Disordered systems at complex temperatures: Phase diagrams, fluctuations and zeros

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- **Complex Random Energy Model: Zeros and Fluctuations.** With Zakhar Kabluchko. Prob. Theor. and Rel. Fields. 2012.
- **Generalized random energy model at complex temperatures.** With Zakhar Kabluchko. Preprint arXiv:1402.2142, 2014.
- **The glassy phase of the complex branching Brownian motion energy model.** With Lisa Hartung. Electron. Commun. Probab. 20 (78), 1–15, 2015.
- **The phase diagram of the complex branching Brownian motion energy model.** With Lisa Hartung. Electron. J. Probab. 23 (127), 27 pp. 2018.

Disordered systems: Hamiltonian as a random field

Fruitfull approach, Derrida (1980):

- Treat $H(\cdot)$ as a **random field**.
- Explore **universality classes**.

- **Q:** What is the **simplest random field**?
- **A:** **“White noise”** (e.g., i.i.d. Gaussian field).

Derrida's random energy model

Derrida (1980): Partition function:

$$\mathscr{Z}_N(\beta) := \sum_{k=1}^{N} \mathrm{e}^{\beta\sqrt{n}X_k}.$$

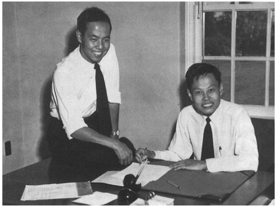
- **Notation:** $n = \log N$.
- $\{X_k\}_{k=1}^{\infty}$ are **i.i.d.** $\mathscr{N}(0,1)$ **random energies**.
- Seemingly unrealistic: No microscopic interactions, no spins, completely random energy levels, ...
- **Q:** Why bother?

Large-volume limit of the log-partition function:

$$p(\beta) := \lim_{N\to\infty} \frac{1}{n} \log \mathscr{Z}_N(\beta) = \begin{cases} 1 + \frac{1}{2}\beta^2, & 0 \le \beta \le \sqrt{2}, \\ \sqrt{2}\beta, & \beta \ge \sqrt{2}. \end{cases}$$

$\Rightarrow$ **phase transition!**

Lee-Yang Program (1952)



Phase Transitions

$\Updownarrow$

Analyticity Breaking of $p_N(\beta)$, as $N \to \infty$

$\Updownarrow$

($\log$ is non-analytic only at zero, $\mathscr{Z}_N(\cdot)$ is an entire function)

$\Updownarrow$

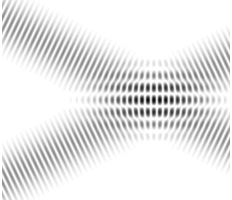
Zeros of $\mathscr{Z}_N(\beta)$, as $N \to \infty$, $\beta \in \mathbb{C}$

accumulate around $\beta_{\mathbf{c}} \in \mathbb{R}$

$\Rightarrow$ **Complex-valued Hamiltonians!**

Interference phenomena

- **Quantum Physics:**
  - Schrödinger equations with random potentials.
  - Path integrals.
  - Quantum Monte Carlo.
  - ...



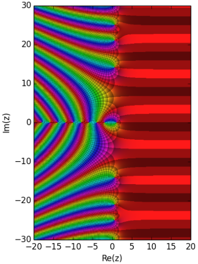
interference

$\Rightarrow$ **Complex-valued Hamiltonians!**

Riemann zeta-function, random matrices and complex random energy models

Riemann's zeta-function:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s \in \mathbb{C} \setminus \{1\}.$$



- $\log|\zeta(1/2 + it + i\omega T)|$, for $\omega \sim \mathrm{Uniform}(0,1)$, $T \to \infty$ behaves like a complex log-correlated field on micro- and mesoscopic scales, see **Fyodorov, Keating (2014)**; **Saksman, Webb (2016); Arguin, Belius, Bourgade (2019)**, ...
- $\log$ of CUE's characteristic polynomial behaves like a log-correlated field

$\Rightarrow$ **Complex-valued Hamiltonians!**

Complex REM

with Zakhar Kabluchko

Define:

$$\mathcal{Z}_N(\beta) := \sum_{k=1}^{N} \mathrm{e}^{\sqrt{n}(\sigma X_k + i\tau Y_k)}, \qquad \beta = (\sigma, \tau) \in \mathbb{R}^2.$$

- $\{(X_k, Y_k)\}_{k=1}^{\infty}$ i.i.d. zero-mean bivariate Gaussian random vectors with

$$\operatorname{Var} X_k = \operatorname{Var} Y_k = 1, \quad \operatorname{corr}(X_k, Y_k) = \rho,$$
$$-1 \le \rho \le 1$$

$\Rightarrow$ **Complex REM.**

Log-partition function

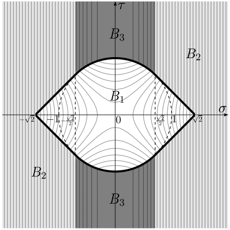
Theorem (Kabluchko and K. 2014)

For every $\beta \in \mathbb{R}^2$ *and any* ρ*, the limit*

$$p(\beta) := \lim_{N\to\infty} p_N(\beta)$$

exists in probability and in L^q*,* $q \geq 1$*, and is explicitly given as*

$$p(\beta) = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B}_1, \quad \textbf{[LNN]} \\ \sqrt{2}|\sigma|, & \beta \in \overline{B}_2, \quad \textbf{[EVT]} \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B}_3. \quad \textbf{[CLT, } \mathrm{Var} > \mathbb{E}^2\textbf{]} \end{cases}$$



N.B.:

- ρ-independent limit.
- The formula was heuristically derived by **Derrida (1991)**.
- Our proof: **fluctuations of** $\mathscr{Z}_N(\beta)$ + **continuous mapping theorem**.

Zeros

Theorem (Zeros)

It holds that

(a) *There are no zeros of $\mathscr{Z}_N$ inside any $K \Subset B_1$ w.h.p., as $N \to \infty$.*

(b) *There is positive density of zeros only in B_3 and on $(\bar{B}_1 \cap \bar{B}_3) \cup (\bar{B}_1 \cap \bar{B}_3)$:*

$$\frac{1}{n} \sum_{\beta \in \mathbb{C}:\ \mathscr{Z}_N(\beta)=0} f(\beta) \xrightarrow[N\to\infty]{P} \frac{1}{2\pi} \int_{\mathbb{C}} f(\beta) \Xi(\mathrm{d}\beta), \quad \forall f \in C_K(\mathbb{C}, \mathbb{R}),$$

where $\Xi := \Delta p$ and therefore

$$\Xi = 2\Xi_3 + \Xi_{12} + \Xi_{13},$$

where

- $\Xi_3 = \text{Lebesgue2D}(B_3)$,
- $\frac{\mathrm{d}\Xi_{13}}{\mathrm{dLebesgue1D}(\bar{B}_1 \cap \bar{B}_3)}(\sigma, \tau) = 2|\tau|$, $(\sigma, \tau) \in \bar{B}_1 \cap \bar{B}_3)$,
- $\Xi_{12} = \text{Lebesgue1D}(\bar{B}_1 \cap \bar{B}_3)$.

Fluctuations of zeros

- For $\beta_0 \in B_3$, $\forall f \in C_K(\mathbb{C}, \mathbb{R})$,

$$\sum_{\beta \in \mathbb{C}:\ \mathscr{Z}_N(\beta)=0} f(\sqrt{n}(\beta - \beta_0)) \xrightarrow[N\to\infty]{w} \sum_{\beta \in \mathbb{C}:\ \mathbb{G}(\beta)=0} f(\beta),$$

  where
  - **Gaussian analytic function**: $\mathbb{G}(t) := \sum_{k=0}^{\infty} \xi_k \frac{t^k}{\sqrt{k!}}$, $t \in \mathbb{C}$.
  - $\xi_0, \xi_1, \ldots$ are i.i.d. $\mathscr{N}_{\mathbb{C}}(0,1)$.
- For $f \in C_K(B_2, \mathbb{R})$,

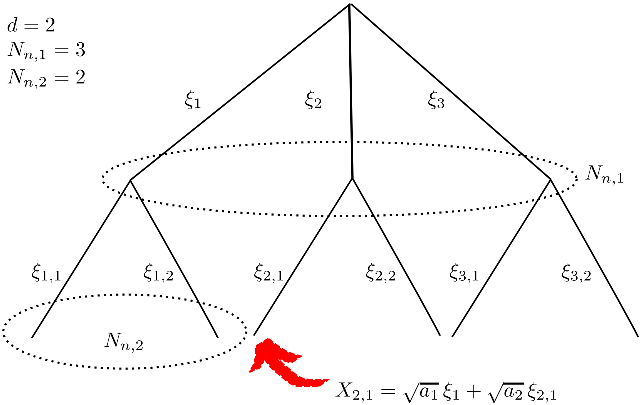
$$\sum_{\beta \in B_2:\ \mathscr{Z}_N(\beta)=0} f(\beta) \xrightarrow[N\to\infty]{w} \sum_{\substack{\beta \in B_2: \\ \zeta_P^{(1)}(\beta/\sqrt{2})=0}} f(\beta) + \sum_{\substack{\beta \in B_2: \\ \zeta_P^{(2)}(\beta/\sqrt{2})=0}} f(-\beta),$$

  where
  - **Poisson ζ-function:** $\zeta_P(\beta) := \sum_{k=1}^{\infty} \frac{1}{{P_k}^\beta}$ with $(P_k)_{k\geq 1}$ being the arrivals of the Poisson process on $\mathbb{R}_+$ with unit intensity.
  - $\zeta_P^{(1)}, \zeta_P^{(2)} \underset{\text{i.i.d.}}{\sim} \zeta_P$.

Beyond the REM universality class

1. **Q: What happens beyond the REM universality class?**
2. **Q: How strong should the correlations be in order to fall out of the REM universality class?**

Generalised Random Energy Model: two levels



Cumulative displacement = energy:

$$X_{\varepsilon_1 \varepsilon_2} = \sqrt{a_1}\,\xi_{\varepsilon_1} + \sqrt{a_2}\,\xi_{\varepsilon_1 \varepsilon_2}.$$

Generalised Random Energy Model: d levels

Define a zero-mean Gaussian random field $\{X_\varepsilon : \varepsilon \in \mathbb{S}_n\}$ by

$$X_\varepsilon = \sqrt{a_1}\,\xi_{\varepsilon_1} + \sqrt{a_2}\,\xi_{\varepsilon_1\varepsilon_2} + \ldots + \sqrt{a_d}\,\xi_{\varepsilon_1\ldots\varepsilon_d}.$$

1. **Number of levels** $d \in \mathbb{N}$.
2. **The variances** of the levels $a_1, \ldots, a_d > 0$ (**energetic parameters**).
3. **The branching exponents** $\alpha_1, \ldots, \alpha_d > 1$ (**entropic parameters**).
4. **Branching numbers** $N_{n,1} = [\alpha_1^n], \ldots, N_{n,d} = [\alpha_d^n]$.
5. **Leaves of the tree**
   $\mathbb{S}_n = \{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{N}^d : 1 \le \varepsilon_1 \le N_{n,1}, \ldots, 1 \le \varepsilon_d \le N_{n,d}\}$.
6. **Random energies**
   $\{\xi_{\varepsilon_1\ldots\varepsilon_m} : 1 \le m \le d, 1 \le \varepsilon_1 \le N_{n,1}, \ldots, 1 \le \varepsilon_m \le N_{n,m}\}$ i.i.d. $\mathcal{N}(0,1)$ r.v.'s.

Phase diagram of the GREM in the complex β plane

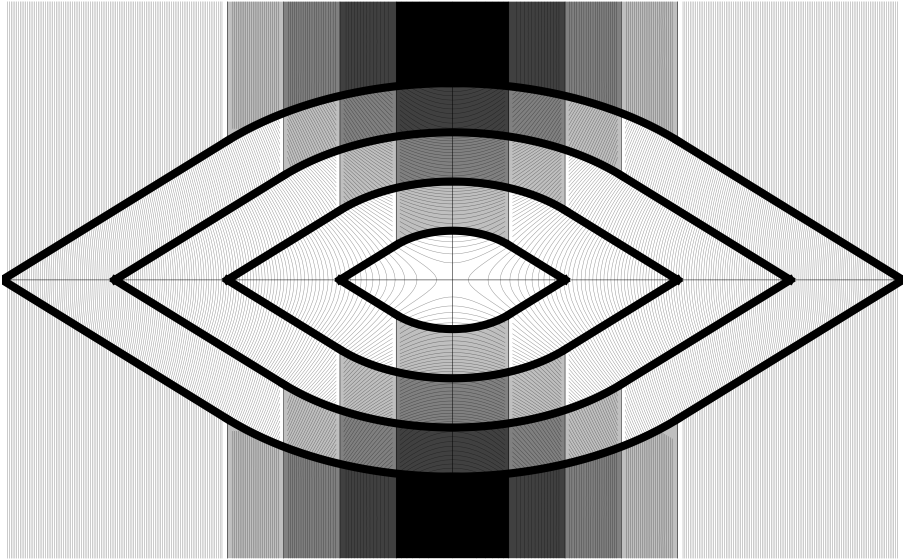
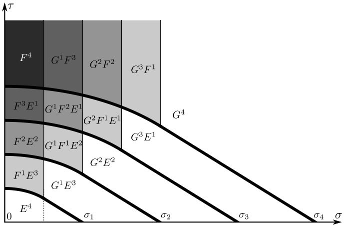


Figure: Phases and zeros. The darker the shading, the more zeros

Phases decoded

The GREM phase transitions (on $\mathbb{R}$) occur at

$$\sigma_k := \sqrt{\frac{2\log \alpha_k}{a_k}}, \quad 1 \le k \le d, \quad \textbf{Assume: } \sigma_1 < \ldots < \sigma_d.$$



$E \rightsquigarrow$ **expectation**,
$F \rightsquigarrow$ **fluctuations**,
$G \rightsquigarrow$ **glassy (extremes)**.

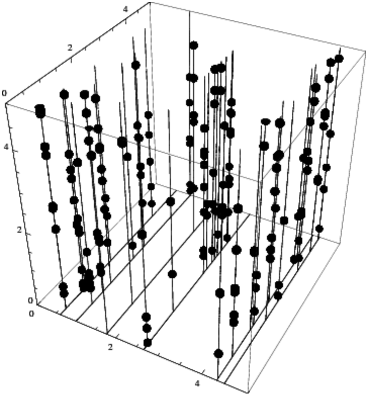
Each level can be in on of the **rescaled REM phases**:

$$G_k := \{\beta \in \mathbb{C} \colon 2|\sigma| > \sigma_k,\ |\sigma| + |\tau| > \sigma_k\},$$
$$F_k := \{\beta \in \mathbb{C} \colon 2|\sigma| < \sigma_k,\ 2(\sigma^2 + \tau^2) > \sigma_k^2\},$$
$$E_k := \mathbb{C} \backslash \overline{G_k \cup F_k},$$

so the rescaled REM phase transition occurs at $\beta = \sigma_k$.

Fluctuations: A limiting object

Poisson cascade $\Pi = \sum_{\varepsilon=(\varepsilon_1,\ldots,\varepsilon_d)\in\mathbb{N}^d} \delta(P_{\varepsilon_1}, P_{\varepsilon_1\varepsilon_2}, \ldots, P_{\varepsilon_1\ldots\varepsilon_d})$, where $\sum_{i=1}^{\infty} \delta(P_{\varepsilon_1\ldots\varepsilon_m i})$ a unit intensity Poisson point process on $(0,\infty)$.



Poisson cascade ζ**-function**: $\zeta_P(z_1,\ldots,z_d) = \sum_{\varepsilon\in\mathbb{N}^d} P_{\varepsilon_1}^{-z_1} P_{\varepsilon_1\varepsilon_2}^{-z_2} \ldots P_{\varepsilon_1\ldots\varepsilon_d}^{-z_d}$.

Fluctuations

Theorem

Let $\beta \in G^{d_1}F^{d_2}E^{d_3}$. Then,

$$\frac{\mathscr{Z}_n(\beta)}{e^{c_n(\beta)}} \underset{n\to\infty}{\Longrightarrow} \begin{cases} 1, & \text{if } d_1 = 0 \text{ and } d_2 = 0, \\ N_{\mathbb{C}}(0,1), & \text{if } d_1 = 0 \text{ and } d_2 > 0, \\ \zeta_P\left(\frac{\beta}{\sigma_1}, \ldots, \frac{\beta}{\sigma_{d_1}}\right), & \text{if } d_1 > 0 \text{ and } d_2 = 0, \\ cS_{\sigma/\sigma_1}, & \text{if } d_1 > 0 \text{ and } d_2 > 0. \end{cases}$$

Here, ζ_P is the Poisson cascade zeta function and S_α is the isotrpic, complex standard α-stable random variable with characteristic function $\mathbb{E}e^{i\mathrm{Re}(S_\alpha \bar{z})} = e^{-|z|^\alpha}$, $z \in \mathbb{C}$, where $\alpha \in (0,2)$.

$$c_{n,k}(\beta) = \begin{cases} \beta\sqrt{na_k}\,u_{n,k}, & \text{if } \beta \in G^k, \\ \frac{1}{2}\log N_{n,k} + a_k\sigma^2 n, & \text{if } \beta \in F^k, \\ \log N_{n,k} + \frac{1}{2}a_k\beta^2 n, & \text{if } \beta \in E^k. \end{cases}$$

$c_n(\beta) = c_{n,1}(\beta) + \ldots + c_{n,d}(\beta)$, $u_{n,k} = \sigma_k\sqrt{na_k}$.

Log-partition function

Theorem

For every $\beta \in \mathbb{C}$, the following limit exists in probability and in L^q, for all $q \geq 1$:

$$p(\beta) := \lim_{n\to\infty} \frac{1}{n} \log |\mathscr{Z}_n(\beta)| = \sum_{k=1}^{d} p_k(\beta),$$

where

$$p_k(\beta) = \begin{cases} |\sigma|\sqrt{2a_k \log \alpha_k}, & \text{if } \beta \in \bar{G}_k, \\ \frac{1}{2}\log \alpha_k + a_k \sigma^2, & \text{if } \beta \in \bar{F}_k, \\ \log \alpha_k + \frac{1}{2} a_k(\sigma^2 - \tau^2), & \text{if } \beta \in \bar{E}_k. \end{cases}$$

Confirms and extends **Takahashi (2011)**.

Proof: Above **fluctuation results** + **continuous mapping theorem**.

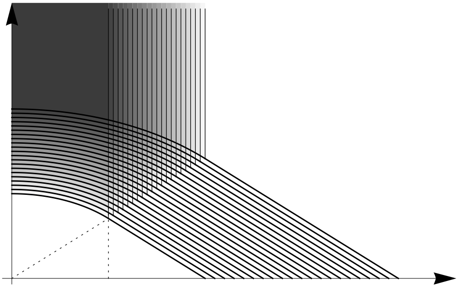
Infinitely deep & wide hierarchies: $d \to \infty$

The continuous GREM (CREM):

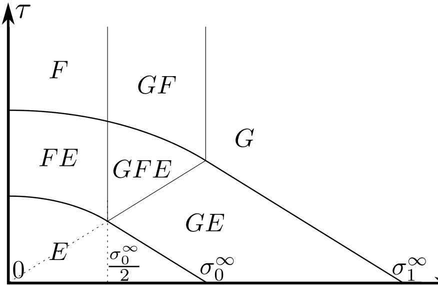
- Let $A\colon [0,1] \to \mathbb{R}$ be an increasing, concave function with $A(0) = 0$.
- Fix also some $\alpha > 1$.
- Consider a GREM with d levels whose parameters $(a_1, \ldots, a_d)$ and $(\alpha_1, \ldots, \alpha_d)$ are given by

$$a_1 + \ldots + a_k = A\left(\frac{k}{d}\right), \quad \log \alpha_k = \frac{1}{d} \log \alpha, \quad 1 \le k \le d.$$

Phase diagram $d = \infty$



Approximating CREM by GREM with many levels.



Conjectured phase diagram of the CREM w.r.t. complex temperatures.

Log-partition function of the CREM

Conjecture (The log-partition function of the CREM)

The log-partition function of the CREM converges to

$$p^{\infty}(\beta) := p_G^{\infty}(\beta) + p_F^{\infty}(\beta) + p_E^{\infty}(\beta), \quad \text{in } L^1,$$

where

$$\begin{aligned}
p_G^{\infty}(\beta) &:= |\sigma|\sqrt{2\log\alpha}\int_0^{\gamma_1}\sqrt{A'(t)}\,\mathrm{d}t, \\
p_F^{\infty}(\beta) &:= \frac{\gamma_2}{2}\log\alpha + (A(\gamma_1+\gamma_2) - A(\gamma_1))\sigma^2, \\
p_E^{\infty}(\beta) &:= \gamma_3\log\alpha + \frac{1}{2}(\sigma^2 - \tau^2)(A(1) - A(\gamma_1+\gamma_2)).
\end{aligned}$$

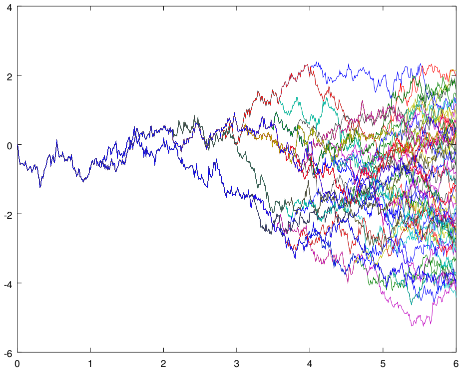
REM borderline

Q: What happens at the borderline of the REM universality class?

- **The glassy phase of the complex branching Brownian motion energy model.** Electron. Commun. Probab. 20 (78), 1–15, 2015
- **The phase diagram of the complex branching Brownian motion energy model.** Electron. J. Probab. 23 (127), 27 pp. 2018.

with **Lisa Hartung**

Branching Brownian motion



- (Supercritical) **Galton-Watson** process: $i_1(t), \ldots, i_{n(t)}(t)$, $t \in \mathbb{R}_+$.
- **Genealogy**: $i_k(s,t)$ is the unique **ancestor** of particle $i_k(t)$ at time $s < t$.
- **Correlations** $k, l \leq n(t)$:

$$\mathbb{E}\left[x_k(s,t)x_l(r,t) \mid \text{Genealogy upto time } t\right] = \underbrace{d(i_k(s,t), i_l(r,t))}_{\text{tree overlap}}, \quad s, r \in [0,t].$$

Complex branching Brownian motion energy model

Partition function:

$$\widetilde{\mathscr{Z}}_{\beta,\rho}(t) := \sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t)+i\tau y_k(t)},$$

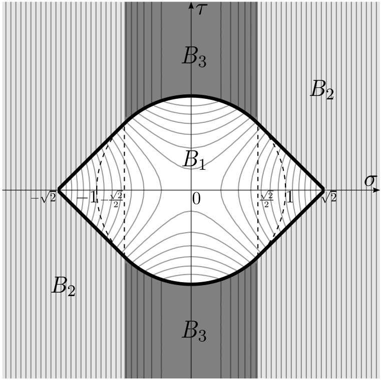
where x, y are BBMs with

- The same genealogy.
- $\mathrm{Cov}(x_k(t), y_k(t)) = \rho t$, $\rho \in [-1, 1]$.

Thechnical assumptions on the Galton-Watson process:

- $\sum_{k=1}^{\infty} p_k = 1$ (none of the particles die);
- $\sum_{k=1}^{\infty} k p_k = 2$ (the expected number of children per particle equals two);
- $K := \sum_{k=1}^{\infty} k(k-1)p_k < \infty$ (finite second moment).

Summary of results

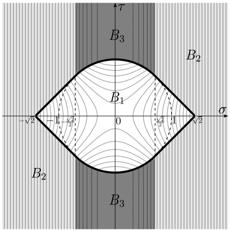


Phase diagram of the BBM energy model

Same phase diagram as in the REM, but markedly different fluctuations:

- B_1: Law of large numbers $\rightsquigarrow$ martingale convergence.
- B_2: Glassy phase $\rightsquigarrow$ EVT for a strongly correlated field.
- B_3: Central limit theorem $\rightsquigarrow$ CLT with a random variance.

Log-partition function



Theorem (Phase diagram)

For any $\rho \in [-1,1]$, the complex ***BBM energy model has the same free energy and the phase diagram as the complex REM****:*

$$\lim_{t\uparrow\infty} p_t(\beta) =: p(\beta) = \begin{cases} 1+\frac{1}{2}(\sigma^2-\tau^2), & \beta \in \overline{B_1}, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B_2}, \\ \frac{1}{2}+\sigma^2, & \beta \in \overline{B_3}, \end{cases}$$

and the convergence holds in probability and (conjecturally) in L^1.

Fluctuations of the partition function

Q: Fluctuations of $\mathscr{Z}_N(\beta)$?

Distribution of the maximum

- Note that $\mathbb{E}[n(t)] = \mathrm{e}^t$.
- Define

$$m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t.$$

Bramson (1978) + **Lalley and Selke (1987)**:

$$\lim_{t\uparrow\infty} \mathbb{P}\left\{ \max_{k\le n(t)} x_k(t) - m(t) \le y \right\} = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}} \right], \quad y \in \mathbb{R},$$

where $C > 0$ is a constant and Z is the a.s. limit of the so-called **derivative martingale**:

$$Z := \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} (\sqrt{2}t - x_k(t))\mathrm{e}^{-\sqrt{2}(\sqrt{2}t - x_k(t))}, \quad \text{a.s.}$$

Extremal process

Arguin, Bovier, Kistler (2013)

$$\mathcal{E}_t := \sum_{k=1}^{n(t)} \delta_{x_k(t)-m(t)}, \quad t \in \mathbb{R}_+$$

converges in law as $t \uparrow \infty$ to the point process

$$\mathcal{E} := \sum_{k,l} \delta_{\eta_k + \Delta_l^{(k)}},$$

where:

(a) $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ are the atoms of a Cox process with **random intensity measure** $CZ\mathrm{e}^{-\sqrt{2}y}\mathrm{d}y$,

(b) $\{\Delta_l^{(k)}\}_{l\in\mathbb{N}} \subset \mathbb{R}$ are the atoms of i.i.d. PP $\Delta^{(k)}$, $k \in \mathbb{N}$ called **clusters** which are i.i.d. copies of

$$\Delta := \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} \delta_{\hat{x}_k(t) - \max_{l\le n(t)} \hat{x}_l(t)}$$

with $\hat{x}(t)$ being BBM $x(t)$ conditioned on $\max_{k\le n(t)} x_k(t) \ge \sqrt{2}t$.

Shape of $\mathscr{E}$

Part 1) (Cox-)Poisson point process with intensity $C\sqrt{2}Z\mathrm{e}^{-\sqrt{2}x}\mathrm{d}x$:

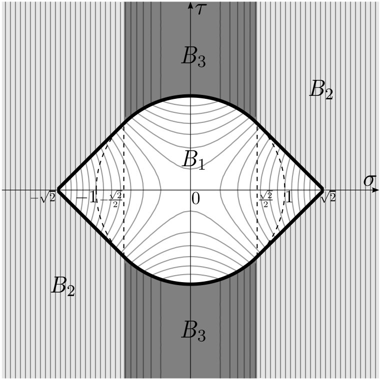
Poisson point process

Part 2) To each point of PPP associate an independent copy of some point process Δ:

Poisson point process + Δ_1

Poisson point process + all clusters

Partition function fluctuations in B_2



Theorem (Partition function fluctuations for $|\rho| = 1$)

For $\beta = \sigma + i\tau \in B_2$, the rescaled ***partition function*** $\mathscr{Z}_{\beta,1}(t) := \mathrm{e}^{-\beta m(t)} \widetilde{\mathscr{Z}}_{\beta,1}(t)$ *converges in law to the r.v.*

$$\mathscr{Z}_{\beta,1} := \sum_{k,l\geq 1} \mathrm{e}^{\beta\left(\eta_k + \Delta_l^{(k)}\right)}, \quad \text{as } t \uparrow \infty.$$

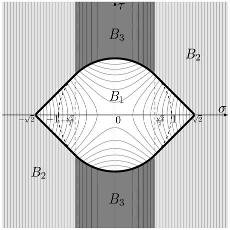
Theorem (Partition function fluctuations for $|\rho| \in (0,1)$)

Let $\beta = \sigma + i\tau \in B_2$ and $|\rho| \in (0,1)$. Then,

- *The rescaled* ***partition function*** $\mathscr{Z}_{\beta,\rho}(t)$ *converges in law to the r.v. $\mathscr{Z}_{\beta,\rho}$, as $t \uparrow \infty$.*
- *Conditionally on Z, $\mathscr{Z}_{\beta,\rho}$ is a* ***complex isotropic $\sqrt{2}/\sigma$-stable r.v.***

Partition function in B_1: Martingale convergence

Denote



$$\mathscr{M}_{\sigma,\tau}(t) := \mathrm{e}^{-t\left(1+\frac{\sigma^2-\tau^2}{2}\right)} \mathscr{Z}_{\beta,\rho}.$$

Theorem (Hartung-K. '17)

Let $\rho \in [-1,1]$ and $\beta \in B_1$.

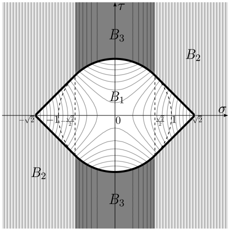
(i) *$\mathscr{M}_{\sigma,\tau}(t)$ is a mean 1* ***martingale****.*

(ii) *For $|\beta| \le 1$, $\mathscr{M}_{\sigma,\tau}(t)$ is in L^2.*
For $|\beta| > 1$, $\mathscr{M}_{\sigma,\tau}(t)$ is in L^p for $p < \sqrt{2}/\sigma$.

$$\lim_{t\uparrow\infty} \mathscr{M}_{\sigma,\tau}(t) = \mathscr{M}_{\sigma,\tau} \text{ a.s. and in } L^1.$$

Partition function in B_1: "CLT" for $|\sigma| < 1/\sqrt{2}$

Similar to phase B_3 a "CLT" holds in B_1 if $\sigma < 1/\sqrt{2}$.



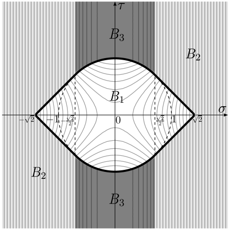
Theorem (Hartung-K. '17)

Let $\beta = \sigma + i\tau$ *with* $|\sigma| < \frac{1}{\sqrt{2}}$ *and* $\rho \in [-1, 1]$.
For $\beta \in B_1$,

$$\frac{\mathscr{M}_{\sigma,\tau}(t+r) - \mathscr{M}_{\sigma,\tau}(r)}{\mathrm{e}^{r(1-\sigma^2-\tau^2)}} \Rightarrow \mathscr{N}\left(0, C_1 \mathscr{M}_{2\sigma,0}\right),$$

as first $t \uparrow \infty$ *and then* $r \uparrow \infty$ *given a realization of* $\mathscr{M}_{2\sigma,0}$ *and for some constant* $0 < C_1 < \infty$.

Partition function in B_3: CLT*



$$B_3 := \{\sigma + i\tau \in \mathbb{C} \colon 2\sigma^2 < 1, \sigma^2 + \tau^2 > 1\}.$$

Intuition: ["CLT"]

Partition function is order of the root of the second moment of the real temperature part (assuming independence)

$$\left(\mathrm{e}^t \mathbb{E}\left(\mathrm{e}^{2\sigma x_k(t)}\right)\right)^{1/2} = \mathrm{e}^{t/2}\mathrm{e}^{\sigma^2 t},$$

where $x_k(t) \sim \mathcal{N}(0,t)$.

Partition function in B_3: CLT*

Theorem (Hartung-K. '17)

Let $\rho \in [-1,1]$ and $\beta \in B_3$. Assume that all moments of the offspring distribution exist.
Then, conditionally on $\mathscr{M}_{2\sigma,0}$,

$$\frac{\mathscr{Z}_{\beta,\rho}}{e^{t\left(\frac{1}{2}+\sigma^2\right)}} \Rightarrow \mathscr{N}\left(0, C_2 \mathscr{M}_{2\sigma,0}\right), \quad \text{as } t \uparrow \infty,$$

where

$$\mathscr{M}_{2\sigma,0} = \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} e^{2\sigma x_k(t)-(1+\sigma^2)t}.$$

Some related results

- **Derrida, Evans, Speer 1993**: independent Re & Im parts of the **complex random energy on a regular tree**, log-partition function, no fluctuations.
- **Barral, Jin, Mandelbrot 2010**: **complex Gaussian multiplicative cascades** ($d = 1$). (Phase I and Phase II, tightness only)
- **Lacoin, Rhodes, Vargas 2015**: **complex Gaussian multiplicative chaos** ($d \geq 2$). (Phase I and Phase III)
- **Meiners and Mentemeier 2017**; **Kolesko and Meiners 2016**: **complex smoothing transforms**.
- **Fyodorov, Hiary and Keating: 2012**; **Arguin, Belius and Bourgade: 2015**; **Saksman and Webb 2016**: **characteristic polynomials of random matrices**, and (probabilistic models of the) **Riemann zeta function**.
- **Hairer and Shen 2016**: renormalization of the **dynamical sine-Gordon model**.

Summary and Outlook

For REM, GREM and for BBM we now know:

- **Fluctuations** of the partition function;
- Distribution of complex **zeros** of the partition function;
- the **log-partition function**;
- the **phase diagram**.

Outlook:

- Free energy of the **randomized ζ-function** at complex temperatures.
- **Complex Gaussian multiplicative chaos** (Phase II).
- Models with **microscopic interactions**.