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GENERATION OF A FRACTAL LANDSCAPE USING NONLINEAR MATHEMATICAL MORPHOLOGICAL TRANSFORMATIONS

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Abstract

A third-order Koch quadric fractal is used to generate a fractal landscape. To generate this artificial landscape, nonlinear mathematical morphological transformations are used to topologically decompose the binary fractal basin. The decomposed regions of prominence are coded with different shades for better perception. The 3D surfaces are generated for the topologically decomposed and coded fractal, which resembles a landscape possessing alluvial fans, of interest to theoretical geomorphologists.

1. INTRODUCTION

The topological characteristics of a third-order Koch quadric fractal[1] are studied here. The study aims to generate a fractal landscape by following set theory-based transformations and coding techniques. A binary fractal basin is decomposed into topologically prominent regions (TPRs) by following nonlinear mathematical morphological transformations.[2] The TPRs will then be coded with shades prior to the generating of a landscape construction of 3D surfaces. This is a maiden attempt which includes morphological rules in the characterization of binary fractal topology for generation of fractal landscape. This paper consists of a total of five sections. In Sec. 2, a brief introduction on mathematical morphological transformations is given. This is followed by their importance in decomposition of fractal into flow direction network

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(FDN) in Sec. 3, decomposition of fractal into TPRs with coding in Sec. 4, and finally a sample study applying the morphology-based framework is shown in Sec. 5.

2. MATHEMATICAL REPRESENTATION OF MORPHOLOGICAL PROCESSES

Most of the mathematical formalism and notations are adopted from Serra.[2] Mathematical morphology based on set theoretic concepts is a particular approach to the analysis of geometric properties of different structures. The main objective is to study the geometrical properties of a natural feature represented as a binary image through the investigation of its microstructures by means of "structuring templates," following Serra's concept.[2] It aims to extract information about the geometrical structure of an object (e.g. water body, basin, channel networks, section of water bodies, etc.) by mathematical morphological concepts. In this, a specific fractal is subjected to transformations by means of another object called the structuring template. The main characteristics of the structuring template are, shape, size, origin and orientation. The topological characteristics of fractal (e.g. spatial distribution, morphology, connectivity, convexity, smoothness and orientation) can be characterized by different structuring templates. According to Matheron's[3] approach, each image object is assumed to contain its boundary, and thus can be represented by a closed subset of Euclidean space. In addition, many structuring templates can be represented by a compact subset of E, so that constraints which correspond to the four principles of the theory of mathematical morphology (such as invariance under translation, compatibility with change of scale, local knowledge and uppersemicontinuty[2]) will be imposed on morphological set transformations (dilation, erosion, opening and closing) for precise extraction of topological information from the fractal.

Dilation, *erosion*, *opening* and *closing* are the simplest quantitative morphological set transformations. These transformations are based on Minkowski set addition and subtraction.[2] The Minkowski set addition of two sets, M and S, is shown as follows [Eq. (1)].

$$M \oplus S = \{m + s : m \in M, s \in S\} = \mathrm{U}_{s\in S} M_s\,. \quad (1)$$

M and S consists of all points c which can be expressed as an algebraic vector addition $c = m + s$, where the vectors m and s respectively belong to M and S.

The Minkowski set subtraction of S from M is denoted as

$$M \Theta S = (M^c \oplus S)^c = \bigcap_{s\in S} M_s\,. \quad (2)$$

Let M be a binary image where the pixels with zeros are marked with a dot for a better legibility. Structuring element S will be moved from top to bottom and left to right by applying the criterion of erosion principle to achieve shrinking. When the rectangle S is centered on one point of the frame of the image M, then it will be truncated and only its intersection with the shape is kept. The discrete binary image M is defined as a finite subset of Euclidean 2D space IR^2. The geometrical properties of a binary image possessing set (M) and set complement (M^c) are subjected to the morphological functions. From geometrical point of view, morphological dilations and erosions are defined as set transformations that expand and contract a set. The morphological operators can be visualized as working with two images. The image being processed is referred to as the fractal while the other image is that of the structuring template. Each structuring template has a designed shape that can be thought of as a probe of the main feature. The three morphological transformations involved in this study are dilation to expand, erosion to shrink, and cascade of erosion-dilation to smoothen the set.

Dilation. Dilation combines two sets using vector addition of set elements. If M and S are sets in Euclidean space with elements m and s, respectively, $m = (m_1, \ldots, m_N)$ and $s = (s_1, \ldots, s_N)$ being N-tuples of element coordinates, then the dilation of M (binary fractal) by S (structuring template) is the set of all possible vector sums of pairs of elements, one coming from M and the other from S. The dilation of a set M with structuring template S is defined as the set of all points m such that S_m intersects M. It is expressed as Eq. (3). It is worth mentioning here that as long as the structuring element is symmetric ($S = \hat{S}$) Minkowski's addition and subtraction are respectively similar to morphological dilation and erosion. The considered structuring template in the present investigation is symmetric.

$$M \oplus S = \{m : S_m \cap M\} = \mathrm{U}_{s\in S} M_s\,. \quad (3)$$

Properties: The small holes inside the image and gulfs on the boundary will be closed by this dilation transformation. This operation enlarges the objects and connects neighboring particles.

Erosion. The erosion of an image M with structuring template S is defined as the set of points m such that the translated S_m is contained in M. It is expressed as Eq. (4).

$$M\Theta S = \{m{:}S_m \subseteq M\} = \bigcap_{s\in S} M_s \tag{4}$$

where $S = \{s{:}s \in S\}$, i.e. S rotated 180° round the origin.

Properties: Isolated points and the small particles will be removed by this operation. It shrinks the other particles, discards peak on the boundary of the object, and disconnects.

The cascades of erosion-dilation and dilation-erosion are respectively termed as the opening and the closing transformations which are also nonlinear transformations.

3. DECOMPOSITION OF BINARY FRACTAL INTO FDN

The region-based technique is used to represent the abstract structure of an object. Morphological skeletons, an example of abstract structures, are used to represent the FDN of the fractal for topological analysis and classification. The process of skeletonization for binary fractal region characterization is sensitive to the wrinkle in the outline. A highly contorted fractal produces more intricate FDNs. The angular points in the successive front-lines indicate the spurs. Successive front-lines indicate the boundaries of all possible eroded sets that can be extracted using the concept of structuring template ranging from smaller to bigger sizes. The width of the angular point in the boundary is smaller compared to that of the successive front-lines of the binary fractal. The structural composition of the binary fractal reflects through the characteristics of angular points in the successive front-lines. The combination of all possible angular points in all possible front-lines produces an aggregated network of arboreal type. The sources of thin dendrites are the dendrites of larger size. In order to segregate the binary fractal into several regions of various orders, an order designation similar to river network order designation can be given. However, this is not within the scope of the study.

3.1 Extraction of FDNs

FDN is a line thinned caricature to summarize the shape, size, orientation and connectivity of the binary fractal. It is defined as the union of maximum possible angular points that can be isolated from all possible successive front-lines of a binary fractal. The FDN of a binary fractal (M), FDN(M), viewed as a subset of IR^2 (2D space), is defined mathematically as

$$\begin{gathered}\mathrm{FDN}_n(M) = (M\Theta S_n)/\{[(M\Theta S_n)\Theta S]\oplus S\}\,,\\ n = 0,1,2,\ldots,N\end{gathered} \tag{5}$$

where M and S represent binary fractal and the structuring template, respectively. In the above expression, the opening of an eroded set is always by means of a structuring template of arbitrary size. Those parts remaining after set difference between the eroded set and its opening are FDN subsets.

$$\mathrm{FDN}(M) = \mathrm{U}_{n=0}^{N}\mathrm{FDN}_n(M) \tag{6}$$

where $\mathrm{FDN}_n(M)$ denotes the nth subset of binary fractal (M). In the above expression, while subtracting from the eroded versions of M, their opening by S retains only the angular points (which are points of the FDN). The union of all such possible points produces FDNs. The FDN patterns and their structural composition depend on the characteristic information of structuring template (S). Asymmetric 2D morphological structuring templates produce irregular FDN. In contrast, symmetric 2D structuring templates produce symmetric FDNs. Structuring templates of bounded type produce only the aggregated type of FDNs. The isolation of angular points in the successive eroded sets of a binary fractal consists of just the isolation of crenulations in a contour. The boundaries of successive eroded sets, also termed as successive front-lines, need to be extracted. Equations (5) and (6) are to extract FDNs.

4. BINARY FRACTAL DECOMPOSITION INTO VARIOUS REGIONS OF PROMINENCE FROM FDN AND CODING TO GENERATE FRACTAL DEM

The topography of the landscape depends on the morphological rule used to decompose the internal topography which varies from region to region. It may be used to decompose the binary fractal into regions of prominence. It is visualized that binary fractal contains several regions of prominence. Each region will be designated according to the prominence with elevation.

Independent of the structuring template used for extraction of FDN, the resulting FDN subsets are able to exactly reconstruct the original binary fractal using Eq. (7). Thus the total information in the original finite binary fractal (M) is equivalent to that in the finite ensemble of all its flow direction subsets $\mathrm{FDN}_n(M)$ together with their corresponding index "n". The sequential phases involved includes generation of a binary fractal (M), the extraction of FDN subsets, the dilation of FDN subsets, and the composition of the basin (M).

$$M = \mathrm{U}_{n=0}^{N} M_n \,, \tag{7}$$

where

$$M_n = [\mathrm{FDN}_n(M) \oplus S_n] \,.$$

The individual FDN subsets ranging from 0 to N of binary fractal (M) are dilated by structuring templates of respective sizes (S_0 to S_n) used to decompose the binary fractal into various regions of prominence (M_0 to M_n). The morphological characteristics of regions of prominence depend on the characteristics of the structuring template (S). In this study, the considered structuring template is a square. The expansion of Eq. (7) after coding of each reconstructed elevation is as follows:

$$\begin{aligned} &\mathrm{FDN}_0(M) \oplus S_0 = M_0^1; \\ &\mathrm{FDN}_1(M) \oplus S_1 = M_1^2; \ldots; \\ &\mathrm{FDN}_{n-1}(M) \oplus S_{n-1} = M_{n-1}^N; \\ &\mathrm{FDN}_n(M) \oplus S_n = M_n^{N+1} \,. \end{aligned} \tag{8}$$

To differentiate various topologically prominent regions, which may have relevance in terms of topographic elevations, nth region is coded by $N+1$. The superscripts describe the color assignment. These coded subregions are represented as elevation regions of several orders.

$$M_0^1 \mathrm{U} M_1^2 \mathrm{U} \cdots \mathrm{U} M_{n-1}^N \mathrm{U} M_n^{N+1} = \text{Fractal DEM} \,. \tag{9}$$

The above expression describes the process of reconstructing the binary fractal from its FDN subsets and then into a fractal digital elevation model (DEM). In the above expression various shades are assigned to various regions of prominence. These decomposed regions of prominence are assumed as various regions of topological prominence. The FDN subsets after respective degree of dilations and coding, are unified systematically.

5. A SAMPLE STUDY

To decompose a binary fractal into several regions of prominence, certain transformations from the field of mathematical morphology (described in the previous sections) are considered. The decomposed binary fractal subsets will be dilated by a specific structuring template to find out the various regions of prominence. In the following, how a binary fractal is decomposed into various regions of prominence is detailed as a sample study.

A binary fractal (Fig. 1) is considered to show a sample study. The FDN (Fig. 2) is extracted according to the procedures detailed in Eqs. (5) and (6). By implementing this procedure, the decomposed FDN subsets of this binary fractal are dilated to the same degree in order to decompose the binary fractal into its regions of prominence. Figure 3 shows the simulated DEM with various regions of topological prominence. A square structuring

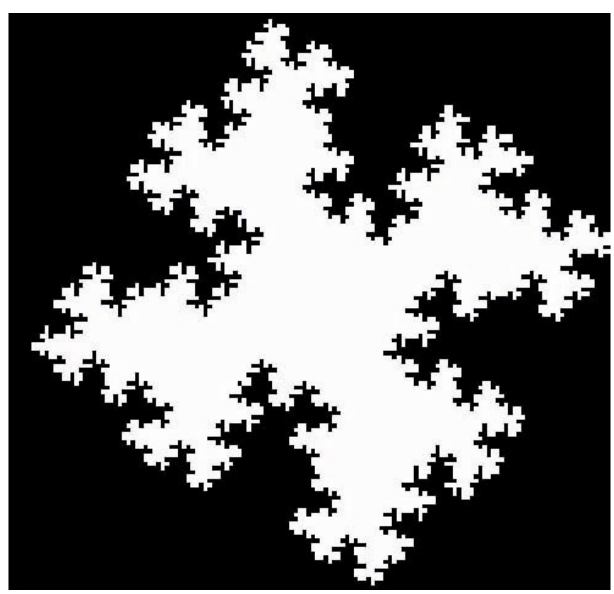


Fig. 1 A third-order Koch quadric binary fractal basin.

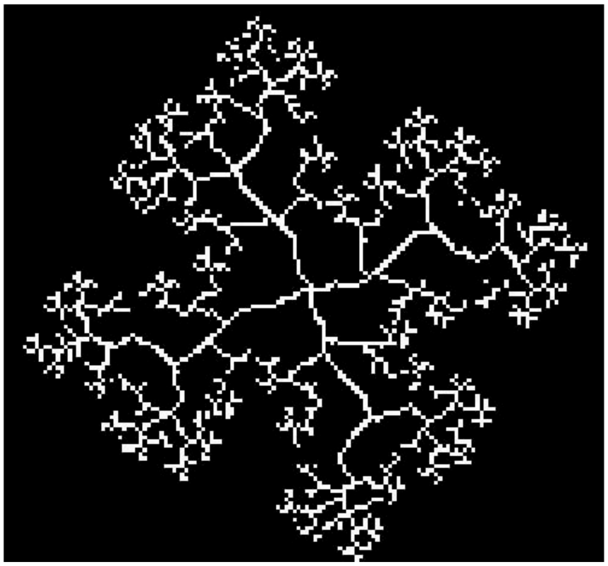


Fig. 2 Fluid FDN extracted from binary fractal basin.

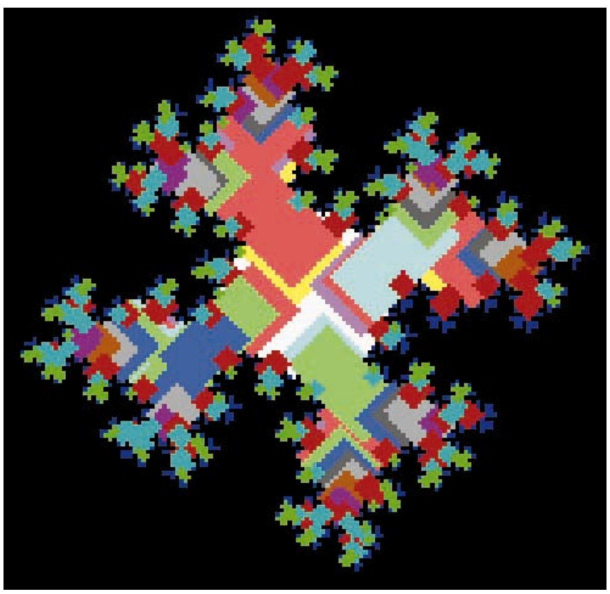
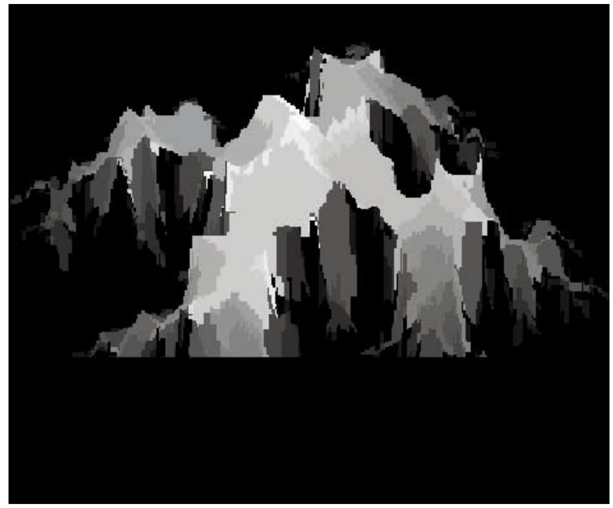
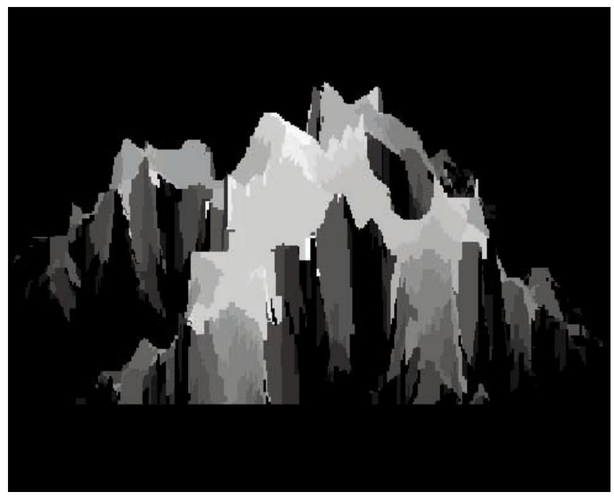


Fig. 3 A binary fractal basin after decomposition into TPRs.



(a)



(b)

Fig. 4 A fractal landscape generated from Fig. 3. Light and dark regions of DEM are visualized as high and low elevations, respectively (vertical exaggeration: **(a)** 5 and **(b)** 7).

template is considered for a similar decomposition. However, other types of structuring templates unravel other topological characteristics of the landscape. In this sample study, the union of dilated and coded FDN subsets starts from $n = N$ to $n = 0$. Various regions indicated by different shades represent various elevation levels in simulated DEM. Individual FDN subsets are dilated to the same degree and coded with respective shades by following the sequential steps depicted in Eqs. (8) and (9), producing a transcendental DEM (Fig. 3) from binary fractal (Fig. 1). The binary fractal basin is decomposed into various TPRs, the surface of which is akin to the fractal landscape. Each of the shaded region is treated as a specific region of elevation in the DEM. Light and dark regions are assumed to represent higher and lower elevations, respectively. The 3D surfaces are plotted for this DEM (Fig. 3) with vertical exaggerations 5 [Fig. 4(a)] and 7 [Fig. 4(b)]. The variations in the fractal landscape topography are subjected to change in the shape and other characteristic information of the structuring template. It is worth while to mention that the morphology of regions of prominence (extracted by decomposing the binary fractal using the procedures detailed sequentially) is liable to vary with changing structuring templates. The precision depends on the design of structuring template. The design of the structuring template can be made by taking into consideration, the morphological characteristics of each elevation level and interrelationships among all the spatially distributed elevation levels from a morphological standpoint; and an asymmetric structuring template (S), where S is not equal to the transpose of S, can also be considered to have more realistic landscapes. The structural variation in the surface topography determines the formation of dendrites which is a natural phenomenon. The topological description of the binary fractal provides a basis for classification of the internal region that is topologically important. This study may be useful to show some meaningful inferences with elevation characteristics. This study is of practical interest to geomorphologists, as the simulated landscape and FDNs are akin to the natural landscape possessing alluvial fans.

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