Allometric power-law relationships in a Hortonian fractal digital elevation model

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[1] We provide a topologically viable model that is geomorphologically realistic from the point of its Hortonity and general allometric scaling laws. To illustrate this, we consider a fractal binary basin, generated in such a way that it follows certain postulates, and decompose it into various coded topologically prominent regions the union of which is defined as geomorphologically realistic Fractal-DEM. We derive two unique topological networks from this Hortonian fractal DEM based on which we derive allometric power-law relationships among the basic measures of decomposed sub-basins of all orders ranging from ω = 1 to ω = Ω. Our results are in good accord with optimal channel networks and natural river basins. *INDEX TERMS:* 1848 Hydrology: Networks; 3250 Mathematical Geophysics: Fractals and multifractals; 3210 Mathematical Geophysics: Modeling; 1824 Hydrology: Geomorphology (1625). **Citation:** Sagar, B. S. D., and T. L. Tien (2004), Allometric power-law relationships in a Hortonian fractal digital elevation model, *Geophys. Res. Lett.*, *31*, L06501, doi:10.1029/2003GL019093.

1. Introduction

[2] Self-affine properties of drainage basin can better describe the fluvial systems on earth [*Tarboton et al.*, 1988; *Rodriguez-Iturbe and Rinaldo*, 1997]. Topological or structural organization of the landscape within such a drainage basin determines two unique geomorphic networks, i.e., loopless channel and loop-like ridge connectivity networks. Former type of network is the source information from which popular Hortonian laws of number and mean channel lengths can be understood. However, the loop-like network, that is the farthest from channel network, in between which the loopless network exists. In other words, stream channels flow between ridges that are Brownian motion-like [*Takayasu*, 1990]. In a spatially distributed Digital Elevation Model (DEM), elevation contours possess Λ and V shaped crenulations: *supremums* and *infimums* that testify the presence of ridge and valley connectivity networks. There is a vast number of network types that follow allometric power-law relationships [*Maritan et al.*, 1996a, 1996b; *Rodriguez-Iturbe and Rinaldo*, 1997; *Banavar et al.*, 1999; *Veitzer and Gupta*, 2000; *Maritan et al.*, 2002; *Banavar et al.*, 2002], out of which the spatial organization of geophysical network is primarily determined by these two types of crenulations. We employ two unique connectivity networks extracted from a Hortonian fractal DEM, the internal topological organization of which is simulated through morphological decomposition procedure to verify the allometric power-laws. Specifically, we employ:

[3] (a) binary morphological erosion and dilation transformations [*Serra*, 1982] and certain logical operations to generate internal topological organization within a basin of defined fractal boundary;

[4] (b) gray level morphological erosion and dilation transformations to extract two unique connectivity networks from the DEM, which is generated at the step (a); and

[5] (c) basic measures computed from these networks and sub-basins decomposed further from this F-DEM to derive allometric power-laws.

2. Fractal DEM and Unique Connectivity Networks

[6] We define the Hortonian fractal Digital Elevation Model (DEM) M of a fluvial basin as a finite subset of two-dimensional space IR^2 that can have values between 0 and 255, each representing spatially distributed elevation region. We simulate this DEM by considering a binary fractal basin (X) that possesses 1s and 0s respectively representing topological space of the basin and its complement. We consider a specific generating mechanism to simulate boundaries of binary fractal basin at different scales by considering two postulates (a) the area of the basin is constant under succession of scale changes, and (b) the length of the channel network should be varied under the succession of scale change to make the basin Hortonian. We decompose this binary fractal basin into topologically prominent regions (TPRs) by employing morphological erosions, dilations, and logical difference and union operations to simulate fractal DEM (F-DEM). The simulation of internal topology of the basin within a defined geometric boundary, referred to as gray level fractal DEM mathematically defined by

$$M = \bigcup_{n=0}^{N} \{\{(X \ominus S_n)\ \{[(X \ominus S_n) \ominus S] \oplus S\}\} \oplus S_n\}, \qquad (1)$$

where, ⊖, ⊕, X, S, and n, respectively are morphological erosion, dilation, binary basin, a discrete rule with certain characteristic information that gets translated over X, and size of this discrete rule. $X \backslash Y$ is the part of X that is not in Y. X and S are sets in Euclidean space with elements x and s, respectively, $x = (x_1,\ldots, x_N)$ and $s = (s_1,\ldots, s_N)$ being N-tuples of element co-ordinates. Five steps in equation (1) include:

[7] (i) Successive erosion frontlines are generated *via* $(X \ominus S_n)$ by increasing the size of structuring element. The

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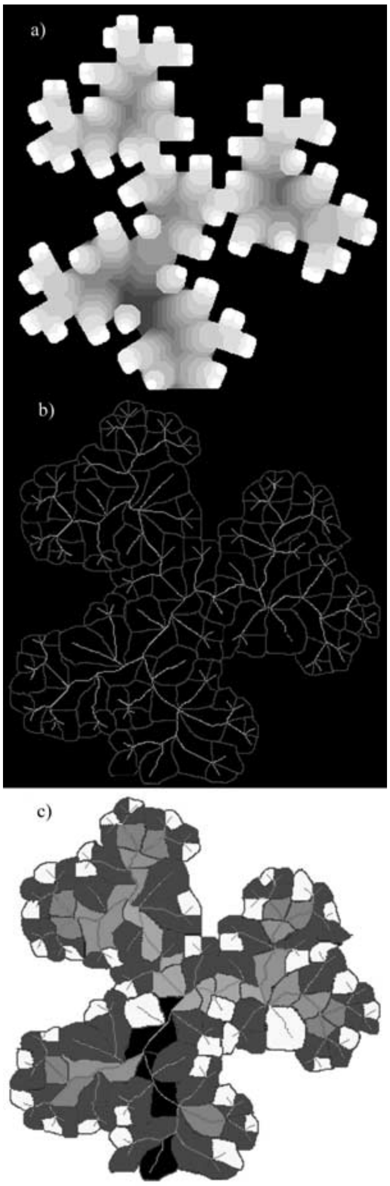

Figure 1. (a). Simulated fractal DEM achieved through morphological decomposition procedure, (b) Loop-like ridge connectivity and loopless channel connectivity networks, and (c) Sub-basins of 6th order basin. See color version of this figure in the HTML.

erosion of X with S is defined as the set of points x such that the translated S_x is contained in X and is mathematically expressed as $X \ominus S = \{x{:}S_x \subseteq X) = \cap_{s \in S} X_s$, where $-S = \{-s{:}s \in S\}$, i.e., S rotated 180° about the origin, and $S_x = \{s + x | s \in S\}$. We perform erosions iteratively to generate erosion frontlines, akin to contours, within a binary fractal basin, where it is obvious to find the existence of crenulations-like *supremums* and *infimums* that are the flow paths of two unique connectivity networks.

[8] (ii) Smoothening of the erosion frontlines is achieved *via* $[(X \ominus S_n) \ominus S] \oplus S$. Here, the dilation combines the eroded version of the eroded binary basin achieved at step (i) and S. The dilation by S is the set of all possible vector sums of pairs of elements, one coming from eroded version and the other from S, and is defined as the set of all points x such that S_x intersects X, mathematically depicted as $X \oplus S = \{x{:}S_x \cap X \neq \emptyset) = \cup_{s \in S} X_s$.

[9] (iii) Various orders of valley connectivity subsets ranging from n = 0 to N are isolated from each erosion frontline by subtracting the resultant information achieved in step (ii) from step (i).

[10] (iv) TPRs are generated by dilating the resultant information, achieved at step (iii) by S_n. This is an iterative procedure till the whole basin is converted into TPRs. Each TPR is assigned a specific shade assuming that spatially distributed TPRs are akin to spatially distributed elevation regions, and

[11] (v) Various orders of coded TPRs thus obtained are combined to achieve the DEM. By employing these sequential steps, we generate a self-affine fractal DEM (Figure 1a).

[12] Cascades of the erosion and dilation transformations possess noninvertible properties. The impact of various characteristic information of S, such as shape, size, origin and direction are redefined from the point of geomorphologic dynamics [*Sagar et al.*, 1998a]. We consider the octagonal S that is symmetric from the point of all characteristic information. Varied topological compositions of DEM can be simulated by changing the characteristics of discrete rule to further visualize realistic and unrealistic landscapes.

[13] Further, we employ gray level morphological erosions and dilations to isolate all *supremums* and *infimums* from all erosion frontlines of all spatially distributed elevation regions of simulated self-affine fractal DEM (Figure 1a) to extract loop-like and loopless ridge and channel connectivity networks. The channel network from this DEM is defined as

$$CH(M) = \bigcup_{n=0}^{N} CH_n(M), \tag{2}$$

where the channel network subsets of nth order $CH_n(M)$ are extracted by following the morphological transformations as $\{(M \ominus S_n)\} \backslash \{[(M \ominus S_n) \ominus S] \oplus S\}$. Although erosions and dilations are represented similarly as in the case of decomposition, these gray scale erosions and dilations are obtained by computing *minima* and *maxima* [see *Serra*, 1982] over M by moving window−the structuring element of size 3 × 3. These transformations are recursively performed on M by increasing the size of structuring element to obtain various orders of channel subsets. The possible channel network exists within F-DEM referred to as fractal-skeletal based channel network (F-SCN) follows Horton's laws [*Sagar et al.*, 1998b; *Sagar and Murthy*, 2000; *Sagar et al.*, 2001; *Sagar et al.*, 2003]. The ridge connectivity network subsets of nth order is defined as

$$RID(M) = \bigcup_{n=0}^{N} \left[CH_n^c(M)\right], \tag{3}$$

where ridge network subsets of nth order ($RID_n(M)$) are obtained by $\{(CH^c \ominus S_n)\} \backslash \{[(CH^c \ominus S_n) \ominus S] \oplus S\}$ in which the complement of $CH(M)$ is used to generate ridge connectivity network. The logical unions of channel and ridge network subsets obtained from M through equations (2) and (3) yield respectively the channel and ridge connectivity

Table 1. Power-Law Values Among Allometric Measures of F-DEM

Relations	Notations	For All Orders	Basin's Order					
			1	2	3	4	5	6
A and L_{mc}	h	0.55	0.502	0.56	0.56	0.55	0.55	0.56
A and P	α	1.35	1.31	1.36	1.41	1.44	1.48	1.46
P and L_{mc}	β	1.39	1.51	1.32	1.28	1.26	1.23	1.23
L_{mc} and L_{ll}	–	0.97	0.92	1.01	1.04	1.03	0.94	0.95
$L_\perp$ and L_{ll}	H	0.95	0.94	0.94	0.96	0.98	0.94	0.98
α and h	–	0.39	0.38	0.41	0.39	0.38	0.37	0.38
H and h	–	1.80	1.87	1.70	1.74	1.77	1.80	1.80
β and h	–	1.34	1.30	1.36	1.41	1.44	1.48	1.46
β and α	–	0.52	0.50	0.55	0.55	0.55	0.55	0.53
2h	D_{Lmc}	1.06	1.00	1.11	1.11	1.10	1.10	1.12
$2/\alpha$	D_P	1.48	1.53	1.47	1.42	1.39	1.35	1.37
D_{Lmc} and D_P	D_P	0.70	0.65	0.75	0.78	0.80	0.81	0.81
$1 + \frac{D_{Lmc}}{1+H}$	–	1.55	1.52	1.57	1.59	1.56	1.57	1.57

networks. The union of these two connectivity networks is depicted diagrammatically (Figure 1b) and mathematically as

$$[RID(M)] \cup [CH(M)] \qquad (4)$$

[14] Equation (4) provides a basis to estimate the contributing area of individual segments of sub-basins of all orders ranging from ω to Ω. The considered F-DEM (Figure 1a) produces 6th order branched channel network that follows Horton's laws. The fractal dimension computed by considering the two topological quantities such as bifurcation and mean channel length ratios yields a value of 1.76, in agreement with natural river basins. The number of decomposed sub-basins (Figure 1c) of respective orders from the simulated 6th order F-DEM include two 5th, five 4th, ten 3rd, 36 2nd, and 86 1st order basins.

3. Basic Measures

[15] We redefine basic measures and discuss from the point of morphological and topological bases. The total channel organization is defined as $\bigcup_{\omega=1}^{\Omega}\{[CH(\omega_{i=1}^{n}, \Omega)]M\}$. Area (A) contributed by segment i of order ω within the basin of order Ω is computed as the area embedded within the ridges that surround a segment i of order ω, $CH(\omega_i, \Omega)$. This contributing area of segment i of order ω, and its corresponding main length in longitudinal direction are respectively $A[CH(\omega_i, \Omega)]$ and $L_{mc}[CH(\omega_i, \Omega)]$. The total channel length and its total contributing area in a basin of order Ω are respectively computed as $\left\{\sum_{\omega_{i=1}^{n}=1}^{\Omega} L[CH(\omega_i, \Omega)]\right\}$ and $\left\{\sum_{\omega_{i=1}^{n}=1}^{\Omega} A[CH(\omega_i, \Omega)]\right\}$. The perimeter (P) of the sub-basin is defined as $\left\{\sum_{\omega_{i=1}^{n}=1}^{\Omega} P[CH(\omega_i, \Omega)]\right\}$, and the total perimeter of all the sub-basins is equivalent to the total ridge length. We compute these basic measures in addition to transverse length ($L_\perp$) and longitudinal length ($L_\|$) for all sub-basins ranging from order 1 to Ω for all segments ranging from 1 to n (Table A1)[1]. These mathematical formulations provide morphological and topological bases to derive the allometric power-law exponents for each channel segment of each order within a basin (M) of order Ω.

[1]Auxiliary material is available at ftp://ftp.agu.org/apend/gl/2003GL019093.

4. Scaling Laws

[16] We employ these measures estimated precisely for all the decomposed self-affine fractal sub-basins ranging from order 1 to Ω for all segments ranging from 1 to n. By employing these measures, we derive several allometric relationships, and find that these exponents follow universal power-law relationships (Table 1). We observe significant deviations among all sub-basins of all segments within a specific stream order basin. We show the allometric relations of A and L_{mc}, A and P, $L_\perp$ and $L_\|$, and $L_\perp$ and L_{mc} (Figure 2) for F-DEM. For basins, it is reported that the relationship is $L_{mc} \sim (A)^h$, with the parameter h as Hack's exponent, i.e., $0.56 < h < 0.6$ [*Hack*, 1957; *Maritan et al.*, 1996b]. Allometric power laws, in particular Hack's law and Horton's law of stream area, are based on the contributing area due to corresponding stream segments. We define a power law akin to Hack's exponent for the sub-basins of order ω by taking the length of a segment i of the basin with order ω, and its contributing area into consideration. From this, we show allometric relationship of sub-basins as $\{L_{mc}[CH(\omega_i, \Omega)]\} \sim \{A[CH(\omega_i, \Omega)]\}^h$, where h is Hack's exponent, $\omega = 1, 2, \ldots, \Omega$, and $i = 1, 2, \ldots, n$. For all the sub-basins, we find a power-law scaling of the form $\{L_\|[CH(\omega_i, \Omega)]\} \sim \{A[CH(\omega_i,$

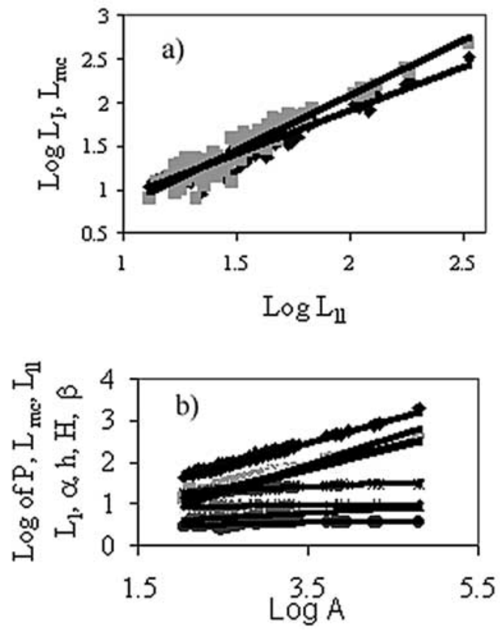

Figure 2. Allometric relationships among basic measures. (a) squares show that the relationship between $L_\|$ and $L_\perp$, and triangles indicate relationship between $L_\|$ and L_{mc}. Note that the former relationship enables the self-affinity of the basin and its sub-basins. (b) Triangles show the area-transverse length relation. The crosses indicate the area-main channel length relationship. The area – perimeter relationship is shown with diamonds, squares show the relationship between area and longitudinal relationship. The area, perimeter and mean length are in units of pixels, and Relationships between the logarithm of area of the basin and α (with open stars), h (with plus), H (with solid stars), and β (with minus). See color version of this figure in the HTML.

Table 2. Scaling Exponents for Several Networks

Network	D_{Lmc}	h	H	$1 + \frac{D_{Lmc}}{1+H}$
Scheideggar	1	2/3	1	5/3
Peono	1	1/2	1	3/2
OCN (Fractal)	1.05	0.56	0.88	1.56
F-SCN	1.06	0.55	0.95	1.54
Tirso (IT)	1.05	0.53	0.94	1.54

$\Omega)]\}^h$ where h is in the range of 0.502–0.56 (see Table 1). We find that the relationship of $h = \frac{1}{1+H}$ is very well fitted for the sub-basins, and further indicates that these sub-basins ranging from ω to Ω are self-affine basins as the $L_\perp \sim L_\|^H$ with $H <$ 1 (Table 1). We show the allometric power-law relationships for main channel lengths L_{mc} vs A, P vs A, L_{mc} vs P, and $L_\perp$ vs $L_\|$ respectively with notations h, α, β, and H. These relations yield the power-law values respectively 0.53, 1.35, 1.38 and 0.95. In several studies, power law relationships are given by considering the similar order (or) higher order Hortonian river basins. However, to investigate the extent of deviations from lower order basins to higher order basin, we provide the relationships among the basic measures of not only higher order basin, but sub-basins within the basin with order Ω as well. It is interesting to note significant deviations in the scaling laws from the lower bound $1 + \frac{D_{Lmc}}{1+H} = \frac{3}{2}$. Infinite topological random networks have asymptotically h = 0.5 [*Veitzer and Gupta*, 2002], and hence $1 + \frac{D_{Lmc}}{1+H} = \frac{3}{2}$. Table 1 contains our estimates for $1 + \frac{D_{Lmc}}{1+H}$ along with other allometric power-law exponents derived from relationships among basic measures of the F-DEM and its decomposed sub-basins with their corresponding main lengths (supporting Figures 4a and 4b)[1]. For better understanding of goodness of the encountered relations, we provide some important graphical representations in Figure 2 (and in supporting Figures 3a–3j)[1]. Further, we consider these exponents to show relationships among exponents derived for Hortonian self-affine fractal DEM. Table 2 depicts comparisons between our estimates and important allometric power-laws derived for Optimal Channel Networks (OCNs) and natural river basins. These estimates are in agreement with exponents estimated for loopless networks such as rivers, exact river networks achieved through computer simulations. Based on these power laws and generalized Horton laws, which are in accordance with universal power laws (Tables 1 and 2), it is inferred that this F-DEM and its corresponding sub-basins are geomorphologically realistic as OCNs and realistic river networks. Table 2 depicts comparative scaling laws that are available for the natural rivers, optimal channel networks with that of allometric power law exponents derived collectively from F-DEM and F-SCNs. From this comparison, it is inferred that F-SCNs and F-DEM also possess fractal properties and universal allometric relationships, as in the case of OCNs and realistic river networks.

5. Conclusions

[17] The allometric relationships of sub-basins decomposed with morphology and topology bases complement recent results derived from OCNs, random self-similar networks (RSNs) in addition to similarities from the point of their Hortonian nature. These results demonstrate the characteristics of sub-basins of a single Hortonian F-DEM through allometric power-law relationships and further indicate F-SCNs, OCNs, and natural rivers are of universality class. Using certain concepts from mathematical morphology [*Serra*, 1982], it is possible to reconstruct the basin and its topology from the network. From such a reconstructed basin, it is also possible to attain the network much similar to the network, based on which basin is reconstructed. The similarities in the OCN, RSN, and the networks extracted from the reconstructed basins provide an approach to relate physical mechanisms with discrete rules, and may facilitate further comparative analysis. We hypothesize that the allometric power-laws derived from basins reconstructed from the various types of networks of various orders, existence of which may be due to different types of landscape organization, provide additional indices. What is most interesting is that the derivation of a discrete rule from the networks and their corresponding basins, whether they are realistic, OCNs or F-SCNs, to accurately reconstruct the landscape organization from network. The investigation of the landscape organizations and their corresponding networks that are (not) following universal power laws may provide potentially valuable insights in understanding different channelization processes.

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