

$$\mathcal{U} := \{x : \alpha_x \geqslant 2 + \kappa\}\,.$$

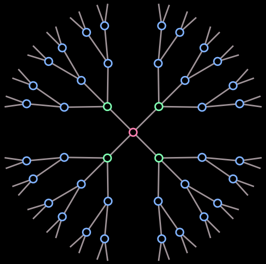
Use the tridiagonal representation of $H := d^{-1/2} A$ around x: write H in the basis $\mathbf{h}_0, \mathbf{h}_1, \mathbf{h}_2, \dots$ obtained by orthogonalizing $\mathbf{1}_x, H\mathbf{1}_x, H^2\mathbf{1}_x, \dots$.

Apply transfer matrix (or orthogonal polynomial) analysis.

Problem: Fluctuations of transfer matrices very hard to control precisely, because $\mathbf{h}_i$ is unwieldy.

Toy model: in a rooted regular tree, the degree depends only on the distance to the root.

Exercise: if $\mathbb{G}|_{B_r(x)}$ is a rooted regular tree, then $\mathbf{h}_i = \mathbf{1}_{S_i(x)}$ for $i \leqslant r$.



- Naive attempt: write H in basis $(\mathbf{1}_{S_i(x)})$ instead of $(\mathbf{h}_i)$, to get an almost tridiagonal matrix.

  Problem: off-tridiagonal matrix is too large.

- More refined attempt: If $\mathbb{G}|_{B_r(x)}$ is a tree, the vector $H^i \mathbf{1}_x$ can be decomposed as a sum over simple walks in $\mathbb{N}$ of length i.

jump left / right $\iff$ terms decreasing / increasing distance from root

- Basis $(\mathbf{h}_i)$: all walks
- Basis $(\mathbf{1}_{S_i(x)})$: only steps to the right

Define basis $(\mathbf{f}_i)$ using walks with at most one step to the left.
For instance,

$$\mathbf{f}_3 = \mathbf{1}_{S_3(x)} + \sum_{y \in S_1(x)} (d\alpha_y - F)\mathbf{1}_y\,, \qquad F \in \mathbb{R}\,.$$

Proposition. Let $r \gg 1$ be suitably chosen. Let M be the matrix H in the basis $(\mathbf{f}_i)_{i=0}^r$. Then

$$\|M - Z_{\mathfrak{d}}(\alpha_x, \beta_x)\| \leqslant d^{-1-c},$$

where

$$\alpha_x = \frac{|S_1(x)|}{d}, \qquad \beta_x = \frac{|S_2(x)|}{|S_1(x)|d},$$

and

$$Z_{\mathfrak{d}}(\alpha, \beta) := \begin{pmatrix} 0 & \sqrt{\alpha} & & & & \\ \sqrt{\alpha} & 0 & \sqrt{\beta} & & & \\ & \sqrt{\beta} & 0 & \sqrt{\mathfrak{d}} & & \\ & & \sqrt{\mathfrak{d}} & 0 & \sqrt{\mathfrak{d}} & \\ & & & \sqrt{\mathfrak{d}} & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \qquad \mathfrak{d} := 1 + \frac{1}{d}.$$

Remark. $\sqrt{d} Z_{\mathfrak{d}}(\alpha, \beta)$ is the tridiagonalization at the root of the rooted regular tree with degree sequence $\alpha d, \beta d, d+1, d+1, \ldots$.

$Z_{\mathfrak{d}}(\alpha_x, \beta_x)$ has a unique eigenvalue $\Lambda_{\mathfrak{d}}(\alpha_x, \beta_x) > 2 + \kappa$, with exponentially decaying eigenvector $(u_i)_{i=0}^r$.

Back to graph $\mathbb{G}$ with

$$\mathbf{y}(x) := \sum_{i=0}^{r} u_i \frac{\mathbf{f}_i}{\|\mathbf{f}_i\|}\,.$$

It is possible to show that

$$\big\|\big(H - \Lambda_{\mathfrak{d}}(\alpha_x, \beta_x)\big)\mathbf{y}(x)\big\| \leqslant d^{-1-c}\,. \tag{1}$$

Step 1 is concluded by analysing the fluctuations of $\Lambda_{\mathfrak{d}}(\alpha_x, \beta_x)$ (of order d^{-1}).

Step 2

Need to ensure:

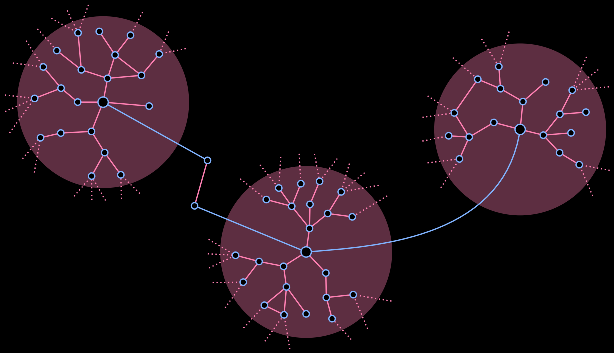
(a) $(\mathbf{y}(x) : x \in \mathcal{U})$ are orthogonal (i.e. $(B_r(x) : x \in \mathcal{U})$ are disjoint).

(b) The high probability bounds hold simultaneously for all $x \in \mathcal{U}$.

(c) The remaining eigenvalues cannot "pollute" the edge of the spectrum.

All of these present significant complications. In fact, (a) and (b) are wrong.

(a) $(B_r(x) : x \in \mathcal{U})$ are disjoint only if either (i) $\mathcal{U}$ is small or (ii) we prune the graph by removing edges to disconnect balls.

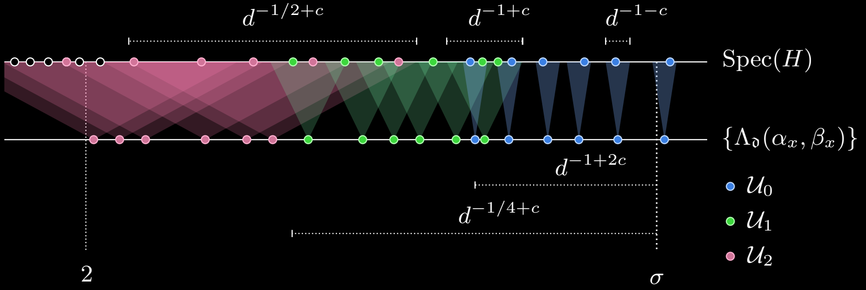


The pruning is potentially deadly, since in general removing even a single edge perturbs an eigenvalue by $O(1/\sqrt{d})$.

We have to prune in places that have a small impact on the extreme eigenvalues: prune only in the neighbourhoods of vertices x whose α_x is far from the top degree.

(b) The estimate (1) is not true simultaneously for all $x \in \mathcal{U}$. Solution: three-scale rigidity argument with the partition $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \mathcal{U}_2$, where $\alpha_x > \alpha_y$ for $x \in \mathcal{U}_i$ and $y \in \mathcal{U}_{i+1}$.

The sets $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2$ are increasing in size, but the accuracy of the estimate (1) deteriorates as i increases.



Block diagonal representation

$$O^{-1}HO = \begin{pmatrix} \mathcal{D}_0 & 0 & 0 & E_0^* \\ 0 & \mathcal{D}_1 & 0 & E_1^* \\ 0 & 0 & \mathcal{D}_2 + \mathcal{E}_2 & E_2^* \\ E_0 & E_1 & E_2 & X \end{pmatrix}$$

where

$$\mathcal{D}_i = \operatorname{diag}(\Lambda_{\mathfrak{d}}(\alpha_x, \beta_x) + O(\xi_i) : x \in \mathcal{U}_i)$$

$$\xi_i + \|E_i\| = \begin{cases} d^{-1-c} & \text{if } i = 0 \\ d^{-1+c} & \text{if } i = 1 \\ d^{-1/2+c} & \text{if } i = 2 \end{cases} \qquad \longleftarrow \text{main estimates}$$

$$\|\mathcal{E}_2\| = O(d^{-1/2+c}) \qquad \longleftarrow \text{pruning}$$

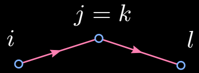
$$\|X\| \leqslant 2 + o(1) \qquad \longleftarrow \text{(c)}$$

(c) Estimate of $\|X\|$ involves two main steps.

1. Quadratic form estimate. $H \leqslant I + Q + \mathrm{o}(1)$ with $Q = \operatorname{diag}(\alpha_1, \dots, \alpha_N)$.

Proof. Define the nonbacktracking matrix $B = (B_{ef})_{e,f \in [N]^2}$ associated with H through

$$B_{(ij)(kl)} := H_{kl} \mathbf{1}_{j=k} \mathbf{1}_{i \neq l} \,.$$



Then, by [Bordenave, Benaych-Georges, K; 2017], $\rho(B) = 1 + \mathrm{o}(1)$.

Next, invoke a general Ihara-Bass-type formula: define the matrices $H(\lambda)$ and $D(\lambda) = \operatorname{diag}(D_x(\lambda))_{x \in [N]}$ through

$$H_{xy}(\lambda) \; := \; \frac{\lambda H_{xy}}{\lambda^2 - H_{xy} H_{yx}} \,, \qquad D_x(\lambda) \; := \; 1 + \sum_u \frac{H_{xu} H_{ux}}{\lambda^2 - H_{xu} H_{ux}} \,. \qquad (2)$$

Then $\lambda \in \operatorname{Spec}(B)$ if and only if $\det(D(\lambda) - H(\lambda)) = 0$.

Show that $H(\lambda) \approx H/\lambda$ and $D(\lambda) \approx I + Q/\lambda^2$. Then a simple continuity argument using $\rho(B) = 1 + \mathrm{o}(1)$ yields the claim. $\square$

2. Local delocalization bound. Let $\lambda \geqslant 2 + \kappa$ be an eigenvalue of H with normalized eigenvector $\mathbf{w} \perp \mathrm{Span}(\mathbf{v}(x) : x \in \mathcal{U})$. Then

$$\sum_x \mathbb{1}_{\alpha_x \geqslant 1+\kappa} w_x^2 = \mathrm{o}(1)\,.$$

Proof. Radial Combes-Thomas-type argument. □

Now we can conclude (c): if $\lambda \geqslant 2 + \kappa$ be an eigenvalue of H with normalized eigenvector $\mathbf{w} \perp \mathrm{Span}(\mathbf{v}(x) : x \in \mathcal{U})$, then

$$\begin{aligned}
\lambda = \langle \mathbf{w}\,, H\mathbf{w} \rangle &\overset{1}{\leqslant} 1 + o(1) + \sum_x \alpha_x w_x^2 \\
&= 1 + o(1) + \sum_x \mathbb{1}_{\alpha_x \leqslant 1+\kappa} \alpha_x w_x^2 + \sum_x \mathbb{1}_{\alpha_x > 1+\kappa} \alpha_x w_x^2 \\
&\leqslant 2 + o(1) + \max_x \alpha_x \sum_x \mathbb{1}_{\alpha_x > 1+\kappa} w_x^2 \\
&\overset{2}{\leqslant} 2 + o(1)\,.
\end{aligned}$$

Part IV: overview of the proof in the delocalized phase

Delocalization follows from local law, controlling Green function

$$G = (H - z)^{-1}$$

for $\operatorname{Im} z \gg N^{-1}$ and $|\operatorname{Re} z| < 2 - \mathrm{o}(1)$.

Schur complement formula yields

$$\frac{1}{G_{xx}} = -z - \frac{1}{d} \sum_{y, \tilde{y} \in S_1(x)} G_{y\tilde{y}}^{(x)} \tag{3}$$

where $(\cdot)^{(x)}$ means vertex x is removed.

Remark. Suppose that all neighbours in $S_1(x)$ are in different connected components of $A^{(x)}$. Then

$$\frac{1}{G_{xx}} = -z - \frac{1}{d} \sum_{y \in S_1(x)} G_{yy}^{(x)}$$

and

$$G_{yy}^{(x)} - G_{yy} = (G_{yy}^{(x)})^2 \frac{1}{d} G_{xx}.$$

The assumption of Remark is badly wrong for $G(N, d/N)$ and the conclusion in general completely wrong for $\operatorname{Im} z \ll 1$.

Key insight of proof: if $\max_{x,y} |G_{xy}|$ is bounded then the conclusions of Lemma 3 remain essentially correct although the justification is completely different.

Hence, we obtain a self-consistent equation for the vector $(G_{xx})_{x \in [N]}$.

More precisely: suppose that $\max_{x,y}|G_{xy}| \leqslant C$.

Step 1. By large deviation estimates for quadratic forms of sparse random vectors [He, K, Marcozzi; 2018] from (3) we obtain

$$\frac{1}{G_{xx}} = -z - \frac{1}{d} \sum_{y \in S_1(x)} G_{yy}^{(x)} + \mathrm{o}(1)$$

with very high probability. (Requires only $d \gg 1$.)

Step 2. Define the error parameter

$$\Psi_x := \frac{1}{d} \sum_{y \in S_1(x)} G_{yy}^{(x)} - \frac{1}{N} \sum_{y \in [N]} G_{yy}^{(x)} .$$

A vertex x is typical if $\Psi_x = \mathrm{o}(1)$.

Key Lemma. With very high probability,

(i) Most vertices are typical.

(ii) Any vertex has few atypical neighbours.

Proof of Lemma is main work, requires $d \gg \sqrt{\log N}$.

Step 3. Obtain self-consistent equation for $(G_{xx})_{x \in [N]}$, which has solution

$$G_{xx} = -\frac{1}{z + \alpha_x m} + \mathrm{o}(1)\,, \qquad m = \text{Stieltjes transform of semicircle law.}$$

The solution is uniformly bounded for $c \leqslant |\mathrm{Re}\, z| \leqslant 2 - c$.

Step 4. Bootstrap in $\mathrm{Im}\, z$ from $\mathrm{Im}\, z = 1$ down to $\mathrm{Im}\, z \gg N^{-1}$.