# Point Set Pattern Matching under Rigid Motion

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# Outline

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# Introduction

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- A problem in pattern matching is to design efficient algorithms for testing how closely a query set $Q$ of $k$ points resembles a sample set $P$ of $n$ points, where $k \leq n$.
- In image processing, computer vision and related applications like finger print matching, point sets represent some spatial features.
- The problem has several variants [Alt and Guibas, 1999] based on:
  - class of allowable transformations
  - exact or approximate matching
  - equal cardinality, subset, or largest common point set matching

# Problem Variants

## Allowable Transformations

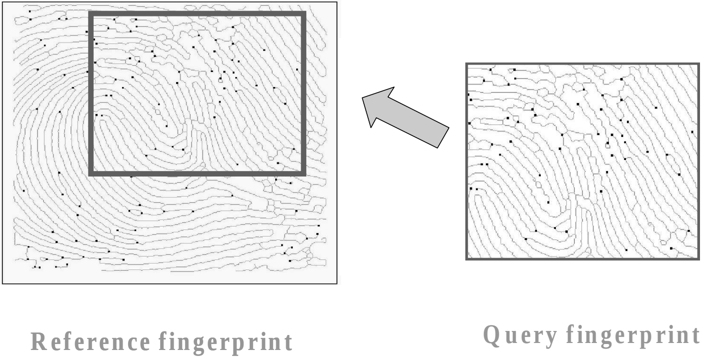
- Translation
- Translation and Rotation - Rigid Motion Transform
- Translation, Rotation and Scaling - Similarity Transform

Distances are preserved in Translation and rigid motion.

## Types of Matching

- Exact matching: points in query set match exactly with points in sample set after the requisite transform.
- Approximate matching: points in query set after the requisite transform go to a neighborhood of points in the sample set.

# Motivation



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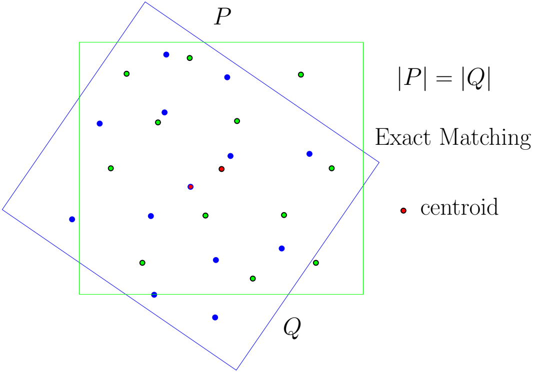


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- Anchor centroids and match distances.
- But anchoring centroids do not help when we match $Q$ with a set of points $P' \subseteq P$.

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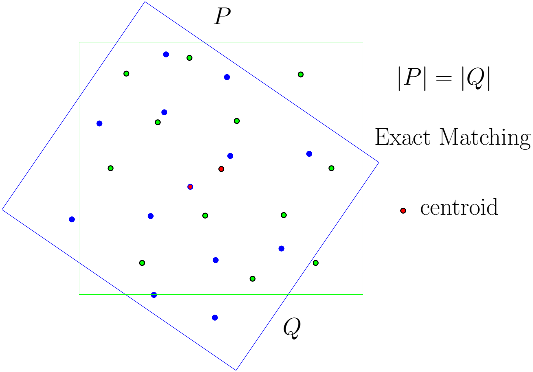


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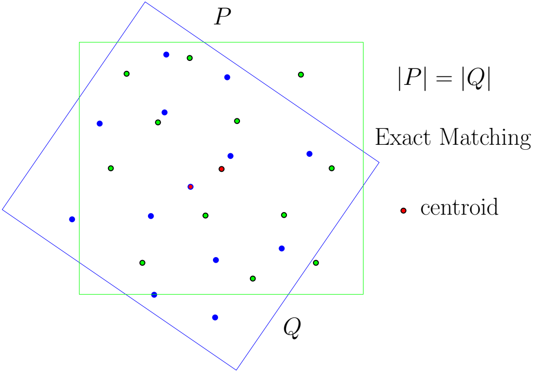


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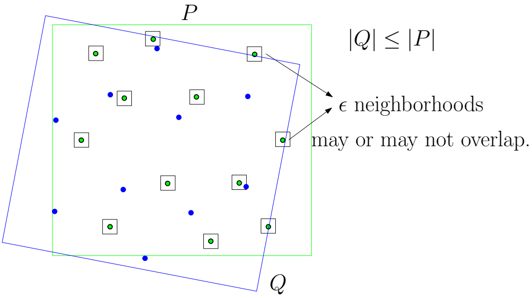
# Approximate Partial Matching under Rigid Motion




Given two point sets $P$ and $Q$ ($|P| = |Q|$), check if there is a bijection $\ell : Q \rightarrow P$ and a congruence $T$, such that $T(q) \in U_\epsilon(\ell(q)), \forall\, q \in Q'$, where $Q' \subseteq Q$, and $U_\epsilon(p)$ denotes the closed $\epsilon$-neighborhood of a point $p \in P$.

# Outline

# Previous Work

## Exact Matching

- Rezende and Lee (1995) proposed an $O(kn^d)$ time algorithm for partial point set pattern matching, where
  $d$ (* dimension of the plane *)
  $n = |P|$ (* size of sample set *)
  $k = |Q|$ (* size of pattern set *).
  It allows translation, rotation and scaling.
- Akutsu, Tamaki and Tokuyama (1998) proposed an $O(kn^{\frac{4}{3}} + \mathcal{A})$ time algorithm for testing the congruence in 2D,
  where $\mathcal{A}$ = time complexity for locating $r$-th smallest distance among a set of $n$ points in 2D.
  = $O(n^{\frac{4}{3}} \log^{\frac{8}{3}} n)$ (on an average).
- Akutsu, Tamaki and Tokuyama (1998) proposed a Las Vegas expected time algorithm of time complexity $O(n^{\frac{4}{3}} \log^{\frac{8}{3}} n + \min(k^{0.77} n^{1.43} \log n, n^{\frac{4}{3}} k \log n))$ using parametric search.

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## Approximate Matching (for $k = n$ case)

- Alt et al. (1988) designed a general algorithm of $O(n^8)$ time that works for overlapping and non-overlapping $\epsilon$-circles and $\epsilon$-boxes.
- They use a geometric fact and bipartite graph matching.
- A valid matching of $Q$ with $P$ exists iff there is a matching where $q_i, q_j \in Q$ are matched exactly to the boundaries of $U_\epsilon(p_\alpha)$, $U_\epsilon(p_\beta)$ of two points $p_\alpha, p_\beta \in P$.
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# Outline

# Exact Point Set Pattern Matching

## The Problem

**Input:** $P = \{p_1, p_2, \ldots, p_n\}$ (* Sample Set *)
$Q = \{q_1, q_2, \ldots, q_k\}$ (* Query Set *),
$k \leq n$.

**Output:** A subset of points in $P$ that are matched with the points in $Q$ if match exists under rigid motion.

## Trivial Algorithm

Choose all possible $\binom{n}{k}$ subset of $P$ and test for a match under rigid motion.

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## An obvious improvement

Preprocessing: Use circular sorting to create a data structure attached with each point in $P$.

Space: $O(n^2)$

Time: $O(n^2)$.

Query:
- Sort the query point set angularly.
- Anchor a pair of query point with each pair of sample point.
- It determines the rotation angle and scaling.
- Match the other query points with a subset of sample points.

Time: $O(k \log k + kn(n-1))$

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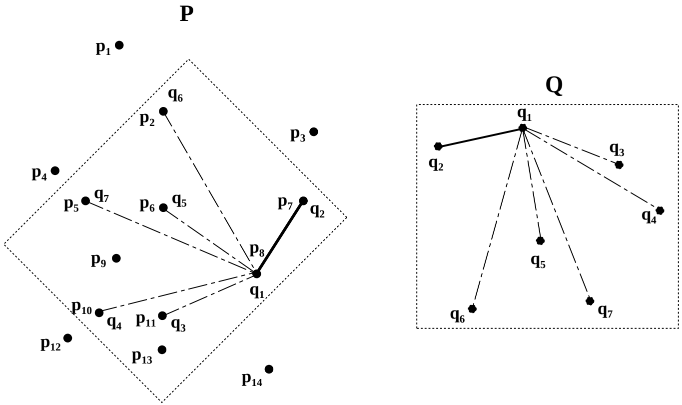
# Demonstration



# An Efficient Algorithm for Rigid Motion

**Fact 1**

All the $\binom{k}{2}$ distances must occur in the $\binom{n}{2}$ distances in $P$

**Fact 2**

[Szekely 1997] In a sample set $P$ of size $n$, the maximum number of equidistant pairs of points is $O(n^{\frac{4}{3}})$ in the worst case.

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## Preprocessing

- Sort the $\binom{n}{2}$ distances of $P$ distances in $P$.
- Create a height balanced binary tree $\mathcal{T}$ with distinct distances.
- Attach an array $\chi_\delta$ with each element $\delta$ of the tree. Its each element is a triple $(p_i, p_j, \psi_{ij})$, where $\psi_{ij}$ is the angle of the line $(p_i, p_j)$ with $x$-axis.
- Each point $p_i \in P$ is attached with a height-balanced binary tree $\mathcal{S}_i$. Its elements are tuples $(r_{ij}, \theta_{ij})$.

Time Complexity: $O(n^2 \log n)$
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## Query

- Take two points $(q_1, q_2)$, and check whether $\lambda(q_1, q_2) \in \mathcal{T}$. If not, report no match found.
- Let $\lambda(q_1, q_2) = \delta$.
- We consider each member $\lambda(p_i, p_j) \in \chi_\delta$.
- Anchor $(q_1, q_2)$ with $(p_i, p_j)$, and search in $\mathcal{S}_i$ for the presence of a match.

## Time Complexity

For each $\lambda(p_i p_j) = \delta$, searching for a match needs $O(k \log n)$ time. So,
Time Complexity: $O(n^{\frac{4}{3}} k \log n)$

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# An Important Note

Though the worst case number of equidistant pairs in a point set of size $n$ is $O(n^{\frac{4}{3}})$, in a random instance actually the number is very less.

| | | | | |
|---|---|---|---|---|
| Number of points (n) | 100 | 200 | 500 | 1000 |
| $\lceil n^{\frac{4}{3}} \rceil$ | 465 | 1170 | 3969 | 10001 |
| Maximum number of Equidistant pairs observed | 4 | 6 | 20 | 53 |

# Experimental Results

| No. of points in sample | No. of points in query | No. of anchoring | CPU time for Rezende and Lee | CPU time for our Algorithm | % savings |
|---|---|---|---|---|---|
| 200 | 50 | 1 | 730.0 | 22.0 | 97.0 |
| | 100 | 1 | 807.4 | 23.4 | 97.0 |
| | 150 | 2 | 867.6 | 31.2 | 96.0 |
| 500 | 100 | 2 | 1905.0 | 32.2 | 98.0 |
| | 200 | 1 | 1937.2 | 23.0 | 99.0 |
| | 400 | 2 | 1952.8 | 32.0 | 98.0 |
| 1000 | 300 | 12 | 4037.6 | 129.0 | 97.0 |
| | 500 | 7 | 4054.0 | 76.0 | 98.0 |
| | 800 | 11 | 4098.0 | 11.0 | 97.0 |

# Outline

# Approximate Point Set Pattern Matching

## Our Effort

We assume that

- the $\epsilon$-neighborhoods are axis-parallel squares of side length $\epsilon$.
- $P$ is well separated, i.e. each pair of points $p, p' \in P$ satisfy either $|x(p) - x(p')| \geq \epsilon$ or $|y(p) - y(p')| \geq 3\epsilon$ or both.

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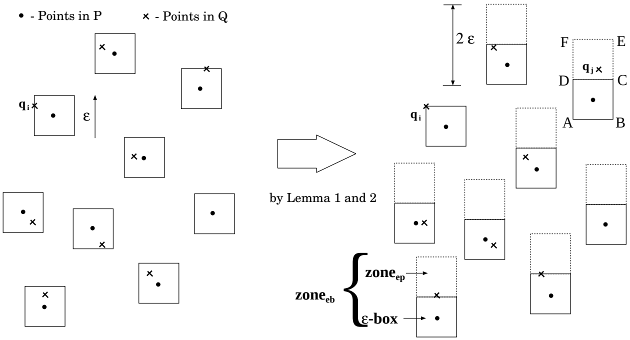
# A Necessary Characterization for a Matching



**Lemma (an iff condition)**

If $\exists$ a transformation $T(Q)$ for the said match, then $\exists$ another transformation $T'(Q)$, such that one point of $Q$ lies on the left boundary of the $\epsilon$-box of a point in $P$, and one point of $Q$ lies on the top boundary of the $\epsilon$-box of a point in $P$.

**Definition**

Consider an $\epsilon$-box $ABCD$ around $p \in P$. The *extended $\epsilon$-box* of $p$ is a $\epsilon \times 2\epsilon$ box formed by attaching another $\epsilon \times \epsilon$ square $CDFE$ above the $\epsilon$-box $ABCD$. $CDFE$ is called the *extended portion* of the $\epsilon$-box.

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## Lemma (an if condition)

If $\exists$ a transformation $T(Q)$ for the said match, $\exists$ another transformation $T'(Q)$, such that

- a point $q \in Q$ lies at the top-left corner of the $\epsilon$-box of a point in $P$.
- at least one point lies in the extended portion of the $\epsilon$-box
- each of the remaining members in $Q$ lie in the extended $\epsilon$-box

# Anchorings for Finding the Transformation

## Anchoring for Translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

- Two unknown parameters, $t_x$ and $t_y$. So, one anchoring is needed.

## Anchoring for Rigid Motion

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

- Three unknown parameters, $t_x$, $t_y$ and $\theta$. So, two anchorings are needed.

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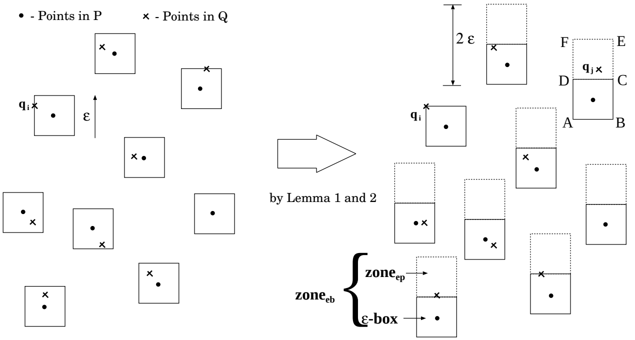
# Matching under Translation



# Translation

## The Algorithm

- Position $q$ at the top-left corner of the $\epsilon$-box of $p$, and check whether each point in $Q \setminus \{q\}$ lies inside the extended box of some point of $P$.
- If the above checking returns false, then no match exists with $q$ at the top-left corner of the $\epsilon$-box of $p$.
- If it returns true, then the points in $Q \setminus \{q\}$ can be partitioned into $Q_1$ and $Q_2$. Each $q_1 \in Q_1$ lies in the $\epsilon$-box of some point in $P$. Each $q_2 \in Q_2$ lies in the extended portion of $\epsilon$-box of some member in $P$.
- Push points in $Q_2$ down such that points in $Q_1$ do not go out.

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# The Analysis

## The Algorithm

- Maintain a planar straight line graph (PSLG) data structure with the $\epsilon$-boxes around each point in $P$.
- Anchoring a point $q \in Q$ with the top-left corner of an $\epsilon$-box, we perform $k$ point location queries in the PSLG. This needs $O(k \log n)$ time.
- Pushing points down take another $O(k)$ time.
- There can be $O(nk)$ anchorings in the worst case.

## Theorem

The worst case time complexity of the approximate matching of $Q$ with a $k$-subset of $P$ in 2D when only translation is considered is $O(nk^2 \log n)$.

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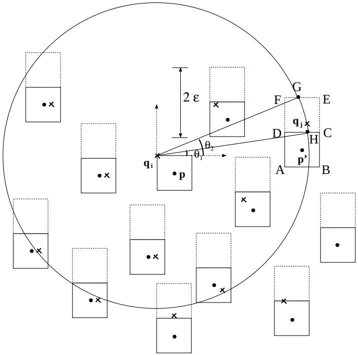
# Outline

# Matching under Rigid Motion



## The Algorithm

- Consider the $k - 1$ concentric circles $\mathcal{C}_{ij}\ \forall\ q_j \in Q \setminus \{q_i\}$. Each circle $\mathcal{C}_{ij}$ intersects some extended $\epsilon$-boxes.
- As $P$ is well-separated, these intersections contribute a set of non-overlapping arcs that define a circular arc graph $G$.
- Each circle $\mathcal{C}_{ij}$ may intersect $O(n)$ boxes making $O(nk)$ nodes in $G$. So, there can be $O(nk)$ cliques of size $k - 1$.
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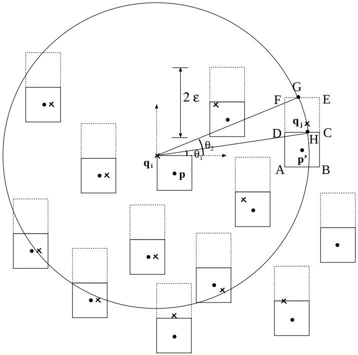
# One of the $k - 1$ concentric circles shown.



# Processing Each Clique

## Processing Cliques

- Each clique $\chi$ represents an angular interval (an arc) $\mathcal{I}^* = [\theta_1^*, \theta_2^*]$. For any angle of rotation $\theta \in \mathcal{I}^*$, each of the $k-1$ point of $Q \setminus q_i$ lies inside $k-1$ disjoint extended $\epsilon$-boxes.
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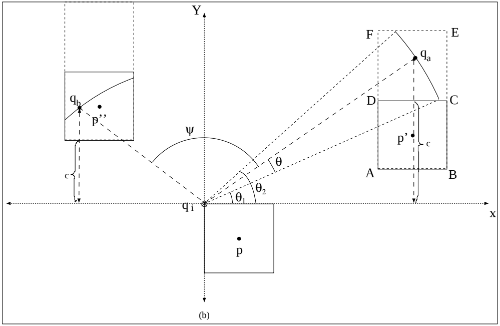
# The Pushing Down



# Homogeneous Splitting

## Homogeneous splitting of $\mathcal{I}^*$

- Consider a $\theta \in \mathcal{I}^*$.
- For each $q \in Q_1$, let $f_q^1(\theta)$ denotes the distance of $q$ from the bottom of the corresponding $\epsilon$-box.
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- The functions $f_q^i(\theta)$ $i = 1, 2$ are like $\overline{q_a q} \sin(\alpha + \theta) - c$.

## Observation

For a given $q \in Q_i$, the above functions are univariate, continuous, and unimodular.

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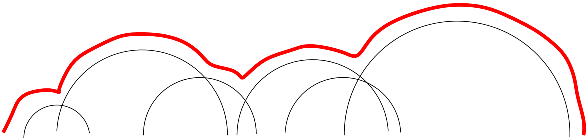
# Envelope of Functions

**Definition($\mathcal{L}(\theta)$ for $\theta \in \mathcal{I}^*$)**

$\mathcal{L}(\theta)$ denotes the lower envelope of $|Q_1|$ functions, namely $f_q^1(\theta)$, $q \in Q_1$.

**Definition($\mathcal{U}(\theta)$ for $\theta \in \mathcal{I}^*$)**

$\mathcal{U}(\theta)$ denotes the upper envelope of $|Q_2|$ functions, namely $f_q^2(\theta)$, $q \in Q_2$.



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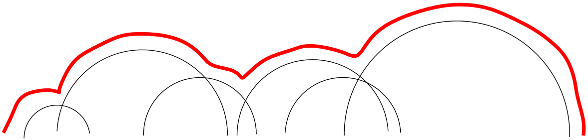

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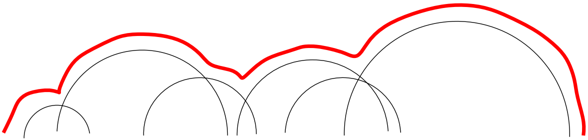

## Minimum Amount of Downward Translation for $Q_2$

At a rotation angle $\theta \in \mathcal{I}^*$, the minimum amount of downward translation required to place the points in $Q_2$ in the corresponding $\epsilon$-box is $max_{q \in Q_2} f_q^2(\theta) = \mathcal{U}(\theta)$.

## Maximum Amount of Downward Translation for $Q_1$

Similarly, the maximum amount of downward translation that may retain all the points in $Q_1$ in its corresponding $\epsilon$-box is $min_{q \in Q_1} f_q^1(\theta) = \mathcal{L}(\theta)$.

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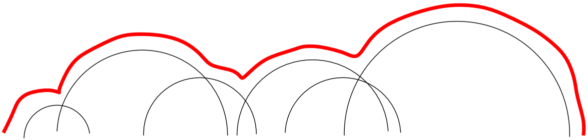

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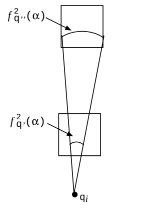
## Definition

A rotation angle $\theta$ is said to be a break-point in $\mathcal{L}$ if $\mathcal{L}(\theta) = f^1_{q_a}(\theta) = f^1_{q_b}(\theta)$ for $q_a, q_b \in Q_1$, $q_a \neq q_b$. Similarly, the break-points of the $\mathcal{U}$ function is defined.

## Lemma

A pair of functions $f^1_{q'}(\theta)$ and $f^1_{q''}(\theta)$ (corresponding to $q', q'' \in Q_1$, $q' \neq q''$) w.r.t. $\theta \in \mathcal{I}^*$ may intersect in at most two points. The same is true for a pair of functions $f^2_{q'}(\theta)$ and $f^2_{q''}(\theta)$ for a pair of points $q', q'' \in Q_2$.



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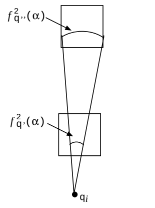


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# Envelope of Functions

## Observation

The collection of functions $\{f_q^1(\theta), q \in Q_1\}$ follows a $(k, 2)$-*Davenport-Schinzel sequence*. The same result is true for the collection of functions $\{f_q^2(\theta), q \in Q_2\}$.

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The maximum number of break-points in the function $\mathcal{L}(\theta)$ is $\lambda_2(|Q_1|) = 2|Q_1| - 1$, and it can be computed in $O(|Q_1| \log |Q_1|)$ time. Similarly, for $\mathcal{U}(\theta)$.

## Moral of Homogeneous Splitting of $\mathcal{I}^*$

$\mathcal{I}^* = [\theta_1^*, \theta_2^*]$ is split into $O(k)$ sub-intervals defined by break-points of $\mathcal{L}(\theta)$ and $\mathcal{U}(\theta)$.

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# Critical Angle

**Definition (Critical Angle)**

If a match is found by a rotation of $Q$ by the angle $\theta^* \in \mathcal{I}^*$ and a vertical downward shift, then $\theta^*$ is said to be a critical angle.

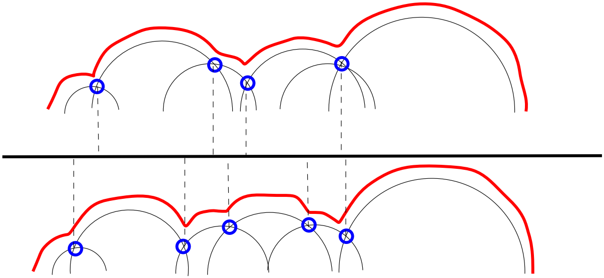
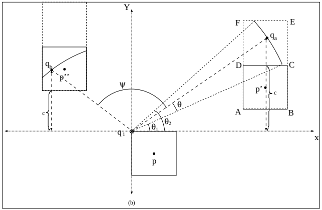


Figure: Envelopes and break points.

# Computation of Critical Angle



(b)

## Critical Angle Computation

For $\theta \in [\theta_1, \theta_2]$, the vertical downward shift will be determined

# Computation of Critical Angle

## Critical Angle Computation

$\Delta_1$ = Minimum amount of downward shift required to bring $q_a$ inside the $\epsilon$-box of $p'$. Thus, $\Delta_1 = \delta(q_i, q_\alpha) sin(\theta + \theta_1) - (y(p') + \frac{\epsilon}{2})$.

$\Delta_2$ = Maximum amount of permissible downward shift keeping $q_b$ inside the $\epsilon$-box of $p''$. Thus, $\Delta_2 = \delta(q_i, q_\beta) sin(\theta + \theta_1 + \psi) - (y(p'') - \frac{\epsilon}{2})$.

- A feasible solution $\theta$ (if it exists) must satisfy $\Delta_1 \leq \Delta_2$.

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# Complexity Analysis

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- Each point of $Q$ needs to be anchored at the top-left corner of the $\epsilon$-box of each point in $P$.
- The nodes of the circular arc graph $G$ are obtained in $O(nk)$ time and the cliques of $G$ can be obtained in $O(nk \log n)$ time.
- While processing a clique, computation of functions $\mathcal{U}$ and $\mathcal{L}$ needs $O(k \log k)$ time.
- Number of elements in $\mathcal{U} \cup \mathcal{L}$ is $O(k)$ in the worst case.
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# The Final Result

**Theorem**

The time complexity of the proposed algorithm for $\epsilon$-approximate matching of $Q$ with a subset of $P$ where the neighbourhood around a point (in $P$) is defined as an $\epsilon$-box, is $O(n^2k^2(\log n + k \log k))$.

# Further reading I


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Thank You!