A panoramic view of modern ergodic theory

On the occasion of the Abel prize to H. Furstenberg and G. A. Margulis

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Hillel Furstenberg

From the Abel prize website

When Hillel Furstenberg published one of his early papers, a rumor circulated that he was not an individual but instead a pseudonym for a group of mathematicians. The paper contained ideas from so many different areas, surely it could not possibly be the work of one man?

Gregory Margulis

Jacques Tits on Margulis

It is not exaggerated to say that, on several occasions, he has bewildered the experts by solving questions which appeared to be completely out of reach at the time.

Armand Borel on Margulis

On more than one occasion, Borel mentioned that Margulis was the first person who caused confusion between two Borels in the Borel measure and the Borel subgroups, by using both Lie theory (or rather algebraic group theory) and ergodic theory simultaneously. He also declared that he was not related to the other Borel.

Ji Lizhen, A Summary of the Work of Gregory Margulis Pure and Applied Mathematics Quarterly Volume 4, Number 1 (Special Issue: In honor of Gregory Margulis, Part 2 of 2) 1–69, 2008

Nostalgia

Early beginnings

Furstenberg, Harry On the infinitude of primes. Amer. Math. Monthly 62 (1955), 353.

Kazhdan, D. A.; Margulis, G. A. A proof of Selberg's hypothesis. (Russian) Mat. Sb. (N.S.) 75 (117) 1968 163–168.

Diverse achievements, common threads

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- *Rigidity*
- *Connections to number theory*
- *Connections to probability*

Abel prize citation

"for pioneering the use of methods from probability and dynamics in group theory, number theory and combinatorics."

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- Fractal methods in ergodic theory,..

- Arithmeticity and Superrigidity theorems

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- Normal subgroup theorem, Margulis Lemma, Bowen-Margulis measure, Random walks on homogeneous spaces..

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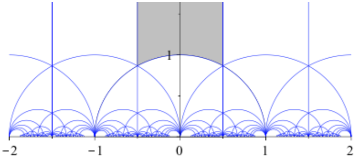
- Horocycle flow: $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$


1
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2

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- Example: the circle is invariant. So is the middle third Cantor set (for $\times_3$)

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- A number is rational if and only if its expansion is eventually periodic

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- $\times_p(0.x_1x_2\dots) = 0.x_2x_3\dots$

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- If A, B are closed and invariant under $\times_2, \times_3$ respectively, do A and B have common structure?

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- $S_{p,q}$ is a semigroup of the natural numbers.

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- Then either $X = \mathbb{T}$ or X is finite (and so $X \subset \mathbb{Q}$).

A Diophantine consequence

- Theorem (Furstenberg): If $S_{p,q}$ is as above and $\alpha \in \mathbb{R}$ is an irrational, then
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- This is a strengthening of (a consequence of) Weyl's equidistribution theorem.

Key idea: disjointness

- Suppose G acts continuously on two spaces X and Y. A joining of (X, G) and (Y, G) is a closed subset $Z \subset X \times Y$ invariant under the diagonal action of G on $X \times Y$ whose projection to the first coordinate is X and to the second coordinate is Y.

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- The two systems (X, G) and (Y, G) are said to be disjoint if the only joining between them is the trivial product joining $Z = X \times Y$.

- Invariant measures: a measure μ on a measurable space $(X, \mathcal{B})$ on which a semigroup G acts is said to be invariant if for every $g \in G$ and any $A \in \mathcal{B}$ we have that $\mu(g^{-1}A) = \mu(A)$

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- Ergodic measures: an invariant measure is said to be ergodic if every set $A \in \mathcal{B}$ which is invariant under every $g \in G$ satisfies that $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

Measure theoretic rigidity contd.

- Conjecture (Furstenberg 1967): Let p, q be two multiplicative independent positive integers. Any Borel measure μ on $\mathbb{T}$ ergodic under the action of $S_{p,q}$ is either Lebesgue measure or an atomic measure supported on finitely many rational points.

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- Furstenberg, Harry Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. Math. Systems Theory 1 (1967), 1–49.

Measure theoretic rigidity continued

- Theorem (Rudolph): Let p, q be relatively prime positive integers, and μ a $S_{p,q}$-ergodic measure on $\mathbb{T}$. Assume that the $\mathbb{Z}_+$-action generated by one of these maps, say $\times_p$, has positive entropy. Then μ is Lebesgue measure.

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- The entropy of Dirac measures supported on periodic orbits is zero. However non-atomic invariant measures can have zero entropy.

The dynamics of $\times_p$ and $\times_q$ do not share any common structure unless there is an obvious <u>algebraic</u> reason.

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- Closely connected to Littlewood's conjecture, arithmetic quantum unique ergodicity

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- Upper density: $\limsup_{n\to\infty} \frac{1}{2n+1} |A \cap \{-n, -n+1, \ldots, n-1, n\}|$

Finitary versions

- Theorem (van der Waerden 1927): Let k and r be positive integers. Then there exists a positive integer $M = M(k, r)$ such that, however the set $\{1, 2, \ldots, M\}$ is partitioned into r subsets, at least one of the subsets contains an arithmetic progression of length k.

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- Furstenberg in 1977 gave a new, completely different proof of Szemerédi's theorem.

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- The shift map sends the sequence $(x_n)_{n\in\mathbb{Z}}$ to $(x_{n-1})_{n\in\mathbb{Z}}$
- Equip $2^{\mathbb{Z}}$ with the product topology where each component $\{0,1\}$ has the discrete topology. By Tychonoff, this is compact.

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- $Q(x, y, z) = x^2 - \sqrt{2}xy + \sqrt{3}z^2$.

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- Raghunathan: Oppenheim's conjecture follows from the statement:
- Any orbit of $\mathrm{SO}(Q)$ on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ is either closed and carries an $\mathrm{SO}(Q)$-invariant probability measure, or is dense.

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- Thereby proving Oppenheim's conjecture.
- In full generality, the conjectures of Raghunathan and Dani were proved in a series of landmark papers by Marina Ratner.

- Let $Q_0(x, y, z) = x^2 + y^2 - z^2$

Dynamics $\longleftrightarrow$ Diophantine analysis

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- Which implies that Q is not irrational.

Quantitative versions of Oppenheim conjecture

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- Conjectured by Piatetski-Shapiro and used by Margulis in his proof of the arithmeticity theorem

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- Conjecture (Littlewood, circa 1930): $\liminf_{n\to\infty} \ n\,\|n\alpha\|\,\|n\beta\| = 0$, Thank You!