Disorder relevance and the random field Ising model

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Based on joint work with Rongfeng Sun

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It may reasonable to assume the irregularities are negligible.

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Examples include

1. Anderson localisation;
2. Directed polymers;
3. Sinai's random walk;
4. Ising model;
5. Pinning models.

Description

Fix a bounded domain: $\Omega \subset \mathbb{Z}^d$,

Inverse temperature: $\beta \geq 0$,

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$$P_\Omega(\sigma) = \frac{1}{Z_\Omega} \exp\left(-\mathcal{H}_\Omega(\sigma)\right)$$

where

$$\mathcal{H}_\Omega(\sigma) = -\beta \sum_{x \sim y} \sigma_x \sigma_y - h \sum_{x \in \Omega} \sigma_x, \quad \text{and} \quad Z_\Omega = \sum_\sigma \exp\left(-\beta \mathcal{H}_\Omega(\sigma)\right)$$

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are the *Hamiltonian* and the *partition function* respectively.

Write P_Ω^+ and P_Ω^- for the model with $+$ and $-$ boundary conditions respectively.

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The limiting measure depends on β, h and, in particular,

$$E_{\mathbb{Z}^d}^+[\sigma_0] \neq E_{\mathbb{Z}^d}^-[\sigma_0] \iff \beta > \beta_c,\ h = 0$$

where

1) if $d = 1$ then $\beta_c = \infty$;
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Is this picture changed by the addition of a small random external field?

Correlation length

A key quantity should be the *correlation length* $\xi(h)$ where

$$|E_{\mathbb{Z}^d}[\sigma_x\sigma_y] - E_{\mathbb{Z}^d}[\sigma_x]E_{\mathbb{Z}^d}[\sigma_y]| \approx \exp\left(-\frac{|x-y|}{\xi(h)}\right)$$

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Key idea: disorder relevance can be understood by observing the pure system.

$\xi(h)$ is a quantity of the pure system in terms of a fixed external field.

Harris criterion

Harris' idea was to use a perturbative argument using coarse graining.

1. Suppose a disordered system has critical inverse temperature $\beta_c(p)$ and consider inverse temperature β close to $\beta_c(p)$. Specifically, so that

$$\xi(p) \approx |\beta - \beta_c(p)|^{-\nu(p)}$$

where $\nu(p)$ is smooth around 0.

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   Divide the system into volumes $\Omega^{(d)}$ with ξ^d spins and correlation length ξ.

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5. Each box has its own critical inverse temperature which should differ from $\beta_c(p)$ proportionally to the standard deviation of the density of defects.

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10 Gives Harris criterion:

- i Disorder relevant: $\nu(0) < 2/d$.
- ii Disorder irrelevant: $\nu(0) > 2/d$.
- iii Disorder marginally relevant: $\nu(0) = 2/d$.

Random field Ising model

Fix a bounded, simply connected domain with piecewise smooth boundary $\Omega \subset \mathbb{R}^d$.
For $a > 0$, define $\Omega_a := \Omega \cap a\mathbb{Z}^d$ and $P_\Omega^a := P_{\Omega_a}$.

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Let $\omega = (\omega_x)_{x \in \Omega_a}$ be i.i.d. with,

- $\mathbb{E}[\omega_x] = 0$;
- $\mathrm{Var}_{\mathbb{P}}(\omega_x) = 1$;
- $\mathbb{E}[e^{u\omega_x}] < \infty$ for all $u \in \mathbb{R}$.

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For $\varepsilon > 0$ define the disordered Hamiltonian by

$$\mathcal{H}_\Omega^\omega(\sigma) = -\beta \sum_{x \sim y} \sigma_x \sigma_y - \sum_{x \in \Omega_a} \varepsilon \omega_x \sigma_x.$$

Recall, previously

$$\mathcal{H}_\Omega(\sigma) = -\beta \sum_{x \sim y} \sigma_x \sigma_y - h \sum_{x \in \Omega_a} \sigma_x.$$

Random field Ising model

For ω fixed we define the *random field Ising model* as

$$P_{\Omega;\varepsilon}^{\omega,a}(\sigma) = \frac{\exp\left(\sum_{x\in\Omega_a} \varepsilon\omega_x\sigma_x\right)}{Z_{\Omega;\varepsilon}^{\omega,a}} P_\Omega^a(\sigma)$$

where

$$Z_{\Omega;\varepsilon}^{\omega,a} = E_\Omega^a\left[\exp\left(\sum_{x\in\Omega_a} \varepsilon\omega_x\sigma_x\right)\right]$$

is the *random partition function*.

These are both random with respect to the disorder.

Disorder

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For $d \geq 3$, β sufficiently large and ε sufficiently small, there is a first order phase transition.

Aizenman, Wehr (1990)

For $d \leq 2$, any $\varepsilon > 0$ and almost every ω, there is a unique infinite volume limit.

Partition functions and free energy

The *quenched free energy* is defined as the rate of exponential growth of the partition function:

$$F(\varepsilon, h) \ := \ \limsup_{a\to 0} \frac{1}{|\Omega_a|} \, \mathbb{E}\left[\log\left(Z^{\omega,a}_{\Omega;\varepsilon}\right)\right].$$

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Discontinuity of the first derivative corresponds to spontaneous magnetisation.

Perhaps, $F(\varepsilon, h)$ or $Z^{\omega,a}_{\Omega;\varepsilon}$ is the right thing to look at.

This is central to the idea of chaos expansions.

Chaos expansion

Supposing $\sigma_x \in \{0,1\}$, using a high temperature expansion

$$
\begin{aligned}
Z^{\omega,a}_{\Omega;\varepsilon} &= E^a_\Omega\left[\exp\left(\sum_{x\in\Omega_a}\varepsilon\omega_x\sigma_x\right)\right] \\
&= E^a_\Omega\left[\prod_{x\in\Omega_a}(1+(e^{\varepsilon\omega_x}-1)\sigma_x)\right]
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&\approx \sum_{k\geq 0}\frac{1}{k!}\sum_{(x_1,\ldots,x_k)\in\Omega_a^d}\varepsilon^k E^a_\Omega\left[\sigma_{x_1}\ldots\sigma_{x_k}\right]\omega_{x_1}\ldots\omega_{x_k}
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&\approx \sum_{k\geq 0}\frac{1}{k!}\varepsilon^k a^{-dk/2}\int\cdots\int_{\Omega^k} E^a_\Omega\left[\sigma_{x_1}\ldots\sigma_{x_k}\right]W(\mathrm{d}\mathbf{x})
\end{aligned}
$$

replacing ω_x with a white noise approximation $a^{-d/2}W(x+\Delta)$ for $\Delta = (-a/2, a/2)^d$.

Spin correlations

This suggests that the model should be disorder relevant if

$$a^{-k\gamma} E^a_\Omega [\sigma_{x_1} \ldots \sigma_{x_k}]$$

converges in L^2 to a non-trivial limit as $a \to 0$ for some $\gamma > 0$.

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Chelkak, Hongler, Izyurov (2015)

Let $d = 2$. For any $k \geq 1$ and $x_1, \dots, x_k \in \Omega$ distinct,

$$\lim_{a \to 0} a^{-\frac{k}{8}} E_{\Omega}^{a,+} \left[\prod_{i=1}^{k} \sigma_{x_i} \right] = \mathcal{C}^k \phi_{\Omega}^{+}(x_1, \dots, x_k).$$

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In dimension 2, we have $\gamma = 1/8$.

Convergence of partition functions

Choose

$$\varepsilon_x^a := \lambda a^{7/8} \qquad \text{for } \lambda > 0,$$

$$\tilde{Z}_{\Omega;\varepsilon}^{\omega,a,+} := \theta_a Z_{\Omega;\varepsilon}^{\omega,a,+} \qquad \text{where } \theta_a := \exp\left(-\frac{1}{2} a^{-1/4} \lambda^2\right).$$

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Caravenna, Sun, Zygouras (2017)

The rescaled partition function $\tilde{Z}^{\omega,a,+}_{\Omega;\varepsilon}$ converges in $\mathbb{P}$-distribution to the *Wiener chaos expansion*

$$\mathcal{Z}^{W,+}_{\Omega;\lambda} = 1 + \sum_{n=1}^{\infty} \frac{\mathcal{C}^n \lambda^n}{n!} \int \cdots \int_{\Omega^n} \phi^+_\Omega(x_1, ..., x_n) \prod_{i=1}^{n} W(\mathrm{d}x_i)$$

where W is *white noise* and ϕ^+_Ω is the *spin correlation function*.

Pure Ising magnetisation field

We define the magnetisation field as

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Camia, Garban, Newman (2015)

Consider the critical Ising model with $+$ boundary. The magnetisation field Φ_Ω^a converges in law to a limiting random distribution Φ_Ω.

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Let $\varphi \in C_c^\infty(\Omega)$ and write $\varphi_x^a := a^{15/8}\varphi(x)$ then

$$E_{\Omega;\varepsilon}^{\omega,a,+}\left[\exp\left(i\left\langle \varphi, \Phi_\Omega^a\right\rangle\right)\right] = \frac{E_\Omega^{a,+}\left[\exp\left(\sum_{x\in\Omega_a}\left(\varepsilon_x^a\omega_x + i\varphi_x^a\right)\sigma_x\right)\right]}{Z_{\Omega;\varepsilon}^{\omega,a,+}} = \frac{Z_{\Omega;\varepsilon,\varphi}^{\omega,a,+}}{Z_{\Omega;\varepsilon}^{\omega,a,+}}.$$

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We want to consider joint convergence of $\tilde{Z}_{\Omega;\varepsilon,\varphi}^{\omega,a,+}$ for $\varphi \in C_c^\infty$.

We should have marginal limits

$$\mathcal{Z}_{\Omega;\lambda,\varphi}^{W,+} = 1 + \sum_{n=1}^{\infty}\frac{\mathcal{C}^n}{n!}\int\cdots\int_{\Omega^n}\phi_\Omega^+(x_1,\dots,x_n)\prod_{j=1}^{n}(\lambda W(\mathrm{d}x_j) + i\varphi(x_j)\mathrm{d}x_j).$$

Convergence of the magnetisation field

Write

$$W^{\omega,a} = a \sum_{x \in \Omega_a} \omega_x \delta_x \quad \text{and} \quad W_\psi^{\omega,a} = \langle W^{\omega,a}, \psi \rangle .$$

Then, $W^{\omega,a}$ converges in $\mathbb{P}$-distribution to white noise W.

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B, Sun (2019+)

Let $A, B \subset C_c^\infty(\Omega)$ be finite. Then, as $a \to 0$,

$$\Big((\tilde{Z}^{\omega,a,+}_{\Omega;\varepsilon,\varphi})_{\varphi \in A}, (W_\psi^{\omega,a})_{\psi \in B} \Big) \to \Big((\mathcal{Z}^{W,+}_{\Omega;\lambda,\varphi})_{\varphi \in A}, (W_\psi)_{\psi \in B} \Big)$$

in $\mathbb{P}$-distribution.

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Let $A, B \subset C_c^\infty(\Omega)$ be finite. Then, as $a \to 0$,

$$\Big((\tilde{Z}^{\omega,a,+}_{\Omega;\varepsilon,\varphi})_{\varphi \in A}, (W_\psi^{\omega,a})_{\psi \in B} \Big) \to \Big((\mathcal{Z}^{W,+}_{\Omega;\lambda,\varphi})_{\varphi \in A}, (W_\psi)_{\psi \in B} \Big)$$

in $\mathbb{P}$-distribution.

In particular, the random distribution $\mu^{\omega,a}_{\Omega,\varepsilon}$ over the magnetisation field Φ^a_Ω converges in $\mathbb{P}$-distribution to a random probability measure $\mu^W_{\Omega,\lambda}$.

Relation to the case without disorder

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Let $\mathcal{F}_N := \sigma\left(\left\{\left\langle \Phi, \mathbf{1}_{B^N_{i,j}} \right\rangle\right\}_{i,j=1}^N\right)$ where $\{B^N_{i,j}\}_{i,j=1}^N$ is a partition of Ω.

It suffices to show that for $\mathbb{P}$-a.e. W,

$$\mathcal{R}_N := \frac{\mathrm{d}\mu^W_{\Omega;\lambda}}{\mathrm{d}\mu^W_{\Omega;0}}\bigg|_{\mathcal{F}_N}$$

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In particular, it suffices to show that

$$\lim_{N\to\infty} \mathbb{E}\left[E_\Omega\left[(\mathcal{R}_N)^{1/2}\right]\right] = 0.$$

Fractional moment method

Let $g_N = g_N(W, \Phi) > 0$ such that, for W fixed, g_N is $\mathcal{F}_N$ measurable.

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\mathbb{E}\left[E_\Omega\left[\mathcal{R}_N^{1/2}\right]\right] &= \mathbb{E}\left[E_\Omega\left[g_N^{1/2}\mathcal{R}_N^{1/2}g_N^{-1/2}\right]\right] \\
&\leq \mathbb{E}\left[E_\Omega\left[g_N\mathcal{R}_N\right]\right]^{1/2}\mathbb{E}\left[E_\Omega\left[g_N^{-1}\right]\right]^{1/2} \\
&\leq \mathbb{E}\left[E_\Omega^W\left[g_N\right]\right]^{1/2}\mathbb{E}\left[E_\Omega\left[g_N^{-1}\right]\right]^{1/2}.
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\end{aligned}$$

Aim to choose g_N such that the first term converges to 0 and the second term is bounded.

For $M_N, K_N \nearrow \infty$, we choose

$$g_N(W, \Phi) := \exp\left(-K_N \mathbf{1}_{\{X_N \geq M_N\}}\right)$$

where

$$X_N = X_N(W, \Phi) \approx \frac{\mathrm{d}\mu_{\Omega;\lambda}^W}{\mathrm{d}\mu_{\Omega;0}^W}$$

is measurable with respect to $\mathcal{F}_N$ for each W fixed.

Ising loops and interfaces

An Ising configuration corresponds uniquely to a loop configuration in the dual graph.

Dobrushin boundary: fix two points $u, v \in \partial\Omega$ and set $\sigma_x = -1$ for x in the boundary arc (u, v) and 1 in the boundary arc (v, u).

There is a unique Ising interface from u to v - a simple curve from u to v with $+1$ on the left and -1 on the right.

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Benoist and Hongler (2016+)

Consider the critical Ising model with $+$ boundary. The set of all Ising loops converges to CLE(3).

FK-Ising clusters

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The limiting magnetisation field Φ_Ω for the pure Ising model is measurable with respect to the macroscopic FK-Ising clusters.

The field can be represented as

$$\Phi_\Omega = \sum_j \eta_j \mu_j^{FK}$$

where η_j are i.i.d. signs and μ_j^{FK} are rescaled area measures associated to FK-Ising clusters.

Thank you for listening

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