Polynomial growth, recurrence and ergodicity for random walks on locally compact groups and homogeneous spaces

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Abstract Let G be a locally compact group, E a homogeneous space of G. We discuss the relations between recurrence of a random walk on G or E, ergodicity of the corresponding transformations and polynomial growth of G or E. We consider the special case of linear groups over local fields.

Keywords: Random walk ; ergodicity homogeneous spaces ; polynomial growth
MSC(2000): 22E30, 31C12, 37A17, 37A50

1 Introduction

Let G be a locally compact and separable group, λ_G be a left Haar measure on G. We denote by μ a given probability measure on G, by G_μ the closed subgroup generated by its support. We study recurrence properties of random walks defined by μ, either on G, or on homogeneous spaces of G. We also discuss ergodicity of associated homeomorphisms on path spaces. This paper can be considered as an introduction to the more complete article [17] where detailed proofs are given. For basic informations and results we refer to the surveys [7], [14] and to the books [6], [12], [29], [30]. We thank F. Ledrappier P. Bougerol, Y. Derriennic and T. Steger for useful comments on ergodicity of bilateral Markov shifts with infinite invariant measure.

2 Polynomial growth

Definition 1 We say that G has polynomial growth of degree at most $d \geq 0$ if for any compact neighbourhood W of e, there exists $C_W > 0$ such that for every $n \in \mathbb{N}$

$$\lambda_G(W^n) \leq C_W n^d.$$

A typical example of polynomial growth is the following : N is a nilpotent compactly generated group, K is a compact group of automorphisms of N and G is a semi-direct product of K and N. Then d can be calculated in terms of the descending series of N (See[10]). We give now a few structural facts.

It is well known that polynomial growth of G implies amenability and unimodularity for G as well as for its closed subgroups. A fundamental result of [9] says that if G is finitely generated with polynomial growth then G is virtually nilpotent. If G is locally compact and compactly generated with polynomial growth then G has a compact normal subgroup K such that G/K is a Lie group (connected or not) [21]. The connected Lie groups with polynomial growth are the groups of rigid type [10]. Rigid type for a connected Lie group G means that for every $g \in G$, the automorphism Adg of the Lie algebra of G has only eigenvalues of modulus one.

3 Transience and recurrence

1) Definitions

We consider the product space $\Omega = G^{\mathbb{N}}$ (resp $\widehat{\Omega} = G^{\mathbb{Z}}$) endowed with the shift θ and the product measure $\mathbb{P} = \mu^{\mathbb{N}}$ (resp $\widehat{\mathbb{P}} = \mu^{\mathbb{Z}}$) and we denote by $X_k(\omega)$ the coordinates of $\omega \in \Omega$ (resp $\omega \in \widehat{\Omega}$). If E is a locally compact G- space and $x \in E$, the random walk on E of law μ, starting from x, is the sequence of random variables $S_n(\omega)x$ ($n \in \mathbb{N} \cup \{0\}$ or $\mathbb{Z}$) defined by :

$$S_0(\omega)x = x$$

$$S_n(\omega)x = X_n(\omega) \cdots X_1(\omega)x \qquad (n > 0)$$

$$S_n(\omega)x = X_{n+1}^{-1}(\omega) \cdots X_0^{-1}(\omega)x \qquad (n < 0)$$

If $E = G$, the asymptotic behaviour of $S_n(\omega)x$ is essentially independent of $x \in G$. This is not the case in general if $E \neq G$ and we will discuss the situation in terms of a Radon measure λ quasi invariant under the μ-action on E. In particular if λ is a μ-invariant measure ($\mu * \lambda = \lambda$) we will consider on $\Omega \times E$ the measure $\widetilde{\lambda} = \mathbb{P} \otimes \lambda$. The skew product $(\Omega \times E, \widetilde{\theta}, \mathbb{P} \otimes \lambda)$, with $\widetilde{\theta}(\omega, x) = (\theta\omega, X_1(\omega)x)$ will play a basic role since $\widetilde{\lambda}$ is $\widetilde{\theta}$-invariant. If $E = G$, we will take $\lambda = \lambda_G$. We will need also to consider the extended bilateral shift $\widehat{\theta}$ on $\widehat{\Omega} \times E$: $\widehat{\theta}(\omega, x) = (\theta\omega, X_1(\omega)x)$ which leads to consider the bilateral random walk $S_n(\omega)$ on E ($n \in \mathbb{Z}$). The Markov operator P on $C_b(E)$, the space of continuous bounded functions on E, which is defined by

$$P\varphi(x) = \int \varphi(gx) d\mu(g),$$

allows to express various quantities of probabilistic interest. For example if $A \subset E$, the expected number of visits of the random walk to A is

$$\int \sum_0^\infty 1_A(S_n(\omega)x) d\mathbb{P}(\omega) = \sum_0^\infty P^k 1_A(x).$$

If $E = G$, we have

$$\sum_0^\infty P^k 1_A(e) = \sum_0^\infty \mu^k(A)$$

where μ^k is the $k-th$ convolution power of μ. We will now mainly restrict to the case $E = G$, and we will come back to the general case in the last section.

Definition 2 We will say that μ or $S_n(\omega)$ is recurrent on G if for every neighbourhood W of e we have $\mathbb{P}-a.e: \ S_n(\omega) \in W$ infinitely often . If $\mathbb{P}-a.e, S_n(\omega)$ escapes to infinity we will say that μ or $S_n(\omega)$ is transient.

It is easy to see that $S_n(\omega)$ is either transient or recurrent. On can show, using Hopf maximal ergodic lemma, that a necessary and sufficient condition for transience of μ is the existence of W, a relatively compact neighbourhood of e, such that $\sum\limits_0^\infty \mu^n(W) < \infty$.

We will assume μ adapted $i.e$ $G_\mu = G$, and also we will exclude the case $G = \mathbb{Z}$ and $\mu = \delta_a (a \in \mathbb{Z})$. The following important concept will be used below

Definition 3 Let f be a Borel function on G. We say that f is left μ-harmonic (resp left μ-superharmonic) if $\int f(hg)d\mu(h) = f(g)$, for any $g \in G$

(resp $f \geq 0$ and $\int f(hg)d\mu(h) \leq f(g)$ for any $g \in G$)

We observe that we could have considered $S'_n(\omega) = X_1(\omega) \cdots X_n(\omega)$, instead of $S_n(\omega)$. But the laws of $S_n(\omega)$ and $S'_n(\omega)$ are equal to μ^n. Then the above implies that recurrence (resp transience) of $S_n(\omega)$ is equivalent to recurrence (resp transience) of $S'_n(\omega)$. Hence we can speak of μ being recurrent (resp transient).

If μ is transient and $u \in C_c^+(G)$, the non negative function $\sum\limits_0^\infty P^k u(x) = \sum\limits_0^\infty (\mu^k * \delta_x)(u)$ is continuous and left-superharmonic.

Then we have the

Proposition 1 Assume μ is as above. Then we have the equivalence

a) $S_n(\omega)$ is recurrent

b) Any continuous left μ-superharmonic function is constant

c) $\widehat{\theta}$ is ergodic with respect to $\widehat{\mathbb{P}} \otimes \lambda_G$.

A natural connection beween $\widetilde{\theta}$-invariant functions and μ-harmonic functions modulo λ_G is as follows.

If $F \in \mathbb{L}^\infty(\Omega \times G, \mathbb{P} \otimes \lambda_G)$ satisfies $Fo\widetilde{\theta} = F$ then the function $f \in \mathbb{L}^\infty(G)$ defined by

$$f(g) = \int F(\omega, g) d\mathbb{P}(\omega)$$

satisfies $\int f(hg)d\mu(h) = f(g)$, $i.e$ f is μ-harmonic mod λ_G. As is well known, this correspondance is bijective. Ergodicity of $\widetilde{\theta}$ amounts to constancy of bounded μ-harmonic functions. Ergodicity of $\widehat{\theta}$ is a stronger property. If $G = \mathbb{Z}^3$ it is easy to construct non trivial $\widehat{\theta}$-invariant functions. However bounded μ-harmonic functions are constant if μ is adapted. Such a construction is as follows. For any $x \in \mathbb{Z}^3$, we denote

$A_x^+ = \{\omega \in \widehat{\Omega} \ \ ; \ \ \exists n > 0, \ \ x + S_n(\omega) = 0\}$

$A_x^- = \{\omega \in \widehat{\Omega} \ \ ; \ \ \exists n \leq 0, \ \ x + S_n(\omega) = 0\}$

$A_x = A_x^+ \cup A_x^-, \quad A = \bigcup\limits_{x \in \mathbb{Z}^3} A_x \times \{x\}$

Then $A \subset \widehat{\Omega} \times \mathbb{Z}^3$ is the set of paths of the random walk which pass through 0 at some time $n \in \mathbb{Z}$; clearly A is $\widehat{\theta}$-invariant. On the other hand, since $S_n(\omega)$ is transient, if $x \neq 0$: $0 < \widehat{\mathbb{P}}(A_x^+) < 1$, $0 < \widehat{\mathbb{P}}(A_x^-) < 1$ (see [27]).

Since $S_n(\omega)$ $(n > 0)$ and $S_n(\omega)$ $(n \leq 0)$ are independent, the complements of A_x, A_x^- are also independent, hence $0 < (1 - \widehat{\mathbb{P}}(A_x^+))(1 - \widehat{\mathbb{P}}(A_x^-)) = 1 - \widehat{\mathbb{P}}(A_x^+ U A_x^-) < 1$. It follows that A is a non trivial $\widehat{\theta}$-invariant set of $\widehat{\Omega} \times \mathbb{Z}^3$.

Definition 4 We say that the group G is recurrent if there exists a probability measure μ on G which is adapted and recurrent.

One can show (see below) that if G is recurrent then G and its closed subgroups are unimodular. Also in this case amenability of G is valid and follows from the constancy of bounded harmonic functions.

2 Examples

a) It is well known that $\mathbb{Z}^2$ is recurrent while $\mathbb{Z}^3$ is transient (see [27]). The free group F_d with d generators $(d \geq 2)$ is transient since it is non amenable (see [8]).

b) Let $\mathbb{G}_2$ be the group of motions of the euclidean plane. We identify $\mathbb{G}_2$ with the group of maps of the form $gz = az + b$ where $z \in \mathbb{C}$ and $|a| = 1$, $b \in \mathbb{C}$. Let $\alpha \notin \mathbb{Q}$ and $a, b \in \mathbb{G}_2$ defined by $az = e^{2i\pi\alpha}z + 1$, $bz = e^{-2i\pi\alpha}z + 1$, $\mu = \frac{1}{2}(\delta_a + \delta_b)$. Then one can show that μ is recurrent, hence $\mathbb{G}_2$ is recurrent. The proof relies on the estimation of the polymer sums $S'_n(\omega)0 = 1 + \sum_1^{n-1} e^{2i\pi s_k(\omega)}$ where $s_k(\omega) = \sum_1^k \varepsilon_k(\omega)$ and $\varepsilon_k(\omega) = \pm 1$ with probability 1/2 ; one gets $\int |S'_n(\omega)0|^2 d\mathbb{P}(\omega) \sim Cn$ with $C > 0$ and this can be shown to imply recurrence (see [12]).

c) Let $G = SL(2, \mathbb{R})$, $\Gamma \subset G$ a cocompact discrete subgroup of G, Γ' a normal subgroup of Γ such that $\Gamma/\Gamma' = \mathbb{Z}$ or $\mathbb{Z}^2$, $E = G/\Gamma'$, $\mu = \delta_a$ with $a = diag\ (e, e^{-1})$. We consider the Haar measure m on E and the action of a on E. The corresponding dynamical system is the time-one geodesic flow on E and can be identified with $(\widehat{\Omega} \times E, \widehat{\theta}, \widehat{\mathbb{P}} \otimes m)$. One can show (see [13], [24], [28]) that this system is ergodic. Here recurrence of the a-action is an easy first step based on Birkhoff ergodic theorem.

4 Growth and recurrence

The following quadratic growth conjecture was stated in [11] : G is recurrent if and only if G has polynomial growth at most 2.

One can show using [21], that if G has polynomial growth of degree at most 2, then G is recurrent (see [23]). Also for the converse one can assume that G is compactly generated. In [19] it was already conjectured that finitely generated recurrent groups have non exponential growth. The general quadratic growth conjecture is based on this idea but is clearly stronger. Also its possible proof can be seen to involve necessarily a group theoretic argument. One can expect the informations of [21] to be sufficient.

Then in the discrete case the quadratic growth conjecture has been settled in [29] using [9] : recurrent finitely generated groups are virtually $\mathbb{Z}$ or $\mathbb{Z}^2$. On the other hand, as shown in [1], recurrent connected Lie groups are closed subgroups of $\mathbb{G}_2$, up to normal compact subgroups. Then, using the analysis developed in [1], [12] and [29], one can solve the quadratic growth conjecture for Lie groups, connected or not. Also, for Lie groups over p-adic fields, the conjecture is solved in [23]. But this leaves unsolved the conjecture if G is totally disconnected. As a first step, it is natural to consider the case where G is a closed subgroup of $GL(V)$ with $V = \mathbb{F}^d$ and $\mathbb{F}$ is a non-archimedean local field. We observe that, if $\mathbb{F}$ has positive characteristic, G is not a Lie group in general. In this case, due to the ultrametric triangular inequality, one can expect a much more simpler analysis and result than in [12] and [29]. This is indeed the case as shown below. Furthermore, in the general case of a recurrent compactly generated and totally disconnected group G, one can expect the following : G has an open compact normal subgroup K such that G/K is recurrent and finitely generated. Then, the corresponding result would follow from [29].

5 Linear groups over local fields

One has the following.

Theorem 1 Assume G is a closed subgroup of a finite product of $\prod_{i \in I} GL(d_i, \mathbb{F}_i)$ where any $\mathbb{F}_i$ is a local field. Then G is recurrent if and only if G has at most quadratic growth.

One will sketch the main ideas of the proof of necessity of quadratic growth in the case $I = \{1\}$ and $\mathbb{F}_1 = \mathbb{F}$ a non-archimedean local field. Also one will assume G compactly generated recurrent and show the existence of a compact normal subgroup K such that G/K is virtually $\mathbb{Z}$ or $\mathbb{Z}^2$. One will need to consider the contraction subgroup C_g of $g \in G$ i.e

$$C_g = \{h \in G; \lim_{n \to \infty} g^n h g^{-n} = e\}$$

If G is a closed linear group it is easy to show that C_g is also closed. Furthermore if $C_g \neq \{e\}$ is closed, the closed subgroup of G generated by g and C_g is not unimodular. Then the two main steps are given by the following propositions.

Proposition 2 Assume G is locally compact and recurrent. Then every closed subgroup of G is unimodular.

It follows from the above remark that, if furthermore $G \subset GL(V)$ is closed, then $C_g = C_{g^{-1}} = \{e\}$

Proposition 3 Assume G is a compactly generated closed subgroup of $GL(d, \mathbb{F})$ where $\mathbb{F}$ is a non archimedean local field. Then, if $C_g = \{e\}$ for any $g \in G$, G has a compact open normal subgroup.

In the proof of this result an important role is played by the following lemmas.

Lemma 1 (See [18])

Assume G is totally disconnected and $g \in G$ satisfies $C_g = C_{g^{-1}} = \{e\}$. Then there exists a compact open subgroup K_g such that $gK_gg^{-1} = K_g$.

This lemma is a consequence of the study of the so called tidy subgroups [2].

Lemma 2 (See [3], [17])

Assume $G \subset GL(d, \mathbb{F})$ is compactly generated, $\mathbb{F}$ is non-archimedean, and any $g \in G$ generates a bounded subgroup. Then G is bounded.

One may observe that, on the real field, even if G is finitely generated, the corresponding statement is false, which gives a negative answer to a question of S. Ulam (see [3]). But, on ultrametric fields, S. Ulam's question has a positive answer. On the other hand, if G is assumed to be closed, the conclusion of lemma 2 is valid for any local field (see [3], [17]).

Let us give the proof of Proposition 2. Let μ be adapted and recurrent on G. Then, if we denote by P the Markov operator on $E = G/H$ associated to μ, we have for any $u \in C_c^+(E), \quad u \neq 0 \ : \ \sum_0^\infty P^k u = +\infty$ on E. Hence, using [20], one sees that there exists a P-invariant Radon measure λ on E. For any $\varphi \in C_c^+(E)$, we consider the non negative function f on G given by $f(g) = g\lambda(\varphi)$. Clearly $f(g) = \int f(gh)d\mu(h)$, i.e f is μ-harmonic on G. Since μ is recurrent and adapted, Proposition 1 implies that f is constant i.e $g\lambda(\varphi) = \lambda(\varphi)$. Since g and φ are arbitrary λ is G-invariant. As is well known from general group theory this implies $\Delta_G(h) = \Delta_H(h)$ for any $h \in H$, where Δ_G (resp Δ_H) is the modular function of G (resp H). If we take for H the closed subgroup generated by $g \in G$, we get $\Delta_G(g) = \Delta_H(g) = 1$; hence G is unimodular. From above it follows $\Delta_H(h) = 1$ for any $h \in H$, hence H is unimodular.

6 Homogeneous spaces

Various examples of recurrent and ergodic behaviours take place (see [5], [17], [12]). Here we only develop one example and formulate a general question.

1) A framework

Let E be a locally compact G-space, P the Markov operator on $C_b(E)$ defined by :

$$P\varphi(x) = \int \varphi(gx)d\mu(g),$$

we consider a P-invariant Radon measure λ on E, i.e $P\lambda = \lambda$. Then we can consider the extended shifts on $\Omega \times E$ and $\widehat{\Omega} \times E$ defined by $\widetilde{\theta}(\omega, x) = (\theta\omega, X_1(\omega)x)$ and $\widehat{\theta}(\omega, x) = (\theta\omega, X_1(\omega)x)$, we endow $\Omega \times E$ and $\widehat{\Omega} \times E$ with their natural structure of Polish spaces so that $\widetilde{\theta}$ is continuous and $\widehat{\theta}$ is a homeomorphim we write $\Omega^- = G^{-\mathbb{N}\cup\{0\}}$. Then the Markov measure $\widetilde{\lambda} = \mathbb{P} \otimes \lambda$ on $\Omega \times E$ is $\widetilde{\theta}$-invariant. One can show that there exists on $\widehat{\Omega} \times E$ a unique Radon measure $\widehat{\lambda}$ which is $\widehat{\theta}$-invariant and has projection $\widetilde{\lambda}$ on $\Omega \times E$.

In the general case of a dynamical system with σ-finite invariant measure, the construction goes back to V.A Rokhlin ([25]). In our special situation, $\widehat{\lambda}$ can be defined as follows. Let P^* be the adjoint operator of P in $\mathbb{L}^2(E,\lambda)$. Since $P\lambda = \lambda$, P^* is also a Markov operator and we denote by $\overset{\vee}{\mathbb{P}}_\lambda$ the corresponding Markov measure on $\Omega^- \times E$. If we denote by θ^- the shift on Ω^-, we can define an extended shift on $\Omega^- \times E$ by $\widetilde{\theta}^-(\omega_-, x) = (\theta^-\omega_-, g_0^{-1}x)$. Then $\overset{\vee}{\mathbb{P}}_\lambda$ is $\widetilde{\theta}^-$-invariant and has projection λ on E, hence we can disintegrate $\overset{\vee}{\mathbb{P}}_\lambda$ as $\overset{\vee}{\mathbb{P}}_\lambda = \int \overset{\vee}{\mathbb{P}}_y \otimes \delta_y d\lambda(y)$, along the fibers $\Omega^- \times \{y\}$. Then one can verify that the measure $\widehat{\lambda} = \int \overset{\vee}{\mathbb{P}}_y \otimes \mathbb{P} \otimes \delta_y d\lambda(y)$ is $\widehat{\theta}$-invariant.

Definitions 5

Let (E, P, λ) be as above. Then one says that (E, P, λ) has property R if for any relatively compact open set $U \subset E$, and $\mathbb{P} \otimes \lambda - a.e$ $(\omega, x) \in \Omega \times U$, we have $S_n(\omega)x \in U$ infinitely often.

If $E = G$, $\lambda = \lambda_G$ and P is as above then property R for (E, P, λ) is equivalent to recurrence of μ on G. Then one has the

Proposition 4

Assume E is not the G-space $\mathbb{Z}$ with $\mu = \delta_g$ acting by translation on $\mathbb{Z}$. Then one has the equivalence

a) Property R is valid for (E, P, λ) and the equation $Pf = f$, $f \in \mathbb{L}^\infty(\lambda)$ implies f is constant.

b) Property R is valid for (E, P, λ) and λ is extremal in the cone of P-invariant Radon measures.

c) The homeomorphism $\widehat{\theta}$ is ergodic on $\widehat{\Omega} \times E$ with respect to $\widehat{\lambda}$.

2) An example

Assume G is semisimple of real rank 1, Γ is a discrete cocompact subgroup and Γ' is a normal subgroup of Γ which satisfies $\Gamma/\Gamma' = \mathbb{Z}$. We endow $E = G/\Gamma'$ with the Haar measure $\lambda = m$ and observe that G/Γ' is an abelian cover of the compact homogeneous space G/Γ. We consider the homeomorphism $\widehat{\theta}$ of $\widehat{\Omega} \times E$ and the measure $\widehat{\mathbb{P}} \otimes m = \widehat{\lambda}$

Proposition 5

With the above notations we assume μ is symmetric with compact support and G_μ is non amenable. Then $\widehat{\theta}$ is ergodic with respect to $\widehat{\mathbb{P}} \otimes m$.

For the proof we use the equivalence of a) and c) in the above proposition. Property R follows easily from the symmetry of μ and the G-invariance of m, using ergodicity of the random walk on G/Γ.

For the study of the equation $Pf = f$, $f \in \mathbb{L}^\infty(E)$, one uses induced unitary representations, as follows.

One observes that $\Gamma/\Gamma' = \mathbb{Z}$ acts on G/Γ' and this action commutes with the G-action. Taking a Borel fundamental domain $\Delta \subset E$ of this action one can identify G/Γ' with $\Delta \times \mathbb{Z}$. One denotes by $x \to g.x$ the transformation of G/Γ defined by $g \in G$. Then for $x \in \Delta \subset E$

one can write : $gx = (g.x)z(gx)$ with $z(gx) \in \mathbb{Z}, g.x \in \Delta$. For any character χ of $\Gamma/\Gamma' = \mathbb{Z}$ one can define a unitary representationn ρ_χ of G in $\mathbb{L}^2(G/\Gamma)$ by

$$\rho_\chi(g)f(x) = f(g^{-1}.x)\chi(z(g^{-1}x))$$

If $\chi(\Gamma) = 1$, i.e $\chi = 0$ we get the natural representation ρ_0 of G in $\mathbb{L}^2(G/\Gamma)$ and we know that ρ_0 restricted to $\mathbb{L}^2_0(G/\Gamma)$ do not contain weakly the identity representation. Since the one dimensional representation of Γ defined by χ do not contain weakly identity, the same is valid for the induced representation ρ_χ from Γ to G (see [22] Prop 1-11 p. 112). Then one can use the non-amenability of G_μ and a result of [25] (see also [4], [16]) to conclude that if $\chi \neq 0$ the operator $\rho_\chi(\mu)$ defined by $\rho_\chi(\mu)f(x) = \int f(gx)\chi[z(gx)]d\mu(g)$ satisfies $||\rho_\chi(\mu)|| < 1$.

Then the same analysis as in [15] can be performed with the natural Fourier decomposition of $\mathbb{L}^2(E)$: we get that the condition $Pf = f$, $f \in \mathbb{L}^\infty(E)$ implies $f = cte$.

3) A question

Let G be a connected Lie group or an algebraic group defined over a local field $\mathbb{F}$, H a closed subgroup, λ a P-invariant Radon measure on $E = G/H$, $x \in supp\lambda$. Assume that (E, P, λ) satisfies property R. Then, is it true that, for any $n \in N$: $\lambda(W^n x) \leq C_W n^2$, where W is a compact neighbourhood of e and $C_W > 0$?

Is it possible to describe geometrically the systems (E, P, λ) if λ is finite ? Is it true that the basic building blocks for such an E are either boundaries of (G, μ) or spaces of the form G/Γ where Γ is a lattice in G ?

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