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Morphological decomposition of sandstone pore–space: fractal power-laws

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Abstract

Morphological decomposition procedure is applied to estimate fractal dimension of a pore–space, which is isolated from a sandstone microphotograph. The fractal dimensions that have been computed by considering various probing rules have precisely followed the universal power-law relationships proposed elsewhere. These results are derived by considering structuring elements such as octagon, square and rhombus that have been used to decompose the pore–space of sandstone image. The radii of the structuring elements are made to increase in a cyclic fashion. To perceive the decomposed pore image, a color-coding scheme is adapted, from which one can identify several sizes of these structuring elements that could be fit into this pore. This exercise facilitates testing of the relationship between the radius of the structuring elements that could be used to decompose the pore at different levels, and the number of decomposed shapes that could be fit into the pore while using the corresponding structuring element. From the number–radius relationship, the fractal dimensions of pore–space estimated, by considering these structuring elements, yield the values of 1.82, 1.76, and 1.79. These values are in conformity with the values arising from estimation of box dimension method, as well as the dimensions of the corresponding pore connectivity networks (PCNs).
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1. Introduction

Pore–spaces of sandstone samples are fractals [1], and are self-similar [1]. It is reported that the percolation theory does not describe the pore–space geometry [2]. For sandstone samples of varied nature such as tight glass sand with #965, and #466, Coconino, and Navajo the range of fractal dimensions is given between 2.57 and 2.87, which suggests that the pore formation processes do not fall within a single universality class [3]. It is also assumed by these researchers that the sandstone pore–space has the fractal properties of the interface implying a simple relationship between the fractal dimension and the porosity of the rock as also opined by Mandelbrot [1]. Several methods have been adapted to compute the fractal dimension of such objects. Packing of objects, pore being one such example, gives an idea of estimating fractal dimensions through power-law relationships. The two seminal studies of recent past based on the concept of packing of objects have been carried out by Manna and Herrmann [4], and Dodds and Weitz [5]. The recent studies [4,5] have emphasized on deriving universal power-law relationships applying scaling theory. A few attempts have been made to estimate the fractal dimension of planar shape by applying mathematical morphological

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transformations [6–13]. In our previous study, a method was proposed and implemented to decompose a fractal object into shapes such as rhombus, octagon, and square [13]. In this study, we have considered the decomposition procedure that utilizes non-overlapping structuring elements with sizes distributed from radius R_{max} to R_{min} in *pore* image. Such a procedure first decomposes the available pore–space with structuring elements of higher radius (R_{max}). Once the filling is performed using R_{max}, it will be followed by a procedure that reduces the radius of the structuring elements and decomposing again. The procedure will repeat until leftover pore is filled with structuring element with minimum radius (R_{min}). This type of decomposition facilitates a procedure to estimate the dimension, akin to fractal dimension, through a power-law relationship between size (or) radius, and number of decomposed and disconnected shapes at a given threshold value. The power-law relationship can be represented by

$$N(r) \propto r^{-\alpha} \quad \text{with } D = \alpha - 1 \tag{1}$$

where N, r, α, and D respectively represent the number of decomposed shapes that are larger than the threshold radius, radius, slope of the regression line, and fractal dimension. In the remainder of the paper, basic introduction to mathematical morphology (Section 2), packing of pore–space through morphological decomposition procedure with its implementation on Appollonian space (Section 3), and results, discussion on the sample pore–space derived from a sandstone image with a scope (Section 4) have been given with mathematical equations and illustrations.

2. Morphological transformations

The discrete binary image is defined as a finite subset of Euclidean two-dimensional space, lR^2 that can have values 0 and 1. Let a digital binary pore M be represented by a set of elements $m \in M$. The image (M) can be decomposed by probing it with various structuring templates to unravel certain complex features of topological nature. Several sequential transformations are involved in this morphological decomposition procedure. These transformations can be visualized as working with two images namely the image being processed (M), and structuring template (S). Each structuring template has a designed shape that acts as a probe. Morphological operations transform M to a new image by a structuring element S. The four basic morphological transformations are defined with respect to structuring element S. These four morphological transformations include *dilation* to enlarge, *erosion* to shrink, and *opening* and *closing* to smoothen [11]. These are represented as Eqs. (2a–d).

$$\text{Dilation: } M \oplus S = \{m + s : m \in M, s \in S\} = \bigcup_{s \in S} M_s \tag{2a}$$

Dilation enlarges the bright spots and neighbouring grey levels will be connected.

$$\text{Erosion:} \quad M \ominus S = \{m - s : m \in M, s \in S\} = \bigcap_{s \in S} M_s \tag{2b}$$

$$\text{Opening:} \quad M \circ S = (M \bigcirc S) \oplus S \tag{2c}$$

$$\text{Closing:} \quad M \bullet S = (M \oplus S) \ominus S \tag{2d}$$

It is worthwhile to mention that the Minkowski's addition and subtraction are similar to morphological dilation and erosion, if considered structuring element $S = \widehat{S}$. These transformations can be carried out in multiscale approach [6,13]. In the multiscale transformations of cascades of erosion–dilation and dilation–erosion are defined with respect to structuring element S with scaling factor n. In the multiscale approach, size of structuring template will be increased from iteration to iteration as shown in Eq. (3).

$$S_n = \underbrace{s \oplus s \oplus s \oplus \cdots \oplus s}_{n\ times} \tag{3}$$

In this paper, we have considered three types of structuring elements that include rhombus, square and octagon (Fig. 1). To perform multiscale processes, size of structuring template will be increased as shown in Eq. (3). The radius of structuring element (R) is one of the important parameters in establishing the power-law relationship. The radii of rhombus, square and octagon structuring elements at various iterations are diagrammatically illustrated (Fig. 1). Consequently, it was observed that the sizes of *square* and *rhombus* structuring elements increase from iteration to iteration as $S_n : R = n$. However, the size of octagon structuring element (S_n) increases with increasing cycle number (n) and its corresponding relationship is given as $S_n : R = n + 2$. Incorporating these basic morphological transformations, a morphological decomposition procedure for packing of pore–space is explained in following section.

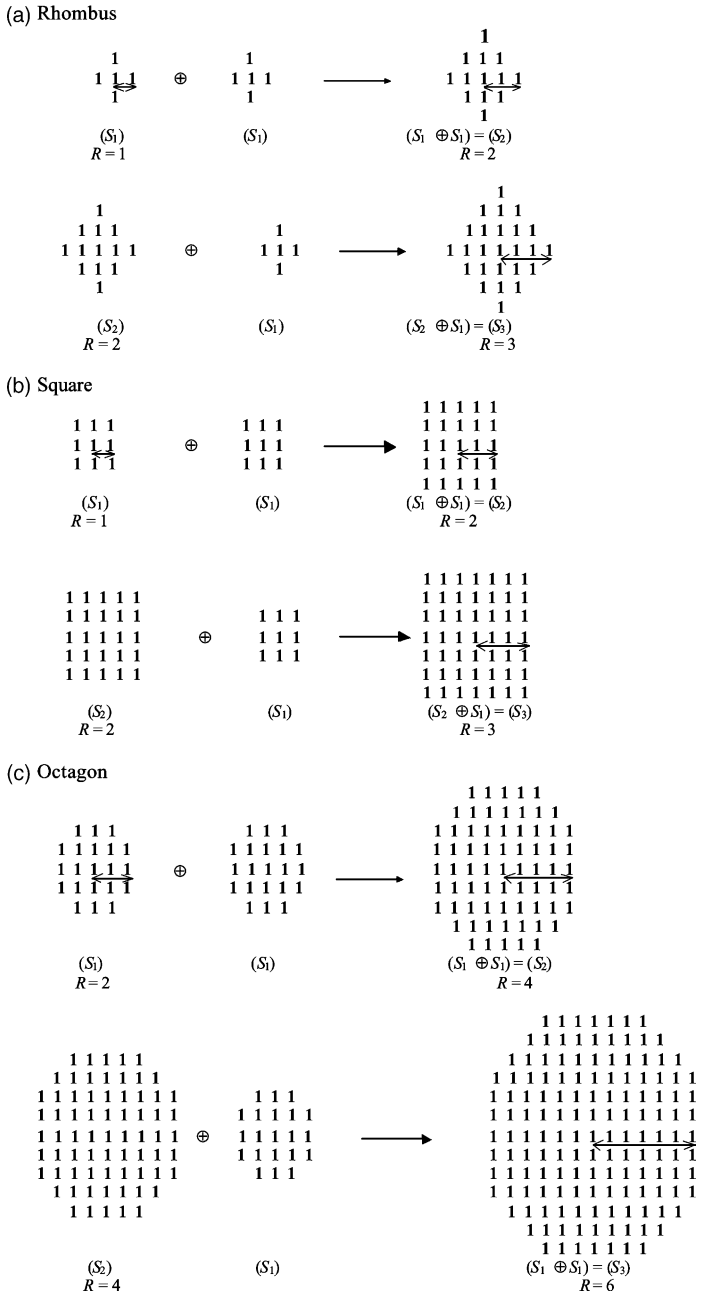

Fig. 1. Various sizes of structuring templates (a) rhombus, (b) square, and (c) octagon.

3. Model for packing of pore: a morphological approach

Tiling space with circles by putting them iteratively in each hole between three mutually touching circles and the circle that tangentially touches all three is known as "Appollonian packing" [4]. Morphological decomposition procedure for packing of fractal object was discussed in our previous study [13]. To estimate the fractal dimension of an Appollonian space (termed as a set) (Fig. 2a), morphological decomposition procedure is adapted. This procedure includes systematic use of *multiscale opening* and certain logical operators. The following set of Eq. (4) enables to show how to decompose a shape by following these morphological transformations.

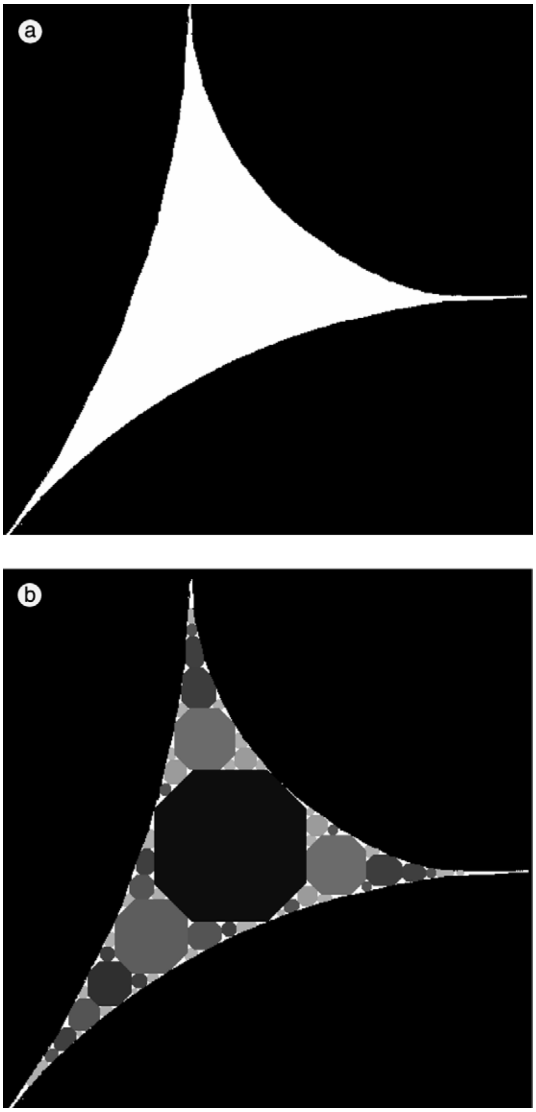

Fig. 2. (a) Appollonian space, and (b) after decomposition by means of octagon.

$$\left.\begin{array}{l} M_2 = M_1 \setminus M_1 \circ S_n \\ M_3 = M_2 \setminus M_2 \circ S_n \\ \qquad \vdots \\ M_n = M_{n-1} \setminus M_{n-1} \circ S_n \quad \text{and} \quad M_{n+1} = M_n \setminus M_n \circ S_n \\ \because \bigcup_{n=0}^{n} M_n = M \quad \text{and} \quad \because M_{n+1} = \phi \\ M_{\text{Decomp}} = (M \circ S_n) \cup (M_1 \circ S_n) \cup (M_2 \circ S_n) \cup \cdots \cup (M_{n-1} \circ S_n) \\ \because M_n \subset M_{n-1} \subset \cdots \subset M_3 \subset M_2 \subset M_1 \subset M \end{array}\right\} \quad (4)$$

After performing n times of multiscale opening on Appollonian space of which the fractal dimension is to be estimated, the *opened* version needs to be subtracted from the original version. This can be achieved by simple logical operation, which is represented with a symbol (\). If $n+1$ iterations require vanishing a set (or shape), n iterations of multiscale opening need to be performed to decompose a main set, and successively achieved subtracted portions of this set. By taking the condition that $n + 1$ iterations of multiscale opening vanishes respective set or the successive subtracted portions into consideration, each subtracted portion is subjected to further decomposition. Number of subtracted portions that may appear while decomposing a set depends on size and shape of a considered set, and of a structuring template with its characteristic information. To have a better understanding of these superficially simple morphological transformations, various steps involved in morphological decomposition procedure are represented in Eq. (4). Implementing these sequential steps by means of octagon, Fig. 2b is resulted. For better perception each level of decomposed region is color-coded. Application of these sequential steps, which are implemented on Appollonian space in this section, is extended to decompose a pore–space isolated from a sandstone image. The results and other power-law relationships are discussed in the next section.

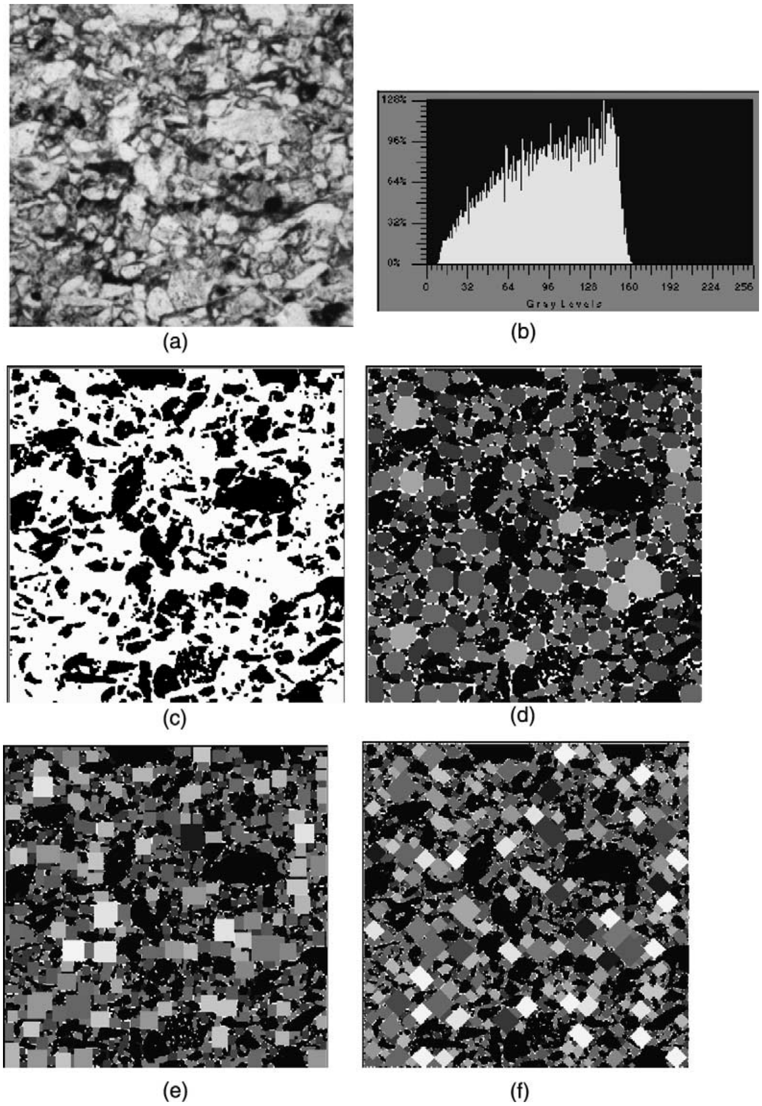

Fig. 3. (a) Sandstone image that contains both grain and pore, retrieved from SEM, (b) histogram and statistics obtained from sandstone image, (c) cross-section of sandstone grain (black in color) and pore (white in color), (d–f) decomposition of isolated pore by means of octagon, square, and rhombus structuring elements.

4. Power-law relationship, sample study, results and scope

To estimate the fractal dimension, a method by employing certain mathematical morphological transformations, which was discussed in Section 3 is used. This straightforward method is applied on a pore–space that is isolated from a sample sandstone image acquired through scanning electron microscope (SEM). As a sample study to implement the framework thus described in Section 3, the decomposition procedure is performed on a pore–space that is isolated from a sandstone image (Fig. 3a). The pore is isolated by a simple thresholding technique. The threshold gray level is taken as a value of 128 (Fig. 3b). This isolated pore image is decomposed into various sizes of octagon, square, and rhombus. This image (Fig. 3c) decomposed by means of octagon, rhombus and square structuring elements are color-coded for better understanding, and shown in Fig. 3d–f. Size distribution histograms of area of number of decomposed shapes and the radius of corresponding structuring element are also plotted (Fig. 4a–c). The number of decomposed patterns of various sizes of octagon, square, and rhombus, and other associated parameters are given in Table 1. It is apparent that the smaller the size of primary pattern that is used to decompose a pore–space, the larger the number of cycles (or) radius is required for decomposition. A power-law relationship as $N(r) \propto r^{-\alpha}$ (with $D = \alpha - 1$) is shown for this pore–space decomposed by means of octagon, rhombus and square structuring elements. The fractal dimensions of this pore–space derived by means of a number–radius relationship by these three types of structuring elements are respectively,

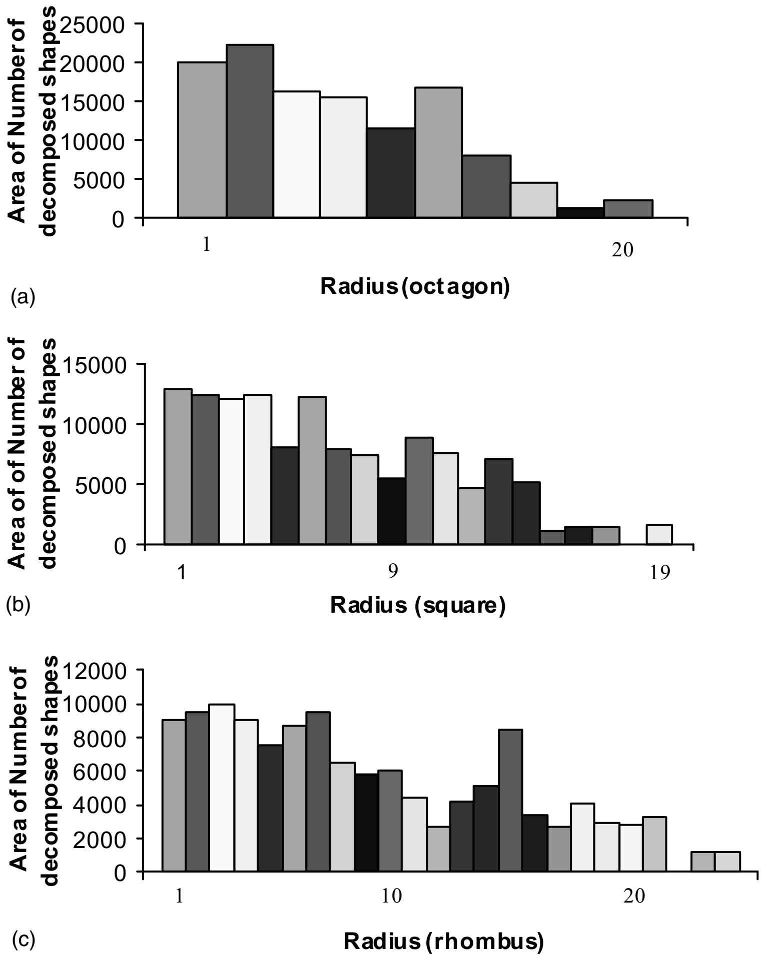

Fig. 4. Histogram plots of area of number of decomposed shapes achieved by structuring elements, (a) octagon, (b) square, and (c) rhombus, versus radii of structuring elements.

Table 1
Fractal dimensions estimated from number–radius power-law relationship

Type of structuring element	Radius (r)	Number of decomposed shapes (N)	Fractal dimension ($D = \alpha - 1$)
Octagon	20	1	1.8261
	18	1	
	16	4	
	14	10	
	12	28	
	10	25	
	8	52	
	6	98	
	4	279	
	2	629	
Square	19	1	1.764
	17	1	
	16	1	
	15	1	
	14	5	
	13	9	
	12	7	
	11	12	
	10	17	
	9	13	
	8	21	
	7	27	
	6	55	
	5	52	
	4	123	
	3	192	
	2	365	
Rhombus	24	1	1.7833
	23	1	
	21	4	
	20	3	
	19	4	
	18	6	
	17	4	
	16	7	
	15	17	
	14	13	
	13	11	
	12	8	
	11	17	
	10	27	
	9	33	
	8	48	
	7	86	
	6	102	
	5	109	
	4	177	

1.8, 1.76, and 1.78 (Fig. 5a–c). It is proved that this power-law of relationship ($N(r) \propto r^{-\alpha}$), the number of cycles required to decompose depend on radius of structuring template. In this case, a size–radius relationship is verified, which is in accord with a universal power-law proposed elsewhere [4,12].

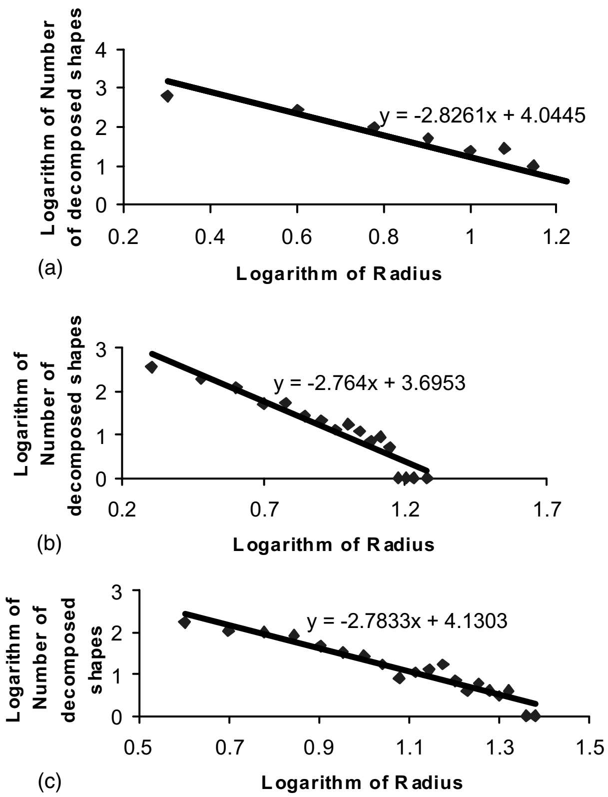

Fig. 5. Graphical plots of logarithms of number of decomposed and disconnected portions of pore versus logarithms of radius of structuring element for (a) octagon, (b) square and (c) rhombus.

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