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Part II : Connection

- Set case
- Function case

Jean Serra

Set Connection

Set Connection :

-> Definition and Properties

-> Derived connections

-> Geodesy and
Reconstruction Opening

Applications :

-> Individual analysis

-> Edge corrections

-> Isolated objects

-> Alignements

Connectivity in Topology (*reminder*)

Topological Connectivity **:** Given a topological space E, set $A \subseteq E$ is connected if one cannot partition it into two non empty closed sets.

A Basic Theorem **:**

If $\{A_i\}$ $i \in I$ is a family of connected sets, then

$$\{ \cap A_i \neq \varnothing \} \Rightarrow \{ \cup A_i \text{ connected} \} \qquad (1)$$

Arcwise Connectivity (more practical for $E = R^n$) **:** A is **arcwise connected** if there exists, for each pair a,b $\in$A, a continuous mapping ψ such that

$$[\alpha, \beta] \in R \qquad \text{and} \qquad \psi(\alpha) = a \ ; \ \psi(\beta) = b$$

This second definition is more restrictive. For the open sets of R^n, both definitions are **equivalent**.

Criticisms

- ***Is* topological connectivity *adapted to Image Analysis ?***

Arcwise connectivity is extensively, and adequately used in notions such as skeletons, watersheds or homotopic mappings.

But :

- Observe that all these algorithms lie on the following intuition:

" A particle is something that one can pick out from a point ; any other point picks out exactly the same particle, or something disjoint ."

Do we really need a topological background to formalise such an intuition ?

- Moreover, planar sectioning (3-D objects) as well as sampling (sequences) tend to disconnect objects and trajectories.

Connection

These criticisms suggest not to take Eq.(1) as a consequence, but as a starting point.

- ***Definition*** **:** Let E be an arbitrary space. We call connected class, or **connection** $\mathcal{C}$ any family in $\mathcal{P}$(E) such that

$$iv)\quad \varnothing \in \mathcal{C}\,;$$

$$v)\quad \forall\, x \in E:\ \{x\} \in \mathcal{C}\,;$$

(*class* $\mathcal{C}$ *contains always the singletons, plus the empty set*)

$$vi)\quad \forall \{A_i\},\ A_i \in \mathcal{C}:\quad \{\cap A_i \neq \varnothing\} \Rightarrow \{\cup A_i \in \mathcal{C}\}$$

(*the union of elements of* $\mathcal{C}$ *whose intersection is not empty is still in* $\mathcal{C}$)

The elements C ∈ $\mathcal{C}$, are said to be ***connected*** .

- Although such a definition does not involve any topological background, both ***topological*** and ***arcwise*** connectivities are particular connections.

Point Connected Opening

Given a set A and a point x∈A, consider the union $\gamma_x(A)$ of all connected components containing x and included in A

$$\gamma_x(A) = \cup\, \{ C : C \in \mathcal{C},\ x \in C \subseteq A \}.$$

- ***Theorem of the point connected opening:*** the family $\{\gamma_x,\ x \in E\}$ is made of openings, called ***point connected opening,*** and such that

$$iv)\ \gamma_x(x) = \{x\} \qquad x \in E$$

$$v)\ \gamma_y(A)\ \text{and}\ \gamma_z(A) \qquad y, z \in E,\quad A \subseteq E\quad \text{are disjoint or equal}$$

$$vi)\ x \notin A \Rightarrow \gamma_x(A) = \varnothing$$

and the datum of a connected class $\mathcal{C}$ on $\mathcal{P}$(E) is ***equivalent*** to such a family. In other words, every $\mathcal{C}$ induces a ***unique family*** of openings satisfying *iv)* to *vi),* and the elements of $\mathcal{C}$ are the invariant sets of the said family $\{\gamma_x,\ x \in E\}$.

Properties of the Connections

Arc generalization: Set X is $\mathcal{C}$-connected iff for all points y and z of X we can find a $\mathcal{C}$-component Y included in X and that contains y and z .

Connection Partitioning Theorem : Let $\mathcal{C}$ be a connection on $\mathcal{P}$(E). For each set A$\in$ $\mathcal{P}$(E) the maximal connected components $\subseteq$A partition A into its connected components. This partition is increasing in that if A$\subseteq$A', then any connected component of A is upper bounded by a connected component of A'.

Lattice of the Connections : The set of the all connections on $\mathcal{P}$(E) is closed under intersection; it is thus a complete lattice in which the supremum of family $\{\mathcal{C}_i ; i \in I\}$ is the least connection containing $\cup\mathcal{C}_i$

$$\inf \{\mathcal{C}_i\} = \cap\mathcal{C}_i \quad \text{and} \quad \sup \{\mathcal{C}_i\} = \mathcal{C}\{\cup\mathcal{C}_i\}$$

Comments

- The above axiomatic and theorem were proposed in1988 by *G.Matheron* and *J.Serra*. They had in mind
  - to formalise the *reconstruction* techniques based on dilations,
  - to make their approach free of any *cumbersome topology of the continuous spaces,*
  - to encompass *more than particles* seen as "one piece objects",
  - to design nice *strong morphological filters.*
- But their approach was basically **set wise** oriented. Now, the major use of filtering concern grey tone and colour images (and their sequences):

Can we derive from connected openings pertinent filters for grey images ?

Do we need dilation based reconstruction algorithms ?

Can we express the notion of a connection for lattices, in general ?

Examples of Connections

In Digital Imagery, the connected components in the senses of the

4- and 8-connectivity (***square*** grid) ,

6-connectivity (***hexagonal*** grid) ,

12-connectivity (***cube-octahedral*** grid) ,

constitute four different connections.

- The ***second generation connections*** by dilation or closing, consider clusters of objects as connected entities.
- Also, the notion extends to numerical and to ***multi-spectral functions.***
- Therefore the previous approach gathers under a ***unique axiomatic*** the various usual meanings of "connectivity", plus new ones (*e.g.* clusters).

Second Connection by dilation

There are two ways to interpret ***clusters of grains*** as connected components, namely via dilations, or via closings (*J. Serra).*

- ***Proposition1*: Let $\delta : \mathcal{P}(E) \to \mathcal{P}(E)$ be an extensive dilation that preserves connection $\mathcal{C}$ (*i.e.* $\delta(\mathcal{C}) \subseteq \mathcal{C}$). Then, the inverse $\mathcal{C}' = \delta^{-1}(\mathcal{C})$ of $\mathcal{C}$ turns out to be a **connection** on $\mathcal{P}(E)$, which is **larger** than $\mathcal{C}$.

[$A' \in \mathcal{C}' \Leftrightarrow \delta(A') \in \mathcal{C}$. The points and $\varnothing$ are in $\mathcal{C}'$ (preservation under δ). Let $A'_i \in \mathcal{C}'$, with $\cap A'_i \neq \varnothing$. We have $\cap\delta(A'_i) \supseteq \cap A'_i \neq \varnothing$, and since $\delta(A'_i) \in \mathcal{C}$, then $\delta(\cup A'_i) = \cup\delta(A'_i) \in \mathcal{C}$, and finally $\cup A'_i \in \mathcal{C}'$.]

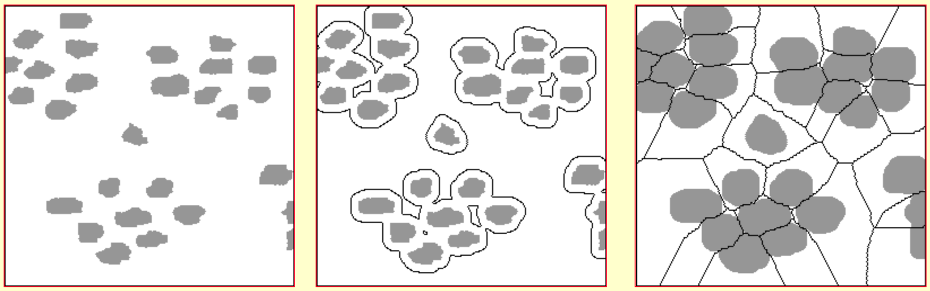
- ***Proposition2*: The $\mathcal{C}$- components of $\delta(A)$, $A \in \mathcal{P}(E)$, are exactly the **images**, under δ, of the $\mathcal{C}'$- components of A .

If γ_x designates the opening of connection $\mathcal{C}$, and ν_x that of $\mathcal{C}'$, we have:

$$\nu_x(A) = \gamma_x \delta(A) \cap A \quad \text{when } x \subseteq A \quad ;$$
$$\nu_x(A) = \varnothing \quad \text{when not.}$$

Application : Search for Isolated Objects

Comment: *One want to find the particles from more than 20 pixels apart. They are the only particles whose dilates of size 10 miss the SKIZ of the initial image.*

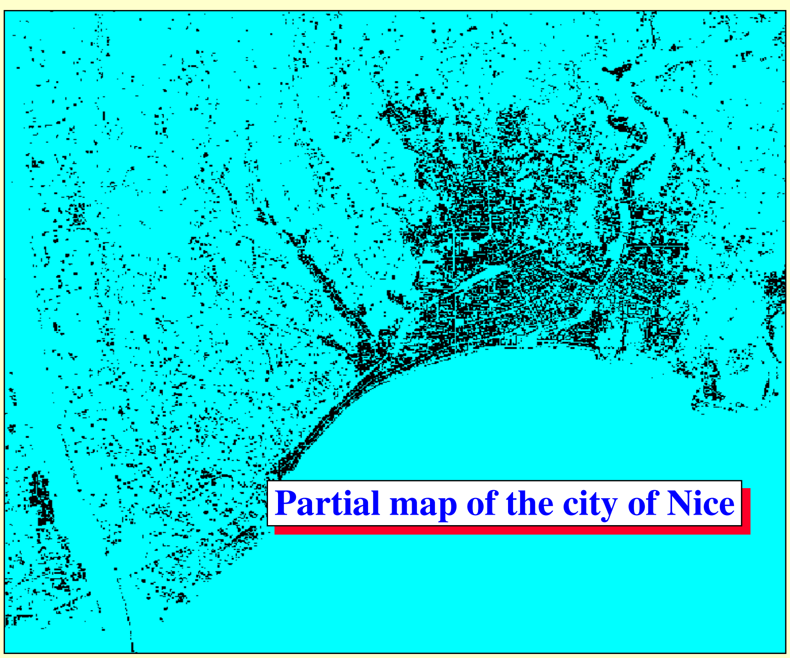


a): Initial Image

b): dilate of a) by disc of radius 10, and new connection

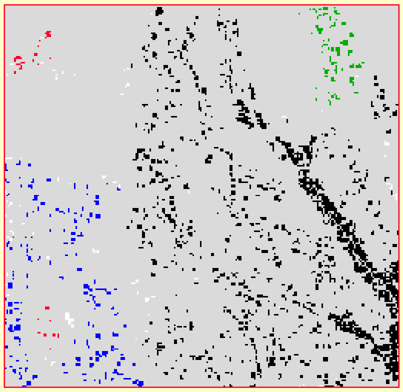
c) : The isolated particles are identical for both connections.

The SKIZ of a) allows to extract them .

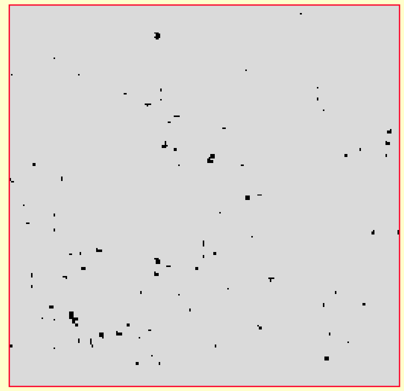


Houses with a Large Garden in Nice

Comment *: Detail of the previous map, where one wish to know the components of the connection by dilation, and, among them, those which are also arwise connected.*

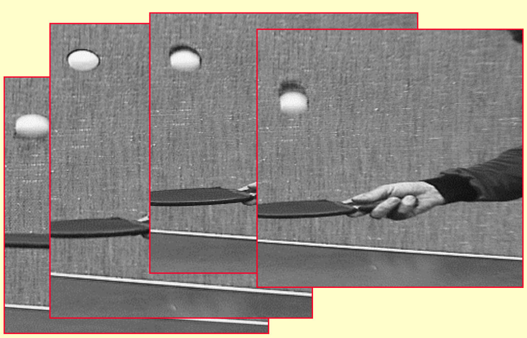


a) Components for the connection by dilation

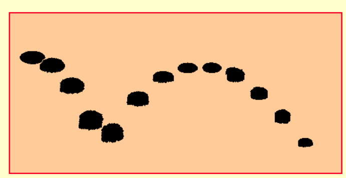


b : Isolated components of a) (according to the above algorithm)

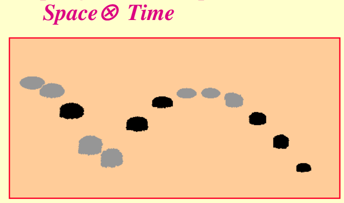
Connections in a Sequence



a) Extracts from an image sequence



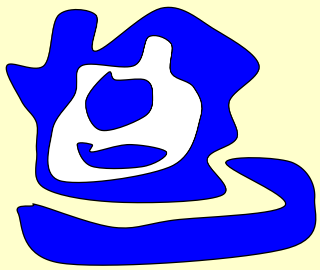
b) representation of the ping-pong ball in the product Space⊗ Time

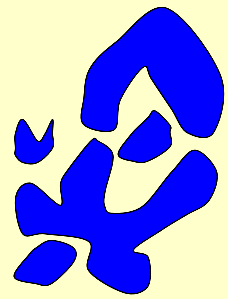


c) Connections after a Space⊗ Time dilation of size 3 (in grey, the clusters)

Second Connection by closing

- ***Proposition* :** Let φ : $\mathcal{P}$(E) → $\mathcal{P}$(E) be closing that preserves connection $\mathcal{C}$ (*i.e.* $\varphi(\mathcal{C}) \subseteq \mathcal{C}$). Then, the inverse image $\mathcal{C}'$ of $\mathcal{C}$ under φ turns out to be a **connection** on $\mathcal{P}$(E), which is **larger** than $\mathcal{C}$.





How many grains for the "clo-holes"closing connection?

How many grains for the closing connection by local convex hull ?

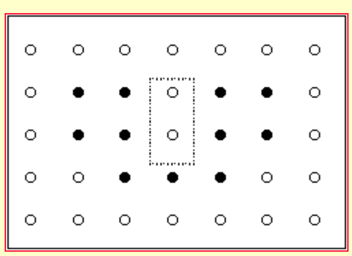
… and for the intersection of the two connections?

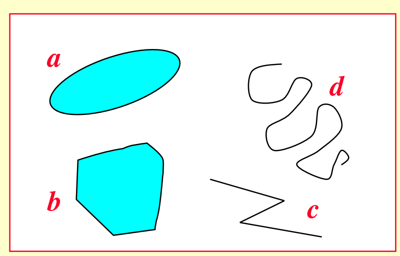
Connection by opening

Let γ be an arbitrary opening. The invariant sets { γ(A), A ∈ $\mathcal{P}$(E)} of γ are closed under union. Hence the family:

$$\mathcal{C} = \{ \gamma(A), A \in \mathcal{P}(E)\} \cup \{ \{x\}, x \in E\}$$

generates a connection (*Ch. Ronse*).

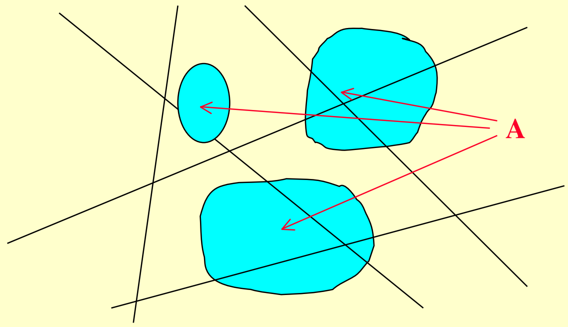




For γ the digital opening by square 2x2, this set is made of four grains:
- the union of the two squares,
- and the three points .

For γ the union of the openings by all segments in R^2, we have :
- one grain = a ∪ b ∪ c
- all points of d as individual grains

Connection by Partitioning



- Given a partition D of space E, all subsets of all classes $\{D(x), x \in E\}$ form a family closed under union. Hence we have the connection *(J.Serra):*

  $\mathcal{C} = \{A \cap D(x), x \in E, A \in \mathcal{P}(E)\} \cup \varnothing$
- The connected component $\gamma_x(A)$, $x \in A$, equals the intersection $A \cap D(x)$ between A and the class of the partition at point x.

Connection Preservation

We say that an operator ψ on $\mathcal{P}(E)$ **preserves connection $\mathcal{C}$** on $\mathcal{P}(E)$ when ψ maps $\mathcal{C}$ into itself, *i.e.* $\psi : \mathcal{C} \to \mathcal{C}$.

***Connected Dilations* :** Let δ: $\mathcal{P}(E) \to \mathcal{P}(E)$ be an extensive dilation. If $\delta(x)$, $x \in E$, is connected, then δ preserves connection $\mathcal{C}$.

Derived Operators : For every dilation δ on $\mathcal{P}(E)$ that preserves connection $\mathcal{C}$, both adjoined erosion ε and opening $\gamma = \delta\varepsilon$ treat the connected components of any $A \subseteq E$ independently of one another.

***Minkowski Addition*:** Let E be R^d or Z^d , equipped with connection $\mathcal{C}$. When A and X belong to $\mathcal{C}$, then $A \oplus X$ is $\mathcal{C}$-connected too.

Comment : This last proposition does not require extensivity. But the first one, which does not assume translation, covers more situations, such as the standard operators, for example.

Geodesic Mappings

- ***How to implement a point connected opening ?***

All connections that we have seen are based on the classical arcwise connection, which is more or less modified. Now, the latter may be obtained by means of ***geodesic operators***.

- ***« Geodesic » Metrics***

In the Euclidean distance, and in its digital versions, the possible obstacles from one point to another are ignored. However, given two points (a,b) in a compact set $X \subseteq R^n$, there exists always a shortest path from a to b that is included in set X (G. Choquet).

This defines a new distance, called ***geodesic*** and restricted to reference X. It generates a wide class of operators . Note that these operations are always **isotropic**, since they bring into play discs and balls only.

Geodesic Distance

Definition The set geodesic distance d_X: $R^n \otimes R^n \rightarrow R$, with respect to reference set X is defined by

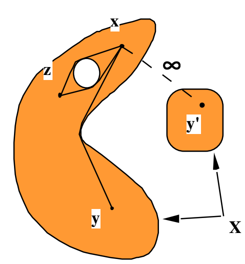
$d_X(x,y)$ = Inf. of the lengths of the paths going from x to and included X;

$d_X(x,y) = + \infty$, if no such path exits.

Properties

1) d_X is a generalized distance, since
   $d_X(x,y) = d_X(y,x)$
   $d_X(x,y) = 0 \Leftrightarrow x = y$
   $d_X(x,z) \leq d_X(x,y) + d_X(y,z)$
2) The geodesic distance is always larger than the Euclidean one;
3) A geodesic segment may not be unique

Examples of geodesics in R^2 :



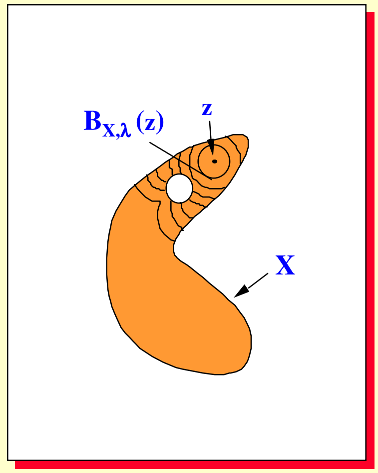
N.B. the portions of geodesics included in the interior of X are line segments

Geodesic Discs

- The notion of geodesic path is seldom used. By contrast, the notion of geodesic discs appears very often:

$$B_{X,\lambda}(z) = \{y,\ d_X(z,y) \leq \lambda\}$$

- When the radius r increases, the discs progress as a wave front emitted from z inside the medium X.
- For a given radius λ, the discs $B_{X,\lambda}$ can be viewed as a set of structuring elements which vary from place to place.



Geodesic Dilation

- The geodesic dilation of size λ of Y inside X is written as follows:

$$\delta_{X,\lambda}(Y) = \cup \{B_{X,\lambda}(y),\ y \in Y\}$$

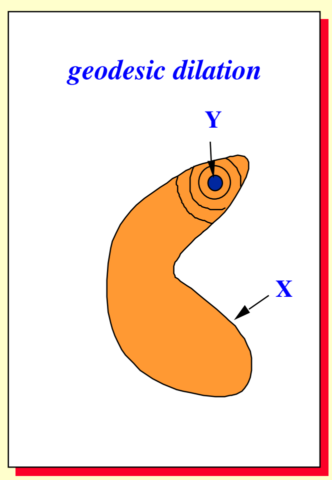
- As λ varies, the $\delta_{X,\lambda}$ generate the following additive semi group

$$\delta_{X,\lambda+\mu} = \delta_{X,\lambda}[\delta_{X,\mu}],$$

(*useful for digital implementation*).

- Note the difference between **geodesic** and **conditional** dilations

$$\delta_{X,\lambda}(Y) \subseteq (Y \oplus B_\lambda) \cap X .$$



Geodesic Erosion

- Any ball being symmetrical, The dualities by adjunction and for the complement are the same.
- But the complement is taken **inside the mask X** (*i.e.* $Y \rightarrow X \setminus Y = X \cap Y^C$), which results in the erosion :

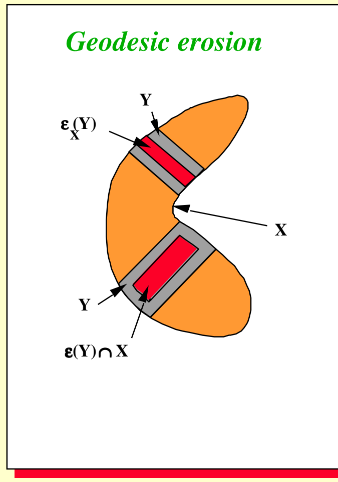
$$\boldsymbol{\varepsilon_X(Y) = X \setminus \delta_X (X \setminus Y)}$$

i.e.

$$\boldsymbol{\varepsilon_X(Y) = \varepsilon (Y \cup X^c) \cap X}$$

where ε stands for Minkowski subtraction.

- Note the difference between and $\varepsilon_X(Y)$ and $\varepsilon (Y) \cap X$.



(Binary) Digital Geodesic Dilation

- In the digital metrics on Z^n , and when $\delta(x)$ stands for the unit ball centered at point x, then the unit geodesic dilation is defined by the relation :

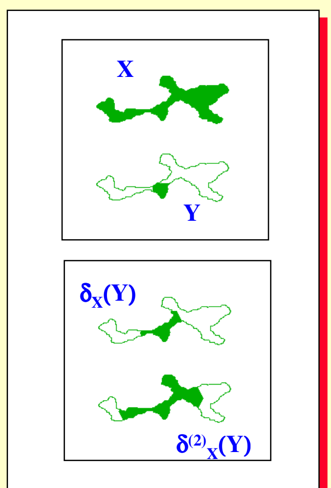
$$\boldsymbol{\delta_X(Y) = \delta (Y) \cap X}$$

- The dilation of size n is then obtained by **iteration** :

$$\boldsymbol{\delta_{X,n}(Y) = \delta^{(n)}{}_X(Y)} \;, \text{ with}$$

$$\delta^{(n)}{}_X(Y) = \delta(\ldots \delta(\delta (Y) \cap X) \cap X \ldots) \cap X$$

- Note that the geodesic dilations are ***not*** translation invariant.



Reconstruction Opening

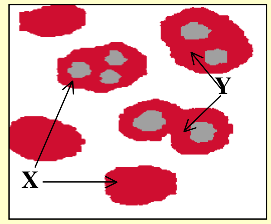
- Given X, the infinite dilation of Y

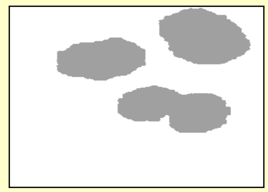
$$\delta_{X,\infty}(Y) = \cup\{ \delta_{X,\lambda}(Y) \,,\ \lambda>0 \}$$

is a ***closing*** ; but if we consider it as an operation on the (now variable) reference set X, for a ***given*** marker Y, then $\delta_{X,\infty}(Y)$ turns out to be the ***reconstruction opening***

$$\gamma^{rec}(X\,;Y) = \cup\{ \delta_{X,\lambda}(Y) \,,\ \lambda>0 \} = \cup\{ \gamma_{y,}(X),\ y \in Y\}$$

of those connected components of set X that contain at least one point of set Y.





N.B.: *As the grid spacing becomes finer and finer, the digital reconstruction opening tends towards the Euclidean one iff X is locally finite union of disjoint compact sets.*

Connection and Reconstruction Opening

The notion of a connection allows to generalize reconstruction openings

1) Call **increasing binary criterion** any mapping c: $\mathcal{P}(E) \to \{0,1\}$ such that:

$$A \subseteq B \quad \Rightarrow \quad c(A) \leq c(B)$$

2) With each criterion c associate **the trivial opening** γ_T: $\mathcal{P}(E) \to \mathcal{P}(E)$

$$\gamma_T(A) = A \quad \text{if} \quad c(A) = 1$$
$$\gamma_T(A) = \varnothing \quad \text{if} \quad c(A) = 0$$

3) By generalizing the geodesic case, we will say that γ^{rec} is **a reconstruction opening** according to criterion c when :

$$\gamma^{rec} = \vee\{\gamma_T\,\gamma_x,\ x \in E\}$$

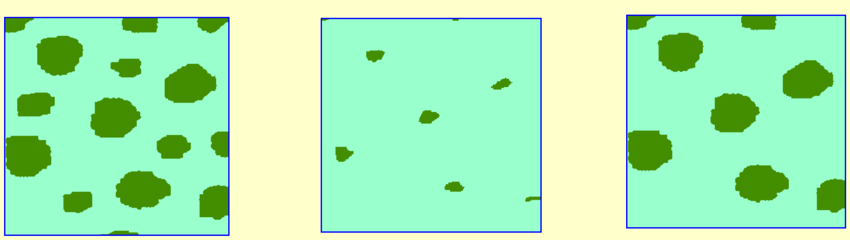
γ^{rec} acts independently on the various components of the set under study, by keeping or removing them according as they satisfy the criterion, or not (*e.g. area, Ferret diameter, volume..*).

Closing by Reconstruction ; Lattices

- ***The closing by reconstruction*** $\varphi^{rec} = \complement\gamma^{rec}\complement$ is defined by duality.
  For example, in R^2, if we take the criterion
  - « ***to have an area*** ≥ 10 », then $\varphi^{rec}(A)$ is the union of A and of the pores of A with an area ≤ 10;
  - or the criterion « ***to hit a given marker M*** », then $\varphi^{rec}(A)$ is the union of A and of the pores of A included in M^c .
- ***Associated Lattices:*** We now consider a family $\{\gamma^{rec}_i\}$ of openings by reconstruction, of criteria $\{ c_i \}$.Their inf $\cap\ \gamma^{rec}_i$ is still an opening by reconstruction, where each grain of A which is left must fulfill all criteria c_i , and where the sup $\cup\ \gamma^{rec}_i$ is the opening where one criterion at least must be satisfied (dual results for the closings). Hence we may state:
- ***Proposition:*** Openings and closing by reconstruction constitute two complete lattices for the usual sup and inf.

Application: Filtering by Erosion-Reconstruction

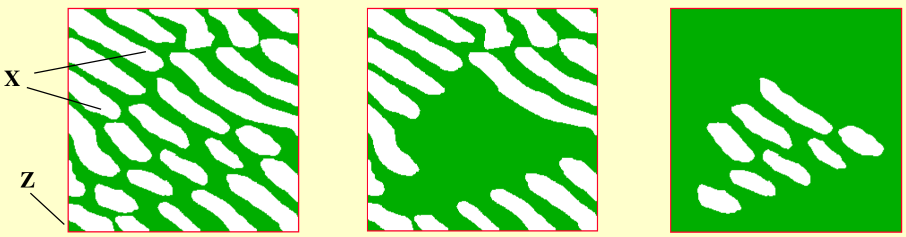
- Firstly, the erosion $X \ominus B_\lambda$ suppresses the connected components of X that cannot contain a disc of radius λ;
- then the opening $\gamma^{rec}(X ; Y)$ of marker $Y = X \ominus B_\lambda$ «re-builts» all the others.



a) Initial image *b) Eroded of a) by a disc* *c) Reconstruction of b) inside a)*

Application: Removal of the edge grains

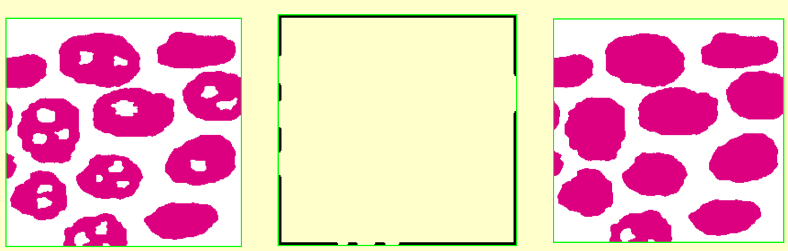
- Let Z be the set of the edges , and X be the grains under study ;
- Set Y is the reconstruction of Z∩X inside set X ;
- the set difference between X and Y extracts the internal particles.



a) Initial image   *b) Particles that hit the edges*   *c) Residue a) - b)*

Application: Holes Filling

Comment : ***efficient algorithm, except for the particles that hit the edges of the field.***



initial image X   *A = part of the edge that hits X^C*   *reconstruction of A inside X^C*

Individual Analysis of Particles

Algorithm

While set X is not empty do

{

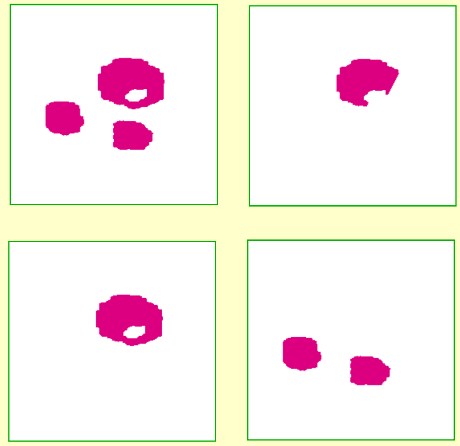
~ p := first point of the video scan;

~ Y := connected component of X reconstructed from p;

~ Processing of Y (and various measurments) ;

~ X := X \ Y

}



Individual extraction of particles

Connectivity and Reconstructions

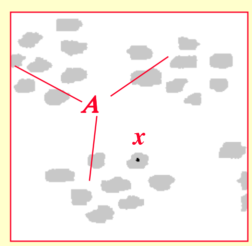
- We saw that if point x is a marker and A a set, the infinite geodesic dilation **$\cup\, \delta_A^{(n)}(x)$** leads to the point connected opening of A at x

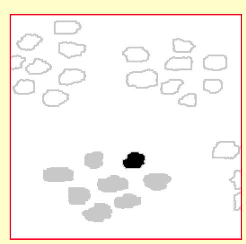
$$\gamma_x(A) = \cup\, \delta_A^{(n)}(x) \qquad (1)$$

- Moreover, when we replace the unit disc δ by that of radius 10, for example, in (1), we yield a second connection, generated by dilation.
- Here two questions arise:

1- If $\delta(x)$ is not a disc, but another set, do we still obtain a ***new derived connection,*** *i.e.* which still ***segments*** set A ?

2- Must we operate by means of ***dilations*** ?





Geodesy et Connections

Curiously, the answer to these questions depends on properties of symmetry of the operators. A mapping $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is ***symmetrical*** when

$$x \subseteq \psi(y) \quad \Leftrightarrow \quad y \subseteq \psi(x)$$

for all points x,y de E.

- ***Theorem*** **:** Let $\psi: \mathcal{P}(E) \to \mathcal{P}(E)$, and let $x \in E$, $A \in \mathcal{P}(E)$. Then the limit iteration

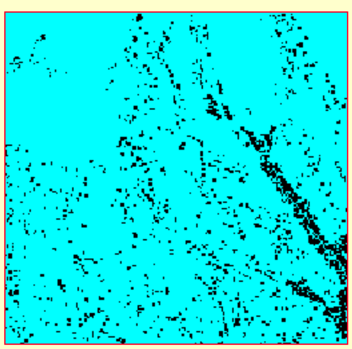
$$\gamma_x(A) = \cup \{\psi_A^{(n)}(x), n > 0\}$$

considered as an operation on A, is a **point connected opening** if and only if ψ is an extensive and symmetrical dilation.

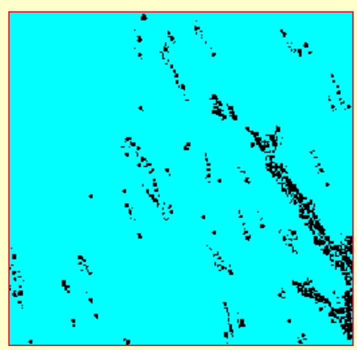
Note that the starting dilation ψ does not need itself to be connected!

Nice : Directional Alignments

Comment ***: Although the structuring element D used for the reconstruction is not connected, it generates a new connection. For display reasons, we the smaller components have been filtered out .***



a) Zone A under study



b) Reconstruction of A from $A \ominus 2B$ by means structuring element D =  ***where each point is a unit hexagon***

Set Connection and Numerical Functions

Concepts :

- -> Extension to functions
- -> Connected operators
- -> Numerical Geodesy
- -> Leveling and self-duality

Applications :

- -> Extrema analysis
- -> Contour preservation
- -> Strong filters
- -> Segmentation

Passage to Numerical Functions

Three passages from binary to grey tone images must be viewed.

- *Connections* We can either :

  \- generalise the concept of a connection to lattices, and find connections which are adapted to numerical functions,

  \- or use functions to induce ***set connections*** on their supports. This simpler (but less powerful) approach will be adopted here.

- *Geodesy*

  It is the simplest one. Dilation and erosion being increasing, it suffices to define numerical operations from binary ones, applied level by level.

- *Applications*

  They are not the same as the binary case. Priority is now given to the processing of the ***extrema*** and to ***contours preservation***.

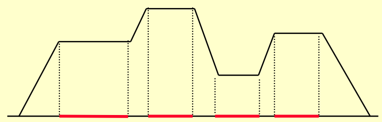
Flat zones connection

Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$ and f be a function on E.

The flat zones of f induce a second connection $\mathcal{C}'$ on E where the connected compenent at point x is

$$\gamma_x(E) = \cup\{ C: C \in \mathcal{C}, x \in C; y \in C \Rightarrow f(y) = f(x) \}.$$

This ***Flat zones connection*** partitions set E into maximal classes on which f is constant



The connected components of $\mathcal{P}(R^1)$ according $\mathcal{C}'$ are either

- the red segments;

- or the points, elsewhere.

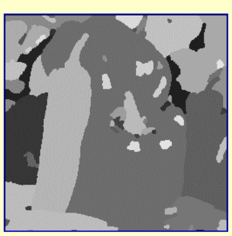
Connected Operators

Definition **:**

- A function operator $\psi : T^E \rightarrow T^E$ is said to be **connected** (for criterion σ) when the partition of E by $\psi(f)$ is larger than that of E by f .

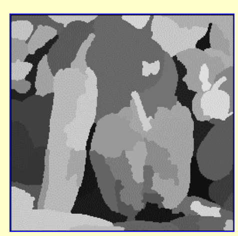
a)

b)



c)

d)



Three mosaic images, due to C. Vachier, obtained by merging the watershed of the gradient of ***a):***

b) by dynamics ; ***c)*** by areas ; ***d)*** by volumes

Flat and Increasing Connected Operators

- From now on, we focus exclusively

  i) on the criterion σ of **flat zones** ;

  ii) and on those operators $\psi : T^E \rightarrow T^E$ that are **connected, flat,** and **increasing**.

***Basic Properties* :**

- Every **binary** connected (resp. and increasing) operator induces on T^E, via the cross sections, a ***unique*** connected (resp. and increasing) operator ;
- In particular, all geodesic implementations extend to the numerical case ;
- The properties of the set case, to be strong filters, to constitute semi-groups, etc.. are transmitted to the connected operators induced on T^E.

Note that a mapping may be anti-extensive on T^E, but extensive on the lattice $\mathcal{D}$ of the partitions (*e.g.*reconstruction openings).

Numerical Geodesic Dilations (I)

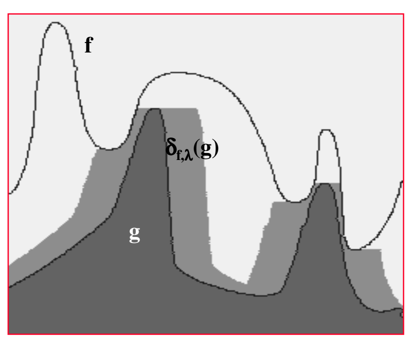
- Let f and g be two numerical functions from $\mathbb{R}^d$ into T, with $g \leq f$.

  The binary geodesic dilation of size λ of each cross section of g inside that of f at the same level induces on g a dilation $\delta_{f,\lambda}(g)$ (*S.Beucher*) .

- Equivalently, (*L.Vincent*) the sub-graph of $\delta_{f,\lambda}(g)$ is the set of those points of the sub-graph of f which are linked to that of g by

  - a non descending path
  - of length $\leq \lambda$.

Numerical geodesic dilation of g with respect to f



Numerical geodesic Dilations (II)

- The digital version starts from the unit geodesic dilation:

  $\delta_f(g) = \inf(g \oplus B, f)$

which is iterated n times to give that of size n

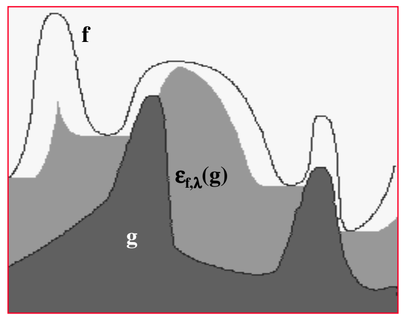
  $\delta_{f,n}(g) = \delta_f^{(n)}(g) = \delta_f(\delta_f \ldots (\delta_f(g)))$.

- The Euclidean and digital erosions derive from the corresponding dilations by the following duality

  $\varepsilon_f(g) = m - \delta_f(m - g)$,

- which is **different** from the binary duality.

Numerical geodesic erosion of f with respect to g :



Numerical Reconstruction

- The reconstruction opening of f from g is the supremum of the geodesic dilations of g inside f, this sup being considered as a function of f:

  $\gamma^{rec}(f\,;g) = \vee\{\delta_{f,\lambda}(g),\ \lambda>0\}$

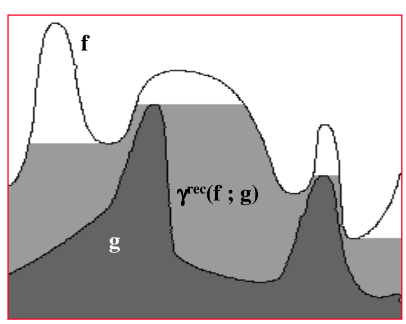
  The dual closing for the negative is

  $\varphi^{rec}(f\,;g) = m - \gamma^{rec}(m-f\,;m-g)$

- The three major applications are :

  - ***swamping***, *or reconstruction of a function by imposing markers for the maxima;*
  - ***reconstruction*** *from an erosion*
  - ***contrast opening*** *,which extracts and filters the maxima.*

Numerical reconstruction of g inside f :



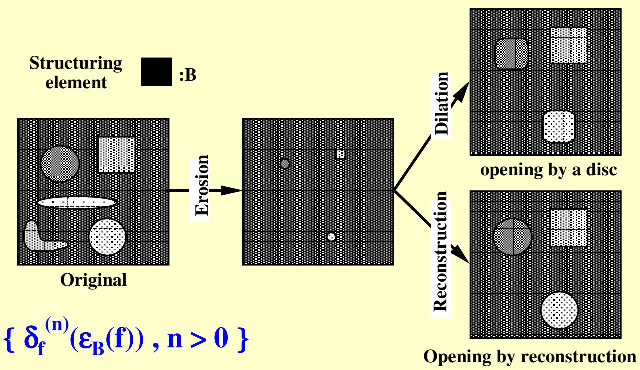
Opening by reconstruction

Goal : contour preservation

Whereas the adjunction opening modifies contours, this transform is aimed to efficiently and precisely reconstruct the contours of the objects which have not been totally removed by the filtering process.

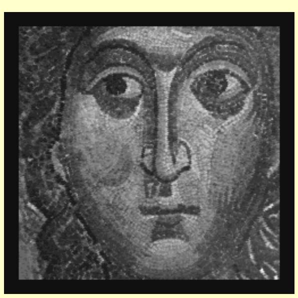
Algorithm

- the mask is the original signal ,
- the marker is an eroded of the mask.



$$\gamma^{rec}(f \,;\, \varepsilon_B(f)) = \vee\{ \delta_f^{(n)}(\varepsilon_B(f)) \,,\, n > 0 \}$$

Opening by reconstruction



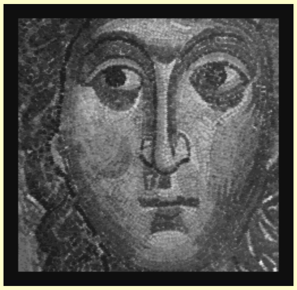
a) Initial image

b) erosion of a)

Reconstruction of image b) inside a)

Opening by reconstruction

Comment: *the same operation on the complement image suppresses the dark small components*



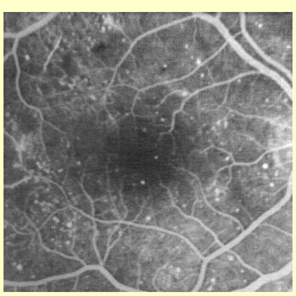
a) Initial image

b) dilation of a)

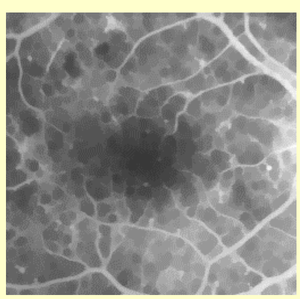
Reconstruction of image b) inside a) (via the complements)

Application to Retina Examination

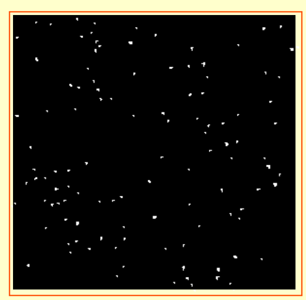
Comment: *The aim is to extract and to localise aneurisms. Reconstruction operators ensure that one can remove* **exclusively** *the small and isolated peaks*



a) Initial image



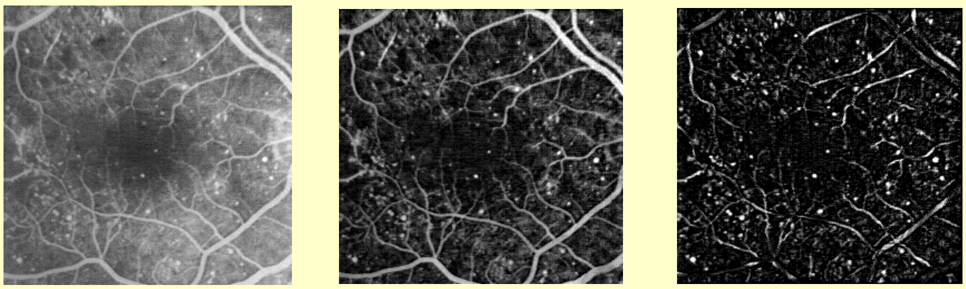
b) closing by dilatation-reconstruction followed by opening by érosion- reconstruction



c) difference a) minus b) followed by a threshold

Comparison with other top-hats

Comment : *Top hat c), better than b) is far from being perfect. Here opening by reconstruction yields a correct solution*



Negative image of the retina. | *Top hat by an hexagon opening of size 10.* | *Top hat by the sup of three segments openings of size 10.*

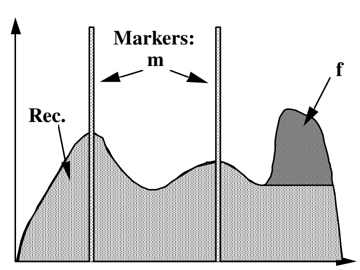
Reconstruction of a Function from Markers

Goal

To remove the useless maxima (or minima) of a function.

Algorithm

- The "marker" is a bi-valued (0,m) function identifying the peaks of interest.
- The reconstruction process result is the largest function ≤ f and admitting maxima at the marked points only. It is called the **swamping** of f by opening



Swamping of f by markers m (by opening)

An Example of Swamping : Contrast Opening

Goal

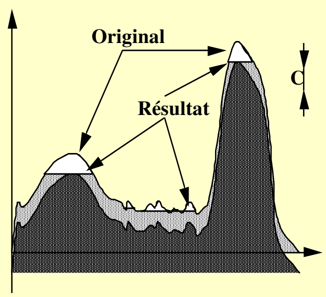
Both morphological and reconstruction openings reduce the functions according to size criteria which work on their cross sections. In opening by **dynamics**, the criterion holds on grey tones contrast .

Algorithm

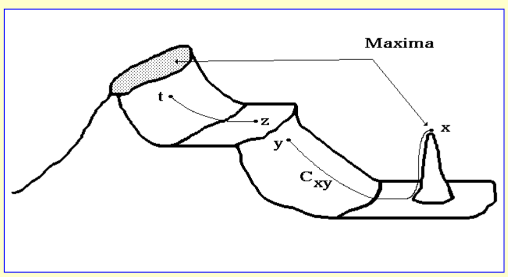
- Shift down the initial function f by constant c;
- Rebuilt f from function f - c, *i.e.*

$$\gamma^{rec}(f\ ;\ f\text{-}c) = \vee\{\ \delta_f^{(n)}(f\text{-}c)\ ,\ n > 0\ \}$$

The associated top-hat extracts all peaks of dynamics ≥ c



Application to Maxima Detection



- The ***maxima*** of a numerical function on a space E are the connected components of E where f is constant and surrounded by lower values.
- Therefore they are obtained by contrast opening of shift c = 1.
- More generally, the residuals associated with a shift c extract the maxima surrounded by a descending zone deeper than c. They are called ***Extended Maxima.***

Strong Filters by Reconstruction

- ***Proposition 1*:** Let γ^{rec} be a reconstruction opening on T^E that does not create pores and φ^{rec} be the dual of such an opening (not necessarily γ^{rec}). Then :

  $\varphi^{rec}\gamma^{rec}$ and $\gamma^{rec}\varphi^{rec}$ are ***strong filters***, and $\varphi\gamma^{rec} \leq \gamma^{rec}\varphi$

  In particular, $I \wedge \gamma^{rec}\varphi^{rec}$ is an ***opening*** (appreciated for its top-hat).

- ***Proposition 2*:** Let $\{\gamma_i^{rec}\}$ and $\{\varphi_i^{rec}\}$ denote a granulometry and a (not necessarily dual) anti-granulometry, then

  - the corresponding alternating sequential filters N_i and M_i are ***strong*** ; and form the ***semi group***

  $$N_j N_i = N_i N_j = N_{sup(i,j)} \quad ; \qquad M_j M_i = M_i M_j = M_{sup(i,j)}$$

  - both operators $\Psi_n = \wedge\{\varphi_i\gamma_i,\ 1 \leq i \leq n\}$ and $\Theta_n = \vee\{\gamma_i\ \varphi_i,\ 1 \leq i \leq n\}$ are ***strong filters***.

A pyramid of connected A.S.F.'s

Flat zones connection :

Each contour is preserved or suppressed, but never deformed ; the initial partition increases under the Pyramid of the successive filterings.

(hexagonal structuring elements)

Initial Image

A pyramid of connected A.S.F.'s

Flat zones connection :

Each contour is preserved
or suppressed,
but never deformed ;
the initial partition
increases under the
Pyramid of the
successive filterings.

ASF of size 1

(hexagonal structuring elements)

A pyramid of connected A.S.F.'s

Flat zones connection :

Each contour is preserved
or suppressed,
but never deformed ;
the initial partition
increases under the
Pyramid of the
successive filterings.

ASF of size 4

(hexagonal structuring elements)

A pyramid of connected A.S.F.'s

Flat zones connection :

Each contour is preserved or suppressed, but never deformed ; the initial partition increases under the Pyramid of the successive filterings.

ASF of size 8

(hexagonal structuring elements)

A pyramid of connected A.S.F.'s

Flat zones connectivity, (i.e. $\varphi = 0$).
Each contour is preserved or suppressed, but never deformed : the initial partition increases under the successive filterings, which are strong and form a semi-group.

Initial Image

ASF of size 1

ASF of size 4

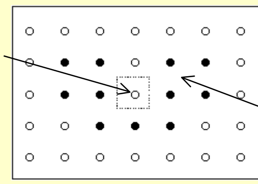
ASF of size 8

(hexagonal structuring elements)

Adjacency

- ***Adjacency:*** Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$. Sets $X,Y \in \mathcal{P}(E)$ are said to be adjacent when $X \cup Y$ is connected, whereas X and Y are disjoint.

  Note that for the digital connection by a 2x2 square opening, the point marker M of the figure is adjacent to no grain of set X, but to X itself.



- ***Adjacency Prevention:*** Connection $\mathcal{C}$ is ***adjacent preventing*** when, for any element $M \in \mathcal{P}(E)$ and any family $\{B_i ; i \in I\}$ in $\mathcal{C}$, to say that M is adjacent to none of the B_i is equivalent to saying that M is not adjacent to $\cup B_i$.

Leveling

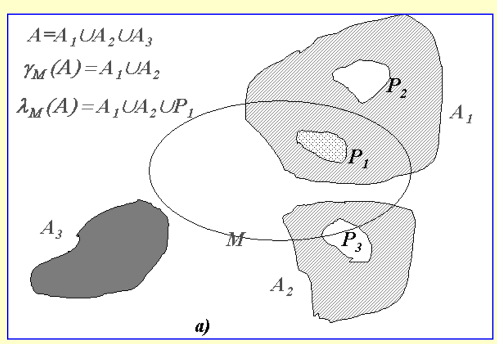
- Given marker M, consider $A \in \mathcal{P}(E)$. Let
  ~ $\gamma_M(A)$ be the union of the grains of A that hit or that are adjacent to M
  ~ $\varphi_M(A)$ be the union of A and of its pores that are included in M and non adjacent to M^c

- **leveling λ** is the ***activity supremum***

$$\lambda = \gamma_M \curlyvee \varphi_M$$

i.e. $\lambda(A) \cap A = \gamma_M \cap A$,
and $\lambda(A) \cap A^c = \varphi_M \cap A^c$.

λ acts as opening γ_M inside A ,
and as closing φ_M inside A^c .



Properties of the Leveling

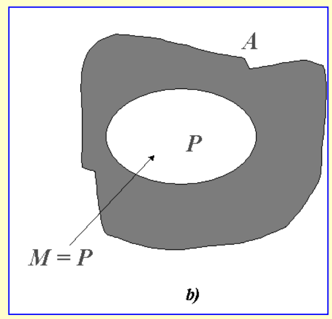
- ***Self-duality**:* The mapping $(A,M) \rightarrow \lambda(A,M)$ from $\mathcal{P}(E) \times \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is self-dual. If M itself depends on A, *i.e.* if $M = \mu(A)$, then λ, as a function of A only, is self-dual iff μ is already self-dual.
- The ***extension to functions*** (via their cross-sections) will be denoted by

$$(f, g) \rightarrow \Lambda\ (f, g)$$

- The leveling is stable because of the adjacency conditions.
- If they are suppressed, we risk to get at the same time

  grain => pore

  and pore=> grain



Properties of the Leveling

Here are a few nice properties of leveling :

- ***Proposition***: The leveling $(A,M) \rightarrow \lambda(A,M)$ is an increasing mapping from $\mathcal{P}(E) \times \mathcal{P}(E) \rightarrow \mathcal{P}(E)$; it admits the equivalent expression:

$$\lambda = \gamma_M \cup (\complement \cap \varphi_M)$$

- ***Proposition**:* The two mappings

  $A \rightarrow \lambda_M(A)$, given M, and $M \rightarrow \lambda_A(M)$, given A,

  are *idempotent* (hence are *connected filters* on $\mathcal{P}(E)$).

- ***Proposition***: Leveling $A \rightarrow \lambda_M(A)$ is a strong filter, and is equal to the commutative product of its two primitives

$$\lambda = \gamma_M \circ \varphi_M = \varphi_M \circ \gamma_M$$

  iff connection C is *adjacency preventing*. Then, λ preserves the *sense of variation* at the grains/pores junctions .

An Example

Initial image *: « Joueur de fifre », by E. MANET*

Markers : *hexagonal alternated filters, (non self-dual)*

Initial image, 83.776 pp
flat zones : 34.835

Marker $\varphi_1 \gamma_2$
flat zones : 53.813

Marker $\gamma_1 \varphi_2$
flat zones : 53.858

Duality for Functions

- If 0 and m stand for the two extreme bounds of the gray axis T, then the set complement operation is replaced by its function analogue $f \rightarrow m - f$ and we have for levelling Λ

$$m - \Lambda (m - f , m - g) = \Lambda (f, g) \qquad (1)$$

which means that $f, g \rightarrow \Lambda(f, g)$ is ***always*** a self dual mapping.

- In addition, if g derives from f by a self-dual operation, *i.e.* $g = g(f)$ with

$$m - g (m - f) = g (f) \qquad (2)$$

(*e.g. convolution, median element*), then levelling $f \rightarrow \Lambda(f, g(f))$ is self-dual.

- Observe that rel.(2) is distinct from that of invariance under complement

$$g (m - f) = g (f)$$

which is satisfied by the module of the gradient, or by the extended extrema, for example, and which does not imply self-duality for $f \rightarrow \Lambda(f)$.

An Example of Duality

Marker: ***extrema with a dynamics $\geq h$ (invariance under complement).***

Initial image
flat zones : 34.835

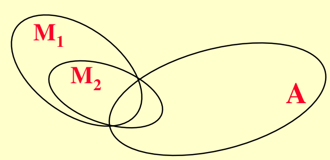
$h = 80$
flat zones : 57.445

Levelling as function of the marker

We now fix set A and study the mapping $M \rightarrow \lambda_A(M)$ as marker M varies. Set A generates on $\mathcal{P}(E)$ the A-activity ordering $\preceq_A$ by the relations

$$\mathbf{M_1 \preceq_A M_2}$$

i.e. if M_1 meets A or is adjacent to A,
then M_2 meets A or is adjacent to A
and if M_2 meets A^c or is adjacent to A^c,
then M_1 meets A^c or is adjacent to A^c.



Proposition : If $M_1 \preceq_A M_2$, then we have

$$\lambda_{\lambda A(M1)}(M_2) = \lambda_{\lambda A(M2)}(M_1) = \lambda_A(M_2)$$

This granulometric pyramid allows to grade markers activities .

An Example of Pyramid

Marker: ***Initial image, where the h-extrema are given value zero (self-dual marker)***

Initial image
flat zones : 34.835

Levelling for h = 50
flat zones : 58.158

Levelling for h = 80
flat zones : 59.178

An Example of Noise Reduction

Marker: ***Gaussian convolution of size 5 of the noisy image***

a) Initial image, plus
10.000 noise points

b) Gaussian
convolution of a)

c) Levelling of a) by b)
flat zones : 46.900