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Part III :

Connective segmentation

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Plan

Segmentation: an intuitive approach ;

Segmentation: counter examples ;

Connective criteria and Segmentation theorem;

Class permanency and local knowledge;

Flat, smooth, and jump connections (segmentations).

Connected operators

Segmentation

- *For human perception,*

  as well as for a data processing,

  an image is basically a mosaic of coloured points,

- *They can be:*

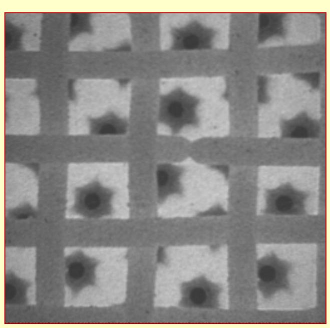
  - the pixels of the digital camera, or of the computer memory,

  - or, for the eye., the response of the retina cones and rod,

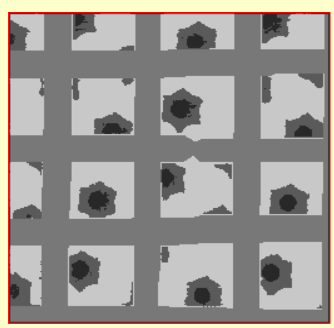
But we rarely see them as such !

Human perception partitioning

For example, when we observe this burner,



Do we see individual pixels ?



Or rather these regions ?

Segmentation in Image Processing

Image segmentation methods, as well as the mathematics they bring with them, appeared during the 60's, in parallel with digital image processing, and have constantly evolved since this time.

When, for the first time, somebody made a threshold, the first image segmentation was born ... and also the most frequently used, even today.

But clearly, it is not the only one and the number of algorithms proposed in the literature about segmentation exceeds several thousands.

Segmentation in Image Processing

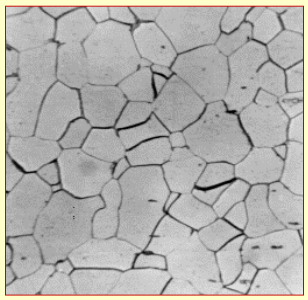
However, these segmentations are usually structured around four main steps:

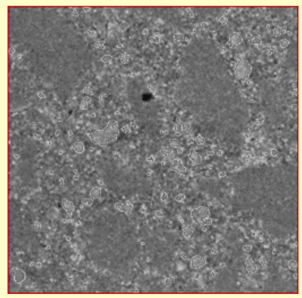
- The datum of a ***perception space***, it may embed the usual physical space and time, and possibly other feature spaces (histograms, textures, directions);
- The choice of some ***criterion*** that translates what we mean by "homogeneous region" ;
- ***partitioning*** of the perception space into zones that are homogeneous according the said criterion ;
- Some ***maximization*** of all possible partitions.

Image Spaces

Also, in the human perception, the meaning helps us in segmenting a house, a face, etc..

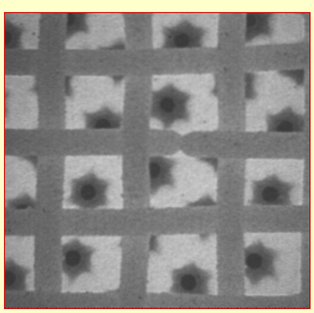
But in image processing, e.g. microscopy, the scenes may be totally meaningless for us, although their quantitative analysis demands we segment them:



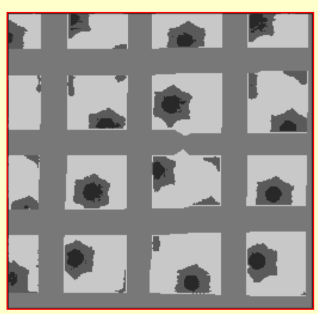


The three steps

We pass from the initial burner to its segmented version in ***three steps***



→



1/ we give ourselves a ***criterion,***

2/ We associate ***partitions*** of the space with this criterion

3/ and we ***take the largest partition***.

1rst step: a Criterion

1st step : we classified the pixels according to the **criterion of flat zones** that we had already in mind
(for other cases, colour, shape, or other criteria could be more convenient).

Criterion : Given a class of $\mathcal{F}$ functions from set E into set T, a criterion σ is a ***binary function*** $\mathcal{F} \otimes \mathcal{P}(E) \rightarrow [0,1]$

$\sigma[f,A] = 1$ when the criterion is satisfied over A
$\sigma[f,A] = 0$ when not

Examples:

- **The flat zones of function f** (i.e. *the zones where the function is constant)*
- **A threshold** (i.e. *the zones where the function is above a given value*

2nd step : a Partition

2nd step : We then replace the pixels by a **partition** of the space into regions, or classes;

Partition: A partition of space E is a mapping D: $E \rightarrow \mathcal{P}(E)$ that associates with each pixel **x** the class **D(x)** to whom it belongs, such that

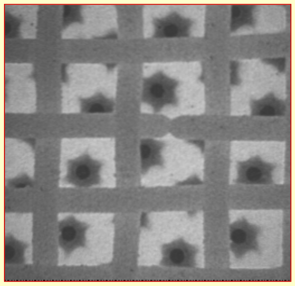
(i) the space is covered : $x \in E \Rightarrow x \in D(x)$

(ii) there is no overlapping: *for all* $(x, y) \in E$,

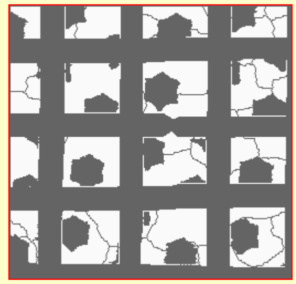
either $D(x) = D(y)$

or $D(x) \cap D(y) = \varnothing$

A flat zones example



a)



b)

a) Gaz burner => extract the whites

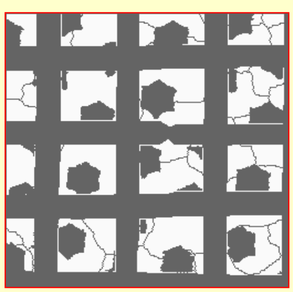
b) One possible partition of the white zones

Is there a larger partition ?

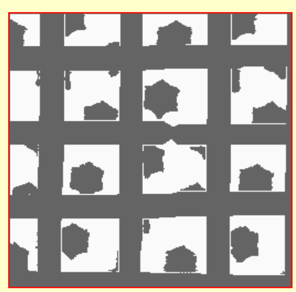
What "larger", or "largest" partition does mean ?

3rd step Maximum partition

3rd step: among all partitions of the space into regions that fulfil the "flat zones" criterion, there exists a ***larger one***



partition into flat zones



Largest partition into flat zones

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Connected operators

Maximum partition

Are we sure that such a largest partition always exists ?

In fact the question is twofold:

1/ Does the “the largest partition” of a family exists ?

=> Yes, because the partitions of a set E do form a complete lattice .

2/ Is the largest partition of a family an element of the said family

=> Who knows ?

Partition Lattice

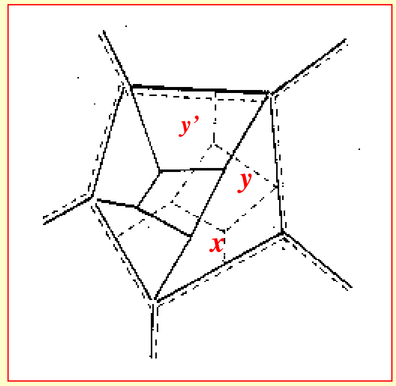
Indeed, we can say that a partition is larger, or is the largest one, because

- The partitions of E form a **complete lattice $\mathcal{D}$** for the ordering according to which
- partition D is smaller than D'

$$D \leq D'$$

when each class of D is included in a class of D'.

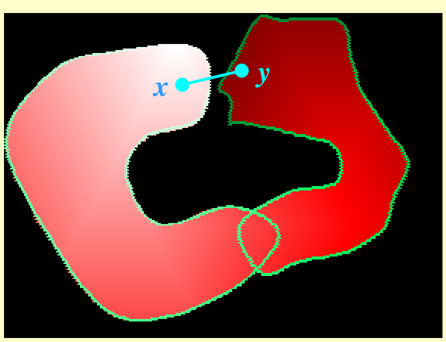
- The largest element of $\mathcal{D}$ is E itself, and the smallest one is the pulverizing of E into all its points.



The sup is the pentagon with common boundaries ;

The inf, simpler, is obtained by intersecting the cells.

1st counter example : Lipschitz Functions



There is no largest partition…

A function f is ***Lipschitz*** of parameter ***k*** when

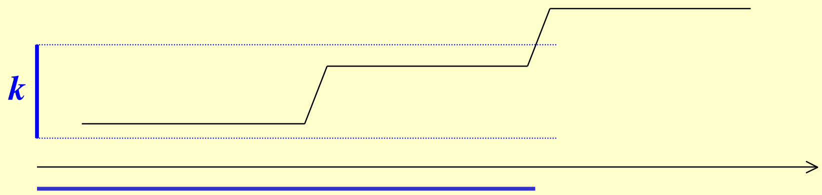
$$|f(x) - f(y)| \leq k\, d(x,y)$$

This one is Lipschitz in the left part

It is also Lipschitz in the right part

BUT NOT INSIDE BOTH !!

Counter example : thick flat zones

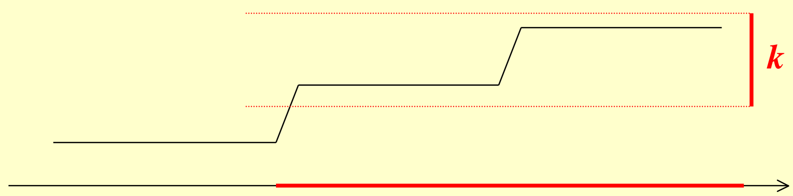


Thick flat zones :

$$x,y \in A \;\Rightarrow\; |f(x)\text{-}f(y)| \leq k$$

=> *The blue segment fulfils the criterion ;*

Counter example : thick flat zones



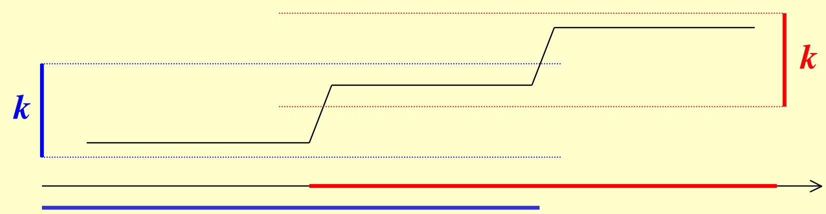
Thick flat zones :

$$x,y \in A \;\Rightarrow\; |f(x)\text{-}f(y)| \leq k$$

=> *The blue segment fulfils the criterion ;*

=> *The red segment also fulfils the criterion ;*

Counter example : thick flat zones



Thick flat zones :

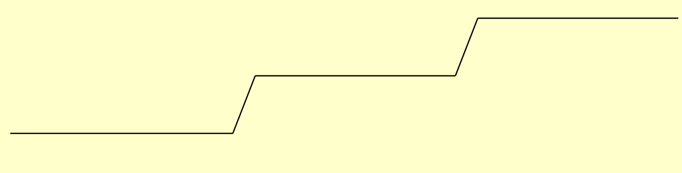
$$x,y \in A \;\Rightarrow\; |f(x)\text{-}f(y)| \leq k$$

=> The blue segment fulfils the criterion ;

=> The red segment also fulfils the criterion ;

BUT THEIR UNION DOES NOT !!

(Thin) flat zones



However, for k=0, we find the usual flat zones :

$$\mathbf{x,y \in A \;\Rightarrow\; |f(x)\text{-}f(y)| \leq 0}$$

Then the largest partition exists and is composed of

- the red segments of the constant regions;

- or the points, elsewhere.

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Segmentation

E and T are two arbitrary sets,
Class $\mathcal{F}$ is a family of functions f: E $\rightarrow$T,
σ is a criterion,

- Given a function $f\in\mathcal{F}$, let $\{D_i(f)\}$ be **the family of all partitions** of set E into homogenous zones of f according to criterion σ.

Segmentation We say that criterion σ **segments** class $\mathcal{F}$ when for each function $f\in\mathcal{F}$,

$\sigma[f,\{x\}] = 1 \quad \forall x$

the family $\{D_i(f)\}$ admits a supremum $\curlyvee\{D_i(f)\}$.

Then the partition $\curlyvee\{D_i(f)\}$ defines the ***segmentation*** of f with respect to σ.

Segmentation

The segmentation $D = \curlyvee\{D_i(f)\}$ decomposes set E into zones

- *that are disjoint and cover the whole space E,*
- *where function f is homogeneous according to criterion σ,*
- *and where the class of the partition at each point is the largest possible one that satisfies criterion σ.*

For using the notion in practice, unless we accept to check all possible partitions, for each function f , we need a theorem that ***links the concept of a segmentation with tools*** we can handle.

The convenient notion turns out to be that of a ***connection.***

Connection (reminder)

- ***Definition of a connection*** **:** Let E be an arbitrary space. We call connected class, or connection $\mathcal{C}$ any family in $\mathcal{P}$(E) such that

$$i)\quad \emptyset \in \mathcal{C}\,;$$

$$ii)\quad \forall\, x \in E : \{x\} \in \mathcal{C}\,;$$

(*class $\mathcal{C}$ contains always the singletons, plus the empty set*)

$$iii)\quad \forall\, \{A_i\},\ A_i \in \mathcal{C} :\quad \{\cap A_i \neq \emptyset\} \Rightarrow \{\cup A_i \in \mathcal{C}\}$$

(*the union of elements of $\mathcal{C}$ whose intersection is not empty is still in $\mathcal{C}$*)

The elements $C \in \mathcal{C}$, are said to be ***connected*** .

- Although such a definition does not involve any topological background, both *topological* and *arcwise* connectivities are particular connections.

Connected opening (reminder)

Given a set A and a point $x \in A$, consider the union $\gamma_x(A)$ of all connected components containing x and included in A

$$\gamma_x(A) = \cup \{ C : C \in \mathcal{C} , x \in C \subseteq A \}$$

- ***Theorem of the point connected opening****:* the family $\{\gamma_x , x \in E\}$ is made of openings, called ***point connected opening***, and such that

iv) $\gamma_x(x) = \{x\}$ $\qquad x \in E$

v) $\gamma_y(A)$ and $\gamma_z(A)$ $\qquad y, z \in E , \quad A \subseteq E$ are disjoint or equal

vi) $x \notin A \Rightarrow \gamma_x(A) = \varnothing$

and the datum of a connected class $\mathcal{C}$ on $\mathcal{P}(E)$ is ***equivalent*** to such a family.

In other words, every $\mathcal{C}$ induces a unique family of openings satisfying *iv)* to *vi)*, and the elements of $\mathcal{C}$ are the invariant sets of the said family $\{\gamma_x , x \in E\}$.

Connective criterion

It remains to introduce the last piece of the puzzle, namely the following connection property for criteria

Connective criterion: A criterion σ is connective when for any family $\{A_i\}$ and any function $f \in \mathcal{F}$ we have :

1- $\sigma[f,\{x\}] = 1$ **for all x ;**

2- $\cap A_i \neq \varnothing$ **and** $\wedge \sigma[f, A_i] = 1 \quad \Rightarrow \quad \sigma[f , \cup A_i] = 1$

In words : when f satisfies σ on A and on B, and when A and B have at least one common point, then f satisfies σ on $A \cup B$.

N.B. Flat zones, zones above a given threshold, etc. are connective, but Lipschitz criterion is not ….

The Segmentation Theorem

***Theorem :** Given a criterion σ on $\mathcal{F}\otimes\mathcal{P}(E)$, the three following statements are equivalent*

1/ criterion σ is connective,

2/ Given $f \in \mathcal{F}$, the class of those sets on which criterion σ is satisfied, forms a connection $\mathcal{C}$,

3/ Criterion σ segments all functions $f \in \mathcal{F}$.

Remarks :

- The concept of a connection is exactly right for the theorem to work
- The theorem links maximal partition (but to get it?) with the connective criterion, a ***more tractable*** notion.

Comments

- Remarkably, we have been able to identify the notion of segmentation with some families of connections without having equipped neither the starting set E, nor the arrival one T, with ***any property***.

- Indeed, theorem opens the way to all applications where ***heterogeneous variables*** are defined over the space. This circumstances arise for example

  - in ***colour imagery***, with the hue, or
  - in ***geomophology*** and ***remote sensing***, where

radiometric data (satellite images) live together with

- physical data (altitude, slope of the ground, sunshine, distance to the sea, etc.)
- and with statistical data (demography, fortunes, diseases, etc.).

Lattice of connective criteria

Theorem : ***Given a function $f \in \mathcal{F}$ the family Σ of all connective criteria on f forms an complete lattice, where the infimum corresponds to the logical intersection of the criteria, and whose smallest element is the partition of space E into its singletons***

Remarks :

- There is no similar logical « OR », as ***the sup of partitions does not involve*** classes where one criterion at least should be fulfilled.
- If set E was not previously provided with a connection, then criterion σ provides E with a connection, $\mathcal{C}$ say.
- Conversely, If space E was initially provided with connection $\mathcal{C}'$ then the intersection $\mathcal{C} \cap \mathcal{C}'$ generates the maximum partition for the intersection of the two constraints.

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Class permanency

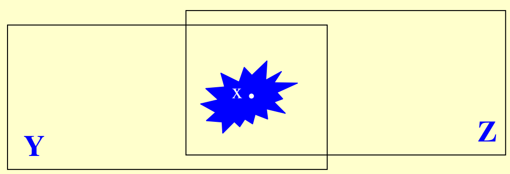
- In the approach by functional optimisation, the contours of a same object ***vary*** with the location of the working mask around it.

- In the lattice approach, things are different. Fix function f and criterion σ. Let Y and Z be two masks included in the whole space E. At point $x \in Y \cap Z$, the two segmentation classes inside Y and Z ***are the same,*** i.e.

$$D_x(Y) = D_x(Z)$$

if and only if ***the object is visible*** in both masks, i.e.

$$D_x(Y) \subseteq Z \subseteq E \quad \text{and} \quad D_x(Z) \subseteq Y \subseteq E.$$



Class permanency

- ***local => global***

From $D_x(Y) \subseteq Z \subseteq E$ and $D_x(Z) \subseteq Y \subseteq E$

We see, by taking **Z = E,** that the class at point x for the whole space E is also the segmentation in the sub-field Y if

$$D_x(E) \subseteq Y$$

- Such a condition is satisfied when :

1/ Criterion σ is C-decreasing for the arcwise connection, i. e.

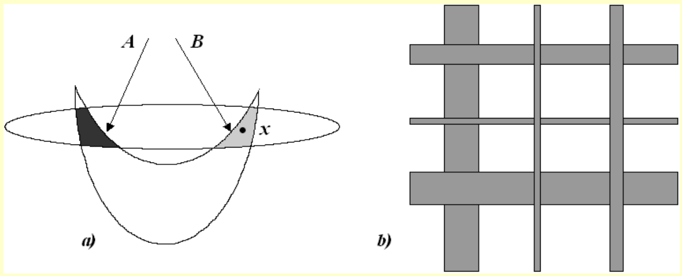
1. The C- classes are arcwise connected

2. If $\sigma(f,A) = 1$ and $B \subseteq A$ with B arc connected, then $\sigma(f,B) = 1$

2/ $D_x(Y)$ is strictly included in Y, i.e. there exists $\varepsilon > 0$ such that

$$D_x(Y) \subseteq Z \ominus \varepsilon B \qquad \textit{(B, open unit ball)}$$

Infimum of connective criteria



- ***Counter-example a) colour image***

  1rst criterion: CC where the green is above threshold v $\Rightarrow$ the ellipse

  2nd criterion: CC where the red is above threshold r $\Rightarrow$ the crescent

- ***Counter-example b) numerical image***

  1rst criterion : $\sigma[f,A]=1 \Leftrightarrow$ union of the vertical lines with one x at least where $f(x)>a$

  2nd criterion : id. for the horizontal lines

Supremum of connective criteria

- ***The logical "OR"*** exists for connective criteria in the following conditions

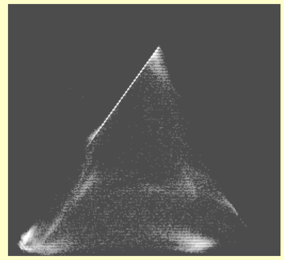
Start from a function $f \in \mathcal{F}$, and let $\{D_i, i \in I\}$ be a family of segmentation partitions that correspond to the connective criteria $\{\sigma_i, i \in I\}$.

- ***Proposition: The supremum $D' = \vee D_i$ is the largest partition of E into classes where one criterion at least is satisfied if and only if each class $D'(x)$ of D' is also a class of one, or more, of the partitions D_i***

- ***Global $\rightarrow$ local :*** A common use of the proposition arises when the non-point class $D_i(x)$ covers point classes only, for all other criteria σ_i, with $i \neq i_0$.

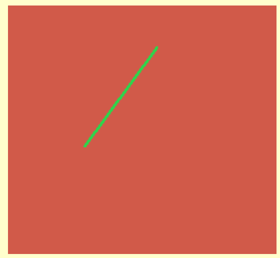
Then we can calculate the maximum class at point x without paying attention to the other classes. The global approach comes back to independent "local" ones.

Marking of “guitar”



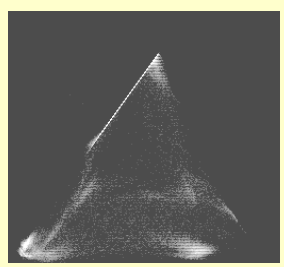
Histogram in L_1 norm

Marked region



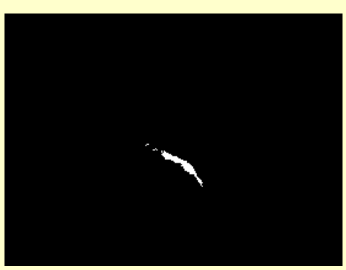
Marker 1

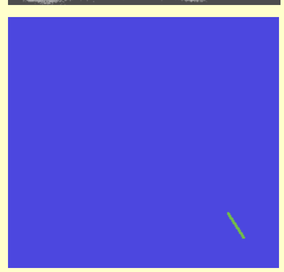
Marking of “guitar”



Histogram in L_1 norm

Marked region

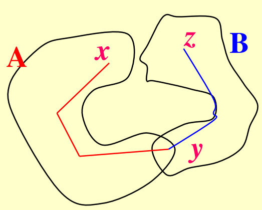


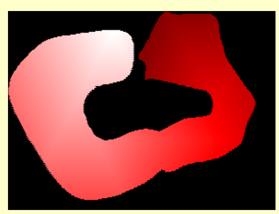


Marker 1

Plan

Smooth Connection





Smooth connection : $E = R^n$, with the *arcwise connection*, and function $f: E \to T$ is fixed. The "*smooth connexion*" is composed

i) of the singletons, plus the empty set ;

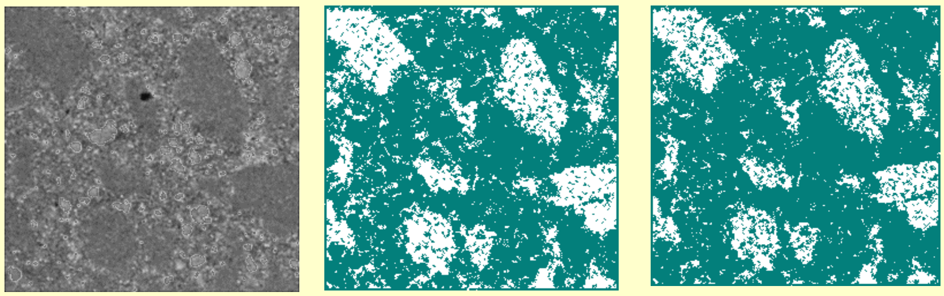
ii) For each $x \in A$, there exists an open ball $B(x) \subseteq A$ where f is Lipschitz

Property : f is k-Lipschitz along all possible paths included in A.

Implementation in Z^2 : erosions by unit cone of slope k

An example of smooth connection

Comment *: the two phases of the micrograph cannot be separated by thresholds. The smooth connections classify them according to their roughness.*



a) Initial image : electron micrograph of concrete

b) smooth connection of slope 7

c) smooth connection of slope 6

(in dark, the singleons)

Jump Connection

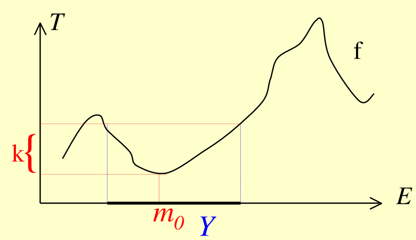
Jump connection : $E = \mathbb{R}^n$, provided with the *arcwise* connection and function f: E →T is fixed. The class $\mathcal{C} \in \mathcal{P}(\mathbb{R}^n)$, composed of

*i)*the singletons plus the empty set ,

*ii)*all connected sets around each minimum, and where the value of f is less than k above the minimum ,

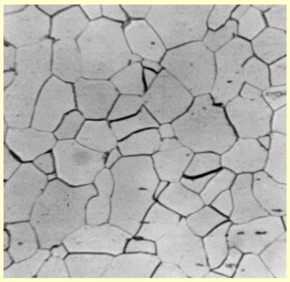
forms a second connection on $\mathcal{P}(\mathbb{R}^n)$, called "*jump connection from minima*"(resp. *maxima*)

One can combine the two connections from maxima and minima

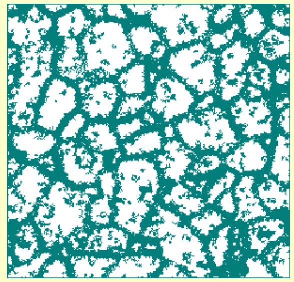


A connected component in the jump connection of range k from the minima .

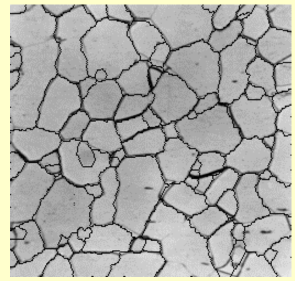
An Example of Jump Connection



a) Initial image: polished section of alumine grains



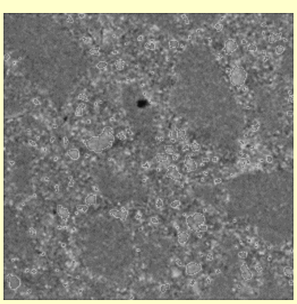
b) Jump connection of size 12 :
- in dark, the point connected components ;
- in white, the other ones



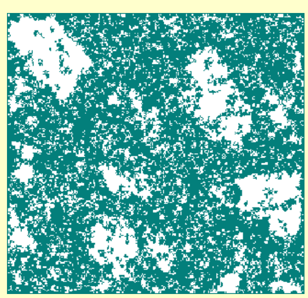
c) Skiz of the set of the dark points of image b)

Roughness and inf of connections

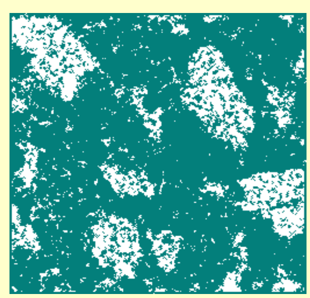
Comment *: Here the jump connection is not satisfactory. But its infimum with the smooth one generates a convenient new connection.*



a) Initial image: rock electron micrograph



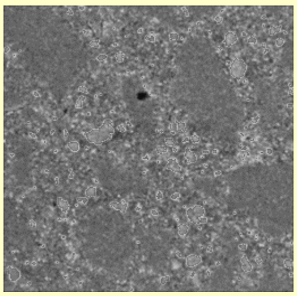
b) Jump connection of amplitude 12



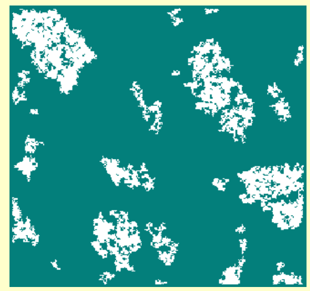
c) smooth connection of slope 6

Roughness and inf of connections

Comment : *Here the jump connection is not satisfactory. But its infimum with the smooth one generates a convenient new connection.*

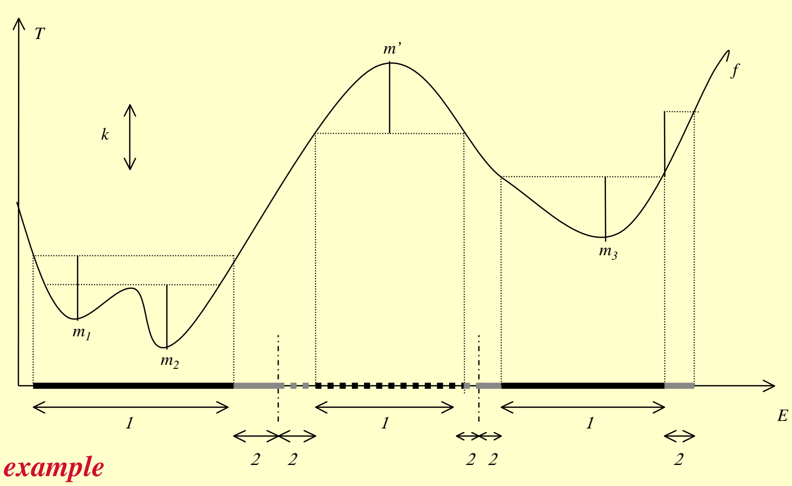


a) Initial image: rock electron micrograph



c) Intersection of the jump (12) and smooth (6) connections .

Iterated jump Connection



An example

Plan

Segmentation: an intuitive approach ;

Segmentation: counter examples ;

Connective criteria and Segmentation theorem;

Class permanency and local knowledge;

Flat, smooth, and jump connections (segmentations).

Connected operators

Connected operators

- Suppose that a first mapping $\psi: \mathcal{F} \to \mathcal{F}$ acts on function f prior to its segmentation . Then the pair $\{\sigma, \psi\}$, considered as a whole, ***defines the criterion***

$$\sigma[\psi(f), A] = 1 \text{ or } 0.$$

- ***Property :*** σ connective $\Rightarrow \{\sigma, \psi\}$ connective (as $\psi(\mathcal{F}) \subseteq \mathcal{F}$)
- ***Definition:*** Operator ψ is said to be ***connected*** when

$$\sigma[f,A] = 1 \quad \Rightarrow \quad \sigma[\psi(f),A] = 1.$$

- ***Property :*** when ψ is connected, the segmentation partition of f according to $\{\sigma, \psi\}$, is larger than that by σ (some classes are clustered)

Hierarchies of connected operators

- ***Increasing semi-groups*** Let ψ and ψ' be two connected operators. We have

$$\sigma[f, A] = 1 \;\Rightarrow\; \sigma[\psi(f), A] = 1 \;\Rightarrow\; \sigma[\psi'\psi(f), A] = 1.$$

i.e. $\psi'\psi$ is connected, and the connection $(\sigma, \psi'\psi)$ contains (σ, ψ).

This suggests to introduce the following ***increasing semi-groups*** $\{\psi_\lambda, \lambda \geq 0\}$ where the product $\psi_\nu = \psi_\lambda \psi_\mu$ acts more than each of its factors, i.e. such that,

1/ $\forall\, \lambda, \mu \geq 0 \quad \Rightarrow \; \nu \geq \sup(\lambda, \mu)$;

2/ $\forall\; \nu \geq \lambda \geq 0$ there exists μ, with $\nu \geq \mu \geq 0$, such that $\psi_\nu = \psi_\lambda \psi_\mu$

- ***Property:*** A semi-group of connected operators is increasing iff the family $\{(\sigma, \psi_\lambda)\ \lambda \geq 0\}$ is totally ordered in the lattice of the connective criteria (or of the corresponding connections).

The segmentations of f by the (σ, ψ_λ) ***increase with λ*** and their contours decreases.

An example of hierarchy

- ***Property :*** When the family of the markers operators is an increasing semi-group then the associated levelings form in turn an increasing semi-group.
- ***Example :*** The markers are both maxima and minima of dynamics > m.

a) b) c)

a) Altea image. b) and c) Levelings of a) by the flat zones criterion. The marker is m_{80} in b) , and m_{230} en c).