A Last Progeny Modified Branching Random Walk

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- A classical example is obtained by taking $\mathbf{Z} = \delta_{\xi_1} + \delta_{\xi_2}$, where ξ_1 and ξ_2 are i.i.d. standard Gaussian.

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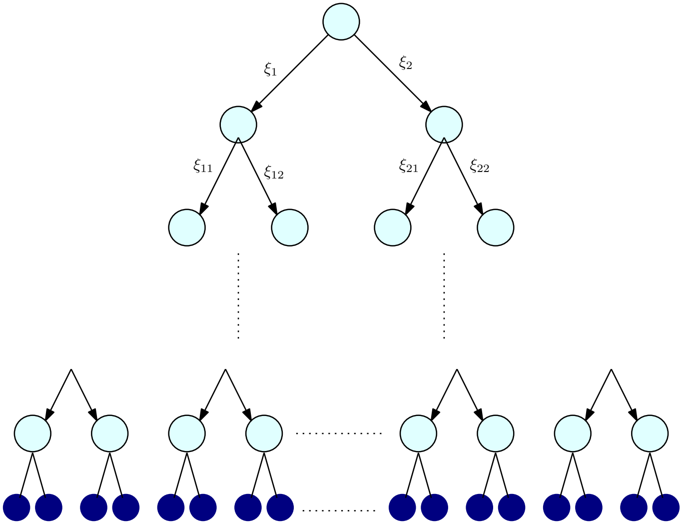

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- Notice that if $\mathbb{T}$ is the genealogical tree of the process, then it is nothing but a *Galton-Watson brunching process* with progeny distribution given by the random variable $N := \mathbf{Z}(\mathbb{R})$, the total mass of the point process $\mathbf{Z}$.

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- For the classical example the tree is nothing but *rooted binary tree* (just as in the figure).
- We will denote by S_v the position of an individual v (say at generation n).
- An interesting statistics related to this process is

$$R_n := \max_{|v|=n} S_v,$$

the so called *right-most position of the BRW*.

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$$\mathbf{cov}\left(S_u, S_v\right) = O\left(|u \wedge v|\right),$$

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- This observation is often referred as "BRW is *log-correlated*".

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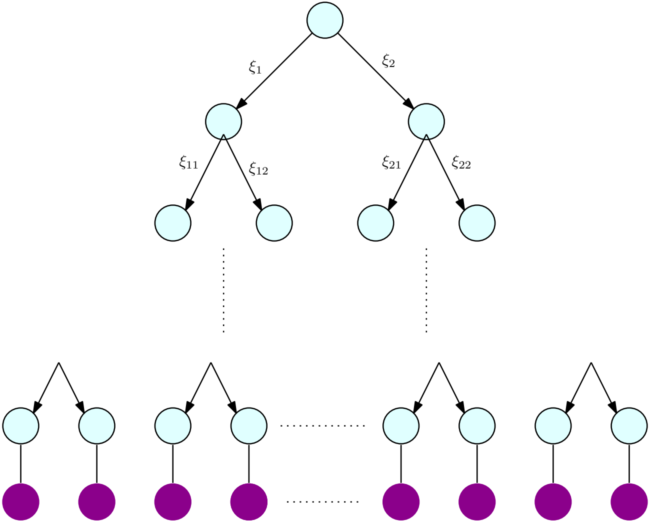

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- We further define
$$R_n^* := \max_{|v|=n} S_v,$$
as the *right-most position of the modified branching random walk*. This is analogous to R_n for the classical BRW.

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   - c. In our model, we will take a specific type of distribution for the X_v's. We will take $X_v = \log \frac{Y_v}{E_v}$, where $(Y_v)_{|v|=n}$ are i.i.d. with some positively supported measure, say, μ and $(E_v)_{|v|=n}$ are i.i.d. Exponential (1)-variables and both sets of variables will be independent.

Notations

- For a point process $\mathbf{Z} = \sum_{j=1}^{N} \delta_{\xi_j}$, we write

$$m(\theta) := \mathbf{E}\left[\int_{\mathbb{R}} e^{\theta x} \mathbf{Z}(dx)\right] = \mathbf{E}\left[\sum_{j=1}^{N} e^{\theta \xi_j}\right],$$

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- Note that ν is a *convex* function.

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- **Assumption 3:** N has finite $(1 + p)$-th moment for some $p > 0$.

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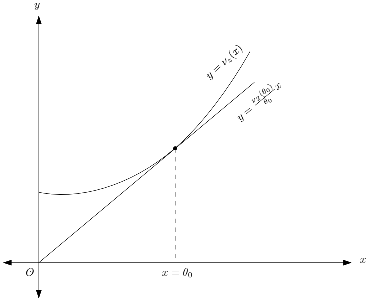
- Note that when $\theta_0 < \infty$, then it is the unique positive constant, such that, the tangent line to the curve $\theta \mapsto \nu(\theta)$ at the point $(\theta_0, \nu(\theta_0))$ passes through the origin.



A Specific Constant θ_0

- Three cases to be considered:
  a. **Boundary Case:** $\theta = \theta_0 < \infty$;
  b. **Below the Boundary Case:** $\theta < \theta_0 \leq \infty$; and
  c. **Above the Boundary Case:** $\theta_0 < \theta < \infty$.

SLLN for R_n^*

Theorem 1 [B. and Ghosh (2020)]

For every non-negatively supported probability $\mu \neq \delta_0$ that admits a finite mean

$$\frac{R_n^*(\theta,\mu)}{n} \xrightarrow{\text{a.s.}} \begin{cases} \frac{\nu(\theta)}{\theta} & \text{if} \quad \theta < \theta_0 \leq \infty; \\ \frac{\nu(\theta_0)}{\theta_0} & \text{if} \quad \theta = \theta_0 < \infty; \text{ and} \\ \frac{\nu(\theta_0)}{\theta_0} & \text{if} \quad \theta_0 < \theta < \infty. \end{cases}$$

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Remark: Note that the almost sure limit remains same as $\frac{\nu(\theta_0)}{\theta_0}$ for the *boundary case* and also in *above the boundary case*.

Boundary Case: $\theta = \theta_0 < \infty$

Theorem 2 [B. and Ghosh (2020)]

Assume that μ admits a finite mean, then there exists a random variable H_∞ such that

$$R_n^* - \frac{\nu(\theta_0)}{\theta_0} n + \frac{1}{2\theta_0} \log n \xrightarrow{d} H_{\theta_0}^\infty + \frac{1}{\theta_0} \log\langle\mu\rangle$$

where $H_{\theta_0}^\infty = \frac{1}{\theta_0}\left[\log D_\infty - \log E + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)\right]$ and D_∞ is a almost sure limit of the *derivative martingale*, namely, $D_n = \frac{1}{m(\theta_0)^n}\sum_{|v|=n}\left(\theta_0 S_v - n\nu(\theta_0)\right)e^{\theta_0 S_v}$ and $E \sim \text{Exponential}(1)$ random variable which is independent of D_∞. Further, $\sigma^2 := \mathbf{E}\left[\frac{1}{m(\theta_0)^n}\sum_{|v|=n}\left(\theta_0 S_v - n\nu(\theta_0)\right)^2 e^{\theta_0 S_v}\right]$.

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where $H_{\theta_0}^\infty = \frac{1}{\theta_0}\left[\log D_\infty - \log E + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)\right]$ and D_∞ is a almost sure limit of the *derivative martingale*, namely, $D_n = \frac{1}{m(\theta_0)^n}\sum_{|v|=n}\left(\theta_0 S_v - n\nu(\theta_0)\right) e^{\theta_0 S_v}$ and $E \sim \text{Exponential}(1)$ random variable which is independent of D_∞. Further, $\sigma^2 := \mathbf{E}\left[\frac{1}{m(\theta_0)^n}\sum_{|v|=n}\left(\theta_0 S_v - n\nu(\theta_0)\right)^2 e^{\theta_0 S_v}\right]$.

Remarks:

- The coefficient for the linear term is exactly same as that of the centering of R_n as proved by Aïdékon [2013].

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Theorem (Aïdékon [2013])

There exists a random variable H^∞ such that

$$R_n - \frac{\nu(\theta_0)}{\theta_0} n + \frac{3}{2\theta_0} \log n \xrightarrow{d} H_\infty,$$

where $H_\infty = \frac{1}{\theta_0}[\log D_\infty - \log E + C]$ and D_∞ is a almost sure limit of the *derivative martingale*, namely, $D_n = \frac{1}{m(\theta_0)^n} \sum_{|v|=n} (\theta_0 S_v - n\nu(\theta_0)) e^{\theta_0 S_v}$ and $E \sim$ Exponential (1) random variable which is independent of D_∞. And C is a constant.

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- So we consider $\theta = \theta_0$ as the *boundary case* and do not further scale the process to make $\theta_0 = 1$.
- It is worth to note here that for the classical BRW the only non-trivial limit happens at $\theta = \theta_0$. But for us all possible parameters values are in principle acceptable.

Below the Boundary Case: $\theta < \theta_0 \leq \infty$

Theorem 3 [B. and Ghosh (2020)]

Assume that μ admits finite mean, then for $0 < \theta < \theta_0 \leq \infty$,

$$R_n^* - \frac{\nu(\theta)}{\theta} n \xrightarrow{d} H_\theta^\infty + \frac{1}{\theta} \log\langle\mu\rangle$$

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Remark: Notice the *Bramson correction* disappears in this case.

Above the Boundary Case: $\theta_0 < \theta < \infty$

Theorem 4 [B. and Ghosh (2020)]

Suppose $\mu = \delta_1$,

$$\frac{R_n^* - \frac{\nu(\theta_0)}{\theta_0} n}{\log n} \xrightarrow{P} -\frac{3}{2\theta_0}.$$

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Remarks:

- The result is imprecise!
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- In fact, we actually discovered that a very simple "coupling" holds between the BRW problem and a more well known/studied process obtained out of a (quite standard) *statistical* technique, know as, the *smoothing transform*.
- But, we are not there yet! What we have are results on this last progeny modified BRW.
- This *coupling* is a *conspiracy* of few *operators*!

Maximum Operator

- Let $\mathbf{Z}$ be the progeny point process with $N := \mathbf{Z}(\mathbb{R}) < \infty$.

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To be interpreted as, each individual produces a random but finitely many offspring (given by $N = \mathbf{Z}(\mathbb{R})$) and they are displaced from the position of the parent according to the points $\xi_1, \xi_2, \cdots, \xi_k, \cdots \xi_N$.

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- With it we can then associate and operator $M_{\mathbf{Z}} : \mathcal{P}(\bar{\mathbb{R}}) \to \mathcal{P}(\bar{\mathbb{R}})$, given by

$$M_{\mathbf{Z}}(\eta) = \text{dist}\left(\max_{1 \le j \le N} (\xi_j + X_j)\right),$$

  where $(X_i)_{i \ge 1}$ are i.i.d. with distribution $\eta \in \mathcal{P}(\bar{\mathbb{R}})$ and are also independent of the point process $\mathbf{Z}$.

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where $(X_i)_{i \ge 1}$ are i.i.d. with distribution $\eta \in \mathcal{P}\left(\bar{\mathbb{R}}\right)$ and are also independent of the point process $\mathbf{Z}$.

- It is then easy to see that $R_n \stackrel{d}{=} M_{\mathbf{Z}}^n(\delta_0)$.

Linear Operator

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- As before let $\mathbf{Z} = \sum_{j=1}^{N} \delta_{\xi_j}$ be the progeny point process.
- With it we can associate another and perhaps more well known operator $L_{\mathbf{Z}} : \mathcal{P}\left(\bar{\mathbb{R}}_+\right) \to \mathcal{P}\left(\bar{\mathbb{R}}_+\right)$, given by

$$L_{\mathbf{Z}}(\mu) = \text{dist}\left(\sum_{1 \le j \le N} e^{\xi_j} Y_j\right),$$

where $(Y_i)_{i \ge 1}$ are i.i.d. with distribution $\mu \in \mathcal{P}\left(\bar{\mathbb{R}}_+\right)$ and are also independent of the point process $\mathbf{Z}$.

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- In probability literature, this was first considered by Biggins [1976] and then by many others, such as, Durrett and Liggett [1983]; Lally and Sellkey [1987]; Liu [2000]; Biggins and Kyprianou [1997, 2004 & 2005]; Hu and Shi [2009]; and Maillard [2009].

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- In statistics, this is essentially *regression* and hence was known to statisticians for long time. It was used in the context of the so called *Non-Parametric Regression*, which was first introduced by Nadarya and Watson [1964].
- It has also appeared in the study of *random algorithms*, for example, in the classical analysis of *Quicksort Algorithm* (e.g. Rösler [1992]).

Link Operator

- Consider a new operator $\mathcal{E} : \mathcal{P}\left(\bar{\mathbb{R}}_+\right) \to \mathcal{P}\left(\bar{\mathbb{R}}\right)$, defined as

$$\mathcal{E}\left(\mu\right) = \text{dist}\left(\log \frac{Y}{E}\right),$$

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- Then ...

Transforming Relations

Theorem 0a (Basic Transforming Relation) [B. and Ghosh (2020)]

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- So using the link operator we can convert a problem related to the maximum operator to a problem on the linear/smoothing operator, which will perhaps be easier to solve.
- In particular, perhaps we can get an easier proof for asymptotic of R_n, the right-most position of a BRW.

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- So we can not immediately use the *General Transforming Relation*.

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- Further, let $\mathbf{Z} = \sum_{j=1}^{N} \delta_{\xi_j}$ be such that, it is independent of all other random variables.

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- $\mathcal{E}$ is in some sense defines a continuous (under weak convergence topology) branch of "logarithm" on the set of positively supported probability measures.
- So *scaling limit* of $L^n_{\mathbf{Z}}$ provides *centering limit* of $M^n_{\mathbf{Z}}$ with logarithmic centering.
- As smoothing transformation is fairly well studied so scaling limit of $L^n_{\mathbf{Z}}$ are known (Biggins and Kyprianou [1997, 2004 & 2005] and Hu and Shi [2009]).

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- So we are trying to do the next best thing, that is to approximate 0 by an "appropriate" sequence of probability measures which are of the form $\frac{1}{\theta}\mathcal{E}(\delta_1)$ and take limit as $\theta \longrightarrow \infty$.
- As discussed earlier, all we need to prove now is

$$R_n^* - \frac{\nu(\theta_0)}{\theta_0} n + \frac{3}{2\theta_0} \log n \xrightarrow{d} \odot$$

for any $\theta > \theta_0$ and then take limit as $\theta \longrightarrow \infty$ to claim the same holds for R_n.

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- To tackle this difficulty we are in the process of studying the *large deviation* properties of the sequence (R_n^*).
- Another approach is to use the *below the boundary* result and use truncation on the displacements. But I guess, this story should be told another day!

Thank you!